Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 11, No. 5, (May 2016), pp. 681–700
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI

Some results on fuzzy bicomplex numbers

Sanjib Kumar Datta, Tanmay Biswas

Received 6 July 2015; Accepted 30 September 2015

Abstract. In this paper we wish to define fuzzy bicomplex numbers and establish some properties of it.

2010 AMS Classification: 08A72, 30G35, 32A30

Keywords: Fuzzy set, Fuzzy number, Fuzzy bicomplex number, Fuzzy bicomplex set.

Corresponding Author: Sanjib Kumar Datta (sanjib_kr_datta@yahoo.co.in)

1. Introduction, definitions and notations

The space of bicomplex numbers $\mathbb{C}_2$ is the first in an infinite sequence of multicomplex spaces which are the generalizations of the space of complex numbers $\mathbb{C}$. We write regular complex number as $z = x + iy$ where x and y are real numbers and $i^2 = -1$.

To start our paper we just recall the following definitions:

Definition 1.1. The set of bicomplex numbers is defined as :

$$\mathbb{C}_2 = \{w : w = p_0 + i_1p_1 + i_2p_2 + i_1i_2p_3, p_k \in \mathbb{R}, 0 \leq k \leq 3\}.$$

Since each elements in $\mathbb{C}_2$ can be written as

$$w = p_0 + i_1p_1 + i_2 (p_2 + i_1p_3)$$

or

$$w = z_1 + i_2z_2,$$

we can express $\mathbb{C}_2$ as

$$\mathbb{C}_2 = \{w = z_1 + i_2z_2 \mid z_1, z_2 \in \mathbb{C}\},$$

where $z_1 = p_0 + i_1p_1$, $z_2 = p_2 + i_1p_3$ and i_1, i_2 are independent imaginary units such that $i_1^2 = -1 = i_2^2$. the product of i_1 and i_2 defines a hyperbolic unit j such that $\mathrm{j}^2 = 1$. The products of all units is commutative and satisfies

$$i_1i_2 = j, \quad i_1j = -i_2, \quad i_2j = -i_1.$$

Definition 1.2. Three kinds of conjugate can be defined on bicomplex numbers. The bicomplex conjugates w^t of w are defined as

$$\text{(i)}\ w^{t_1} = \bar{z_1} + i_2\bar{z_2}, \quad \text{(ii)}\ w^{t_2} = z_1 - i_2 z_2, \quad \text{and}\ \text{(iii)}\ w^{t_3} = \bar{z_1} - i_2\bar{z_2},$$

where the bar $\left(^{-}\right)$ denotes the complex conjugate in $\mathbb{C}$.

Definition 1.3. With each kind of conjugate, one can define a specific bicomplex modulus in the following manner:

$$\begin{aligned} \text{(i)}\ |w|_{t_1}^2 &= w.w^{t_1} = \left(|z_1|^2 - |z_2|^2\right) + 2i_2\Re\left(z_1\bar{z_2}\right), \\ \text{(ii)}\ |w|_{t_2}^2 &= w.w^{t_2} = z_1^2 + z_2^2 \text{ and} \\ \text{(iii)}\ |w|_{t_3}^2 &= w.w^{t_3} = \left(|z_1|^2 + |z_2|^2\right) + 2j\Im\left(z_1\bar{z_2}\right) . \end{aligned}$$

Definition 1.4. For a bicomplex number $w = z_1 + i_2 z_2$, the norm denoted as $||w = z_1 + i_2 z_2||$ is defined in the following manner:

$$\begin{aligned} \|z_1 + i_2 z_2\| &= \left(|z_1|^2 + |z_2|^2\right)^{\frac{1}{2}} \\ &= \left(\frac{|z_1 - i_1 z_2|^2 + |z_1 + i_1 z_2|^2}{2}\right)^{\frac{1}{2}} . \end{aligned}$$

When $w = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3$, for $p_k \in \mathbb{R}, 0 \leq k \leq 3$ then

$$\|w\| = \left(p_0^2 + p_1^2 + p_2^2 + p_3^2\right)^{\frac{1}{2}} .$$

Remark 1.5. We observe that the bicomplex conjugates $w^{t_k} \mid k = 1, 2, 3$ of w satisfy the following properties :

$$\begin{gathered} (1)\quad w^{t_k^{t_k}} = w\ ,\ (2)\ \left(w_1 \pm w_2\right)^{t_k} = w_1^{t_k} \pm w_2^{t_k}\ , \\ (3)\ \left(w_1 \cdot w_2\right)^{t_k} = w_1^{t_k} \cdot w_2^{t_k},\ (4)\ \left(\frac{w_1}{w_2}\right)^{t_k} = \frac{w_1^{t_k}}{w_2^{t_k}}, \\ (5)\ |w|_{t_k} = \left|w^{t_k}\right|_{t_k} \text{ and } (6)\ \|w\| = \left\|w^{t_k}\right\|. \end{gathered}$$

The idea of fuzzy subset μ of a set X was primarily introduced by L.A. Zadeh [9] as a function $\mu : X \to [0, 1]$. Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. Among the various types of fuzzy sets, those which are defined on the universal set of real numbers or complex numbers under certain conditions, be viewed as fuzzy real numbers or fuzzy complex numbers respectively. Several researchers have done extensive works in the field of fuzzy complex numbers. For references one can see [2, 3, 4, 8, 9]. Now we wish to give a suitable definition of fuzzy bicomplex number in the following manner:

Definition 1.6. A fuzzy set w_f may be defined by its membership function $\mu\left(w \mid w_f\right)$ which is a mapping from the bicomplex numbers $\mathbb{C}_2$ into $[0, 1]$ where w is a regular bicomplex number as $w = z_1 + i_2 z_2$, is called a fuzzy bicomplex number if it satisfies the following conditions:

1. $\mu\left(w \mid w_f\right)$ is continuous,

2. An α-cut of w_f which is defined as $w_f^{\alpha} = \{w \mid \mu(w \mid w_f) > \alpha\}$, where $0 \leq \alpha < 1$, is open, bounded, connected and simply connected and
3. $w_f^1 = \{w \mid \mu(w \mid w_f) = 1\}$ is non-empty, compact, arc wise connected and simply connected.

If w_f^{α} where $0 \leq \alpha < 1$ is open and connected then it is automatically arc wise connected. The simply connected assumption is to assure that w_f^{α}, $0 \leq \alpha \leq 1$, will not contain any holes. w_f^1 being non-empty means that all fuzzy bicomplex numbers are normalized i.e., $\mu(w \mid w_f) = 1$ for some w.

In the sequel we now present some definitions that are also used in this paper.

Definition 1.7. The bicomplex conjugates $w_f^{t_k} \mid k = 1, 2, 3$ of w_f may be defined as

$$\begin{aligned}
\text{(i)}\ \mu\left(w \mid w_f^{t_1}\right) &= \mu\left(w^{t_1} \mid w_f\right), \\
\text{(ii)}\ \mu\left(w \mid w_f^{t_2}\right) &= \mu\left(w^{t_2} \mid w_f\right) \text{ and} \\
\text{(iii)}\ \mu\left(w \mid w_f^{t_3}\right) &= \mu\left(w^{t_3} \mid w_f\right).
\end{aligned}$$

where $w^{t_k} \mid k = 1, 2, 3$ are the bicomplex conjugates of w stated in Definition 1.2.

The bicomplex conjugates $w_f^{t_k} \mid k = 1, 2, 3$ of a fuzzy bicomplex number w_f is also a fuzzy bicomplex number because the mappings $w = z_1 + i_2 z_2 \to w^{t_1} = \bar{z_1} + i_2 \bar{z_2}$, $w = z_1 + i_2 z_2 \to w^{t_2} = z_1 - i_2 z_2$ and $w = z_1 + i_2 z_2 \to w^{t_3} = \bar{z_1} - i_2 \bar{z_2}$ are continuous.

Definition 1.8. The modulus $|w_f|_{t_k} \mid k = 1, 2, 3$ of a fuzzy bicomplex number w_f may be defined by

$$\begin{aligned}
\text{(i)}\ \mu\left(|w|_{t_1} \mid |w_f|\right) &= \sup\left\{\mu(w \mid w_f) \mid |w|_{t_1} = \left[\ \left(|z_1|^2 - |z_2|^2\right)\}\ \ + 2i_2 \Re\left(z_1 \bar{z_2}\right)\right]^{\frac{1}{2}}\right\}, \\
\text{(ii)}\ \mu\left(|w|_{t_2} \mid |w_f|\right) &= \sup\left\{\mu(w \mid w_f) \mid |w|_{t_2} = \left[z_1^2 + z_2^2\right]^{\frac{1}{2}}\right\} \text{ and} \\
\text{(iii)}\ \mu\left(|w|_{t_3} \mid |w_f|\right) &= \sup\left\{\mu(w \mid w_f) \mid |w|_{t_3} = \left[\ \left(|z_1|^2 + |z_2|^2\right)\ \ + 2j \Im\left(z_1 \bar{z_2}\right)\right]^{\frac{1}{2}}\right\}.
\end{aligned}$$

Definition 1.9. The norm $\|w_f\|$ of a fuzzy bicomplex number w_f may be defined in the following manner:

$$\mu(r \mid \|w_f\|) = \sup\{\mu(w \mid w_f) \mid \|w\| = r\},$$

where r is the norm of w.

If we consider $f(w_1, w_2) = y$ be any mapping from $\mathbb{C}_2 \times \mathbb{C}_2$ into $\mathbb{C}_2$, then we may extend f to $\bar{\mathbb{C}_2} \times \bar{\mathbb{C}_2}$ into $\bar{\mathbb{C}_2}$ using the extension principle where $\bar{\mathbb{C}_2}$ denotes the space of fuzzy bicomplex numbers. So we may write $f\left(w_{f_1}, w_{f_2}\right) = y_f$ if

$$\mu(y \mid y_f) = \sup\{\Lambda(w_1, w_2) \mid f(w_1, w_2) = y\},$$

where

$$\Lambda(w_1, w_2) = \min\left\{\mu(w_1 \mid w_{f_1}),\ \mu(w_2 \mid w_{f_2})\right\}.$$

One obtains $y_f = w_{f_1} + w_{f_2}$ or $y_f = w_{f_1} \cdot w_{f_2}$ by using $f(w_1, w_2) = w_1 + w_2$ or $f(w_1, w_2) = w_1 \cdot w_2$ respectively. For subtraction, we first define $-w_f$ in the following manner:

$$\mu(w \mid -w_f) = \mu(-w \mid w_f) \ ,$$

and then set

$$w_{f_1} - w_{f_2} = w_{f_1} + (-w_{f_2}) \ .$$

We also define the reciprocal w_f^{-1} of w_f as

$$\mu(w \mid w_f^{-1}) = \mu(w^{-1} \mid w_f) \ .$$

Now we consider some open surface centered at $0 \ \ (0 + i_1 0 + i_2 0 + i_1 i_2 0)$ disjoint from w_f^0 . If w_f^0 is not bounded away from zero, then w_f^{-1} remains undefined. When 0 belongs to *supp* (w_f) , then *supp* $\left(w_f^{-1}\right)$ will not be bounded and by our definition of fuzzy bicomplex numbers, w_f^{-1} will not be fuzzy bicomplex number.

For the division of two fuzzy bicomplex numbers w_{f_1} and w_{f_2}, we may write

$$\frac{w_{f_1}}{w_{f_2}} = w_{f_1} \cdot w_{f_2}^{-1}.$$

Next we wish to give an alternative definition of fuzzy bicomplex number in terms of fuzzy complex numbers in the following way:

Definition 1.10. If z_{f_1} and z_{f_2} are any two fuzzy complex numbers with membership functions $\mu(z_1 \mid z_{f_1})$ and $\mu(z_2 \mid z_{f_2})$ respectively, then

$$w_f = z_{f_1} + i_2 z_{f_2}$$

is a fuzzy bicomplex number with membership function

$$\mu(w \mid w_f) = \min\left(\mu(z_1 \mid z_{f_1}) \ , \ \mu(z_2 \mid z_{f_2})\right)$$

where $w = z_1 + i_2 z_2$.

In this paper we wish to establish some few results related to fuzzy bicomplex numbers on the basis of its definitions stated above.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 ([1]). *If z_f be any fuzzy complex number then $|z_f|^\alpha = \left|z_f^\alpha\right|$ where $0 \leq \alpha \leq 1$ and $|z_f|$ is a truncated real fuzzy number.*

Lemma 2.2 ([5]). *If M and N be any two real fuzzy numbers then $(M+N)^\alpha = M^\alpha + N^\alpha$ and if $M \geq 0, N \geq 0$ then $(M.N)^\alpha = M^\alpha . N^\alpha$.*

3. Main results

In this section we present the main results of the paper.

Theorem 3.1. *Let $w_{f_1}, w_{f_2}, ...w_{f_n}$ be any n number of fuzzy bicomplex numbers. Also let $B = w_{f_1} + w_{f_2} + ... + w_{f_n}$. Then for any α, $0 \leq \alpha \leq 1$,*

$$A^{\alpha} = B^{\alpha} \ .$$

holds where

$$A^{\alpha} = \left\{ w_1 + w_2 + ... + w_3 \mid (w_1, w_{2,}...w_n) \in w_{f_1}^{\alpha} \times w_{f_2}^{\alpha} \times .... \times w_{f_n}^{\alpha} \right\} \ .$$

Proof. **Case I.** Let us suppose $0 \leq \alpha < 1$.

Now let

$$w_1 + w_2 + ... + w_n = w \in B^{\alpha} \ . \tag{3.1}$$

Then it follows from (3.1) that

$$\Lambda (w_1, w_{2,}...w_n) > \alpha \text{ as } \mu (w \mid B) > \alpha \ .$$

This implies that

$$\mu \left(w_1 \mid w_{f_1}\right) > \alpha,\ \mu (w_2 \mid w_{f_2}) > \alpha, \cdots \text{ and } \mu (w_n \mid w_{f_n}) > \alpha$$

which implies that

$$(w_1, w_{2,}...w_n) \in w_{f_1}^{\alpha} \times w_{f_2}^{\alpha} \times .... \times w_{f_n}^{\alpha}$$

$$\text{i.e., } \ w \in B^{\alpha} \Rightarrow w \in A^{\alpha} \ . \tag{3.2}$$

Again let us suppose that

$$w_1 + w_2 + ... + w_n = w \in A^{\alpha} \ . \tag{3.3}$$

Therefore from (3.3) we obtain that

$$\mu \left(w_1 \mid w_{f_1}\right) > \alpha,\ \mu (w_2 \mid w_{f_2}) > \alpha, \cdots \text{ and } \mu (w_n \mid w_{f_n}) > \alpha$$

which implies that

$$\Lambda (w_1, w_{2,}...w_n) > \alpha \ .$$

Therefore from above it follows that $\mu (w \mid B)$ also exceeds α and so

$$w \in A^{\alpha} \Rightarrow w \in B^{\alpha} \ . \tag{3.4}$$

Thus from (3.2) and (3.4) we get that

$$A^{\alpha} = B^{\alpha} \text{ for } 0 \leq \alpha < 1 \ . \tag{3.5}$$

Case II. Let $\alpha = 1$.

We may find n number of fuzzy bicomplex numbers $w_1, w_{2,}...$ and w_n so that $w_1 + w_2 + ... + w_n = w$ and $w \in B^1$.

We can also find w_{1m} in *supp* $\left(w_{f_1}\right)$, w_{2m} in *supp* $(w_{f_2}) \cdots$ and w_{nm} in *supp* $\left(w_{f_n}\right)$ so that $w_{f_{1m}} + w_{f_{2m}} + ... + w_{f_{nm}} = w$ where $m = 2, 3, 4, \cdots\cdots$ and

$$\Lambda (w_1, w_2, ..., w_n) > 1 - \frac{1}{m} \ .$$

Since the supports are compact we may choose a subsequence $w_{1m_k} \to w_1$, $w_{2m_k} \to w_2 \cdots$ and $w_{1n_k} \to w_n$ with $w_1 + w_2 + ... + w_n = w$ and $\Lambda (w_1, w_2, ..., w_n) \geq 1$ because Λ is continuous.

This implies that

$$w_1 \in w^1_{f_1},\, w_2 \in w^1_{f_2} \ \cdots w_n \in w^1_{f_n}.$$

So

$$w \in B^1 \Rightarrow w \in A^1 \ . \tag{3.6}$$

Also let

$$w_1 + w_2 + ... + w_n = w \in A^1 \ ,$$

which implies that

$$\Lambda\left(w_1, w_2, ..., w_n\right) = 1 \ .$$

Therefore from above it follows that

$$\mu\left(w \mid B\right) = 1$$

$$\text{i.e., } w \in A^1 \Rightarrow w \in B^1 \ . \tag{3.7}$$

Now from (3.6) and (3.7) we obtain that

$$A^\alpha = B^\alpha \text{ for } \alpha = 1 \ . \tag{3.8}$$

Thus the theorem follows from (3.5) and (3.8) . □

Theorem 3.2. *Let $w_{f_1}, w_{f_2}, ...w_{f_n}$ be any n number of fuzzy bicomplex numbers. Also let us suppose $M^\alpha = \left\{w_1 \cdot w_2 \cdot .... \cdot w_n \mid (w_1, w_2, ...w_n) \in w^\alpha_{f_1} \times w^\alpha_{f_2} \times .... \times w^\alpha_{f_n}\right\}$ and $D = w_{f_1} \cdot w_{f_2} \cdot .... \cdot w_{f_n}$, then for $0 \le \alpha \le 1$*

$$M^\alpha = D^\alpha \ .$$

Proof. **Case I.** First of all we assume that $0 \le \alpha < 1$.

Also let $w \in M^\alpha$.

Therefore we may find n number of fuzzy bicomplex numbers w_1, w_2 ... w_n so that $w_1 \cdot w_2 ... \cdot w_n = w$.

Now we obtain from above that

$$\mu\left(w_k \mid w_{f_k}\right) > \alpha \text{ for } k = 1, 2 \ ...n$$

which implies that

$$\Lambda\left(w_1, w_{2,} ...w_n\right) > \alpha \ .$$

Thus it follows from above that $\mu\left(w \mid D\right)$ exceeds α and so

$$w \in M^\alpha \Rightarrow w \in D^\alpha \ . \tag{3.9}$$

Again suppose that $w \in D^\alpha$ where $w = w_1 \cdot w_2 ... \cdot w_n$.

Then from above we get that

$$\Lambda\left(w_1, w_{2,} ...w_n\right) > \alpha \text{ since } \mu\left(w \mid D\right) > \alpha$$

$$\text{i.e., } \mu\left(w_k \mid w_{f_k}\right) > \alpha \text{ for } k = 1, 2 \ ...n \ ,$$

which implies that

$$\left(w_1, w_{2,} ...w_n\right) \in w^\alpha_{f_1} \times w^\alpha_{f_2} \times .... \times w^\alpha_{f_n}$$

$$\text{i.e., } \ w \in D^\alpha \Rightarrow w \in M^\alpha \ . \tag{3.10}$$

Thus from (3.9) and (3.10) we get that

$$M^{\alpha} = D^{\alpha} \text{ for } 0 \leq \alpha < 1 \ . \tag{3.11}$$

Case II. Let $\alpha = 1$.

Now let $w_1 \cdot w_2 ... \cdot w_n = w \in M^1$ where w_1, w_2 ...and w_n are n number of fuzzy bicomplex numbers.

Hence

$$\Lambda(w_1, w_{2,} ... w_n) = 1 \ .$$

So we get from above that

$$\mu(w \mid D) = 1$$

$$\text{i.e., } w \in M^1 \Rightarrow w \in D^1 \ . \tag{3.12}$$

Also suppose that

$$w_1 \cdot w_2 \cdot ... \cdot w_n = w \in D^1 \ .$$

Therefore we can find w_{1m} in $supp\left(w_{f_1}\right)$, w_{2m} in $supp\left(w_{f_2}\right)$ $\cdots$ and w_{nm} in $supp\left(w_{f_n}\right)$ so that $w_{1m} \cdot w_{2m} \cdot ... w_{2m} = w$ where $m = 2, 3, 4, \cdots\cdots$ and

$$\Lambda(w_{1m}, w_{2m}, ..., w_{nm}) > 1 - \frac{1}{m} \ .$$

As the supports are compact, we may choose a subsequence $w_{1m_k} \to w_1$, $w_{2m_k} \to w_2$ ... and $w_{nm_k} \to w_n$ with $w_1 \cdot w_2 \cdot ... \cdot w_n = w$ and $\Lambda(w_1, w_2, ..., w_n) \geq 1$ because Λ is continuous.

Thus

$$w_k \in w_{f_k}^1 \text{ for } k = 1, 2, ..., n \ \cdot$$

So

$$w \in D^1 \Rightarrow w \in M^1 \ . \tag{3.13}$$

Now from (3.12) and (3.13) we get that

$$M^{\alpha} = D^{\alpha} \text{ for } \alpha = 1 \ . \tag{3.14}$$

Therefore the theorem follows from (3.11) and (3.14) . □

Remark 3.3. In view of Theorem 3.1 and Theorem 3.2 it can also be said that

$$\begin{aligned} \left(w_{f_1} + w_{f_2} + ... + w_{f_n}\right)^{\alpha} &= w_{f_1}^{\alpha} + w_{f_2}^{\alpha} ... + w_{f_n}^{\alpha} \text{ and} \\ \left(w_{f_1} \cdot w_{f_2} \cdot ... \cdot w_{f_n}\right)^{\alpha} &= w_{f_1}^{\alpha} \cdot w_{f_2}^{\alpha} \cdot ... \cdot w_{f_n}^{\alpha} \text{ for } 0 \leq \alpha \leq 1 \ . \end{aligned}$$

Also for bicomplex conjugates $w_f^{t_k} \mid k = 1, 2, 3$ of a fuzzy bicomplex number w_f,

$$\left(w_f^{t_k}\right)^{\alpha} = \left(w_f^{\alpha}\right)^{t_k} \text{ holds for any } \alpha \text{ with } 0 \leq \alpha \leq 1 \ .$$

Theorem 3.4. *let us suppose* $B = w_{f_1} + w_{f_2} + ... + w_{f_n}$ *or* $B = w_{f_1} \cdot w_{f_2} \cdot .... \cdot w_{f_n}$ *where* $w_{f_1}, w_{f_2}, ...$ *and* w_{f_n} *are any* n *number of fuzzy bicomplex numbers. Also suppose* b_m $\left(b_m \in B^0\right)$ *converges to* b *and* $\mu(b_m \mid B)$ *converges to* ρ *in* $[0, 1]$. *Then* $\mu(b \mid B) \geq \rho$ $\cdot$

Proof. Suppose $B = w_{f_1} + w_{f_2} + ... + w_{f_n}$.

Now for every $\varepsilon\,(> 0)$ there exists w_{1m} in $w^0_{f_1}$, w_{2m} in $w^0_{f_2}$ ... and w_{nm} in $w^0_{f_n}$ so that $w_{1m} + w_{2m} + ... + w_{nm} = b_m$ and

$$\mu\,(b_m \mid B) \geq \Lambda\,(w_{1m}, w_{2m}, ..., w_{nm}) > \mu\,(b_m \mid B) - \varepsilon\ .$$

Now all the w_{1m}, $w_{2m}, ..., w_{nm}$ and b_m belong to compact sets.
So we may choose a subsequence so that $w_{1mk} \to w_1$, $w_{2mk} \to w_2, ...\ w_{nmk} \to w_n$ and $b_{mk} \to b$ where $w_1 + w_2 + ... + w_n = b$ and obviously

$$\rho \geq \Lambda\,(w_1, w_2, ..., w_n) > \rho - \varepsilon\ ,$$

because Λ is continuous.
As ε is arbitrary, we have from above that

$$\rho = \Lambda\,(w_1, w_2, ..., w_n)\,,$$

which implies that

$$\mu\,(b \mid B) \geq \rho\ .$$

This proves the first part of the theorem.

Analogously one may easily prove the second part of the theorem for $B = w_{f_1} \cdot w_{f_2} \cdot .... \cdot w_{f_n}$ and hence the proof is omitted. □

Theorem 3.5. *Let $w_{f_1}, w_{f_2}, ...w_{f_n}$ be any n number of fuzzy bicomplex numbers . Also let $B = w_{f_1} + w_{f_2} + ... + w_{f_n}$ or $B = w_{f_1} \cdot w_{f_2} \cdot .... \cdot w_{f_n}$. Then for any $\alpha\,(0 \leq \alpha < 1)\,,\ B^\alpha$ is open.*

Proof. Let $B = w_{f_1} + w_{f_2} + ... + w_{f_n}$.

Also let $b \in B^\alpha$ for any α, $0 \leq \alpha < 1$.

Now in view of Theorem 3.1, we get that $(w_1, w_2, ...w_n) \in w^\alpha_{f_1} \times w^\alpha_{f_2} \times .... \times w^\alpha_{f_n}$ where $b = w_1 + w_2 + ... + w_n$.

Now in view of Definition 1.4, $w^\alpha_{f_2}, w^\alpha_{f_3}, ..., w^\alpha_{f_n}$ are all open.

So we can choose an open interval $O\,(w_{2,}, \varepsilon)$, $O\,(w_{3,}, \varepsilon)$, ... and $O\,(w_{n,}, \varepsilon)$ centered at $w_{2,}$, $w_{3,}$.. and w_n respectively with radius $\varepsilon > 0$.

Therefore it is natural that $O\,(w_{2,}, \varepsilon)$, $O\,(w_{3,}, \varepsilon)$, ... and $O\,(w_{n,}, \varepsilon)$ contained in $w^\alpha_{f_2}, w^\alpha_{f_3}, ...$ and $w^\alpha_{f_n}$ respectively.

So the set $w_1 + O\,(w_{2,}, \varepsilon) + O\,(w_{3,}, \varepsilon) ... + O\,(w_{n,}, \varepsilon)$ is an open set containing b.

Also in view of Theorem 3.1, the set $w_1 + O\,(w_{2,}, \varepsilon) + O\,(w_{3,}, \varepsilon) ... + O\,(w_{n,}, \varepsilon)$ wholely1. Wholly inside B^α .

Therefore B^α is open.

Hence the first part of the theorem follows.

Similarly one may easily prove the second part of the theorem for $B = w_{f_1} \cdot w_{f_2} \cdot .... \cdot w_{f_n}$ and hence the proof is omitted. □

Theorem 3.6. *If $w_{f_1}, w_{f_2}, ...w_{f_n}$ be any n number of fuzzy bicomplex numbers, then*

$$(i)\ \ w_{f_1} + w_{f_2} + ... + w_{f_n}\ \ \textit{and}$$
$$(ii)\ \ w_{f_1} \cdot w_{f_2} \cdot ... \cdot w_{f_n}$$

are also fuzzy bicomplex numbers.

Proof. Let us suppose that $P = w_{f_1} + w_{f_2} + ... + w_{f_n}$.

We have to show that $\mu(p \mid P)$ is continuous by arguing that $p_n \to p$ implies $\mu(p_n \mid P) \to \mu(p \mid P)$.

It suffices to choose that p_n in P^0 .

Since $\mu(p_n \mid P)$ belongs to $[0,1]$ there is a subsequence $\mu(p_{n_k} \mid P)$ converging to some ρ in $[0,1]$.

We know that {cf. p. 31, [6]}

$$\liminf \mu(p_n \mid P) \leq \rho \leq \limsup \mu(p_n \mid P) \ .$$

Also Theorem 3.5 implies that

$$b \mid \mu(p \mid P) \leq t$$

is closed for all t.

Therefore $\mu(p \mid P)$ is lower semicontinuous and it follows that {cf. p. 74, [7]}

$$\liminf \mu(p_n \mid P) \geq \limsup \mu(p \mid P) \ .$$

However from Theorem 3.4 we obtain that

$$\mu(p \mid P) \geq \rho \ .$$

Hence

$$\liminf \mu(p_n \mid P) = \rho = \mu(p \mid P) \ .$$

Therefore there is a subsequence $\mu\left(p_{n_j} \mid P\right)$ converging to $\limsup \mu(p_n \mid P)$ ([6], p. 32).

Also Theorem 3.4 implies that

$$\liminf \mu(p \mid P) \geq \mu(p_n \mid P) \ .$$

Therefore

$$\liminf \mu(p_n \mid P) = \mu(p \mid P) = \limsup \mu(p_n \mid P) \ .$$

So in view of ([6], p. 31) we have

$$\liminf \mu(p_n \mid P) = \mu(p \mid P)$$

and $\mu(p \mid P)$ is continuous.

In view of Theorem 3.1, it can be easily shown that P^α, $0 \leq \alpha \leq 1$ is bounded because it is the sum of n numbers of bounded sets.

Also from Theorem 3.5 we get P^α is open for all $0 \leq \alpha \leq 1$ and P^1 is closed because $\mu(p \mid P)$ is continuous.

Finally we argue that P^α is connected, arc wise connected and simply connected for $0 \leq \alpha \leq 1$.

Now $w_{f_i}^\alpha, i = 1, 2, 3...n$ are connected, arc wise connected and simply connected and therefore $w_{f_1}^\alpha \times w_{f_2}^\alpha \times .... \times w_{f_n}^\alpha$ is also connected, arc wise connected and simply connects for $0 \leq \alpha \leq 1$.

Also from Theorem 3.1 we get that B^α is the continuous image of $w_{f_1}^\alpha \times w_{f_2}^\alpha \times .... \times w_{f_n}^\alpha$, it follows that $P^\alpha = B^\alpha$ is also connected, simply connected and arc wise connected for all $0 \leq \alpha \leq 1$.

Thus we have shown that P satisfies all the conditions to be a fuzzy bicomplex number.

Hence the first part of the theorem follows.

The proof of the second part of the theorem is similar to the first part and so it is omitted. □

Corollary 3.7. *Suppose* w_{f_1} *and* w_{f_2} *be any two fuzzy bicomplex numbers . Then (i)* $w_{f_1} - w_{f_2}$ *and (ii)* $\frac{w_{f_1}}{w_{f_2}}$ *are also fuzzy bicomplex numbers.*

Proof. As w_{f_2} is a fuzzy bicomplex number,
then $-w_{f_2}$ is also a fuzzy bicomplex number.
Therefore $w_{f_1} - w_{f_2} = w_{f_1} + (-w_{f_2})$ is also a fuzzy bicomplex number.
This proves the first part of the corollary.

Since the mapping $w_2 \to w_2^{-1}$,$w_2 \neq 0$ is continuous and $\left(w_{f_2}^{-1}\right)^{\alpha} = \left(w_{f_2}^{\alpha}\right)^{-1}$ for any α, $0 \leq \alpha \leq 1$, we see that $w_{f_2}^{-1}$ is a fuzzy bicomplex number as w_{f_2} is a fuzy bicomplex number.

Thus $\frac{w_{f_1}}{w_{f_2}}$ is a fuzzy bicomplex number.
Hence the second part of the corollary follows. □

Remark 3.8. In view of Theorem 3.1 and Corollary 3.7, it can also be said that

$$\begin{aligned}
\text{(i)} \quad \left(w_{f_1} - w_{f_2}\right)^{\alpha} &= w_{f_1}^{\alpha} - w_{f_2}^{\alpha} \text{ and} \\
\text{(ii)} \quad \left(\frac{w_{f_1}}{w_{f_2}}\right)^{\alpha} &= \frac{w_{f_1}^{\alpha}}{w_{f_2}^{\alpha}} = w_{f_1}^{\alpha} \cdot \left(w_{f_2}^{\alpha}\right)^{-1} \text{ for } 0 \leq \alpha \leq 1,
\end{aligned}$$

where w_{f_1} and w_{f_2} are any two fuzzy bicomplex numbers .

Remark 3.9. In view of Remark 3.3 and Theorem 3.6, one can easily prove that addition and multiplication of fuzzy bicomplex numbers are associative and commutative. The bicomplex numbers zero and $1+i_1 0+i_2 0+i_1 i_2 0$ are the additive identity and multiplicative identity respectively and there is no additive inverse or multiplicative inverse. Also the addition and multiplication of fuzzy bicomplex numbers are defined from the extension principle, the operations of addition and multiplication will enjoy the same basic properties which have applied for real fuzzy numbers or fuzzy complex numbers.

Theorem 3.10. *The modulus* $|w_f|_{t_k} \mid k = 1, 2, 3$ *of a fuzzy bicomplex number* w_f *is a fuzzy complex number .*

Proof. A fuzzy complex number z_f with membership function $\mu(z \mid z_f)$ is specified by $(z_1/z_2\ ,\ z_3/z_4)$ where (1) $z_1 < z_2 \leq z_3 < z_4$, (2) $\mu(z \mid z_f)$ is continuous and increasing from zero to one on $[z_1, z_2]$, (3) $\mu(z \mid z_f)$ is one on $[z_2, z_3]$, (4) $\mu(z \mid z_f)$ is continuous and decreasing from one to zero on $[z_3, z_4]$ and (5) $\mu(z \mid z_f) = 0$ outside (z_1, z_4). In the degenerate case when $z_1 = z_2 = z_3 = z_4 = z$, z_f is a complex number z.

Now we notice the following cases for the modulus $|w_f|$ of a fuzzy bicomplex number w_f :

(1) If $0 + i_1 0 + i_2 0 + i_1 i_2 0 \in w_f^0$ and $0 + i_1 0 + i_2 0 + i_1 i_2 0 \notin w_f^1$, then $z_1 = 0$ and $\mu\left(0 \mid |w_f|_{t_k}, k = 1, 2, 3\right) \in (0, 1)$.

(2) If $0+i_1 0+i_2 0+i_1 i_2 0 \in w_f^1$ but $w_f^1 \neq \{0+i_1 0+i_2 0+i_1 i_2 0\}$, then $z_1 = z_2 = 0$ and $\mu\left(0 \mid |w_f|_{t_k}, k=1,2,3\right) = 0$.

(3) If $w_f^1 = \{0+i_1 0+i_2 0+i_1 i_2 0\}$, then $z_1 = z_2 = z_3 = 0$.

(4) If $0+i_1 0+i_2 0+i_1 i_2 0$ is not in w_f^0, then $|w_f|_{t_k}, k=1,2,3$ is a fuzzy complex number with $z_1 \geq 0$.

Thus $|w_f|_{t_k}, k=1,2,3$ is a fuzzy complex number. □

Theorem 3.11. *The norm $\|w_f\|$ of a fuzzy bicomplex number w_f is a truncated real fuzzy number.*

Proof. Let

$$\begin{aligned} n_1 &= \inf\left\{\|w_f\| \mid w \in w_f^0\right\}, \\ n_2 &= \inf\left\{\|w_f\| \mid w \in w_f^1\right\}, \\ n_3 &= \sup\left\{\|w_f\| \mid w \in w_f^1\right\} \text{ and} \\ n_4 &= \sup\left\{\|w_f\| \mid w \in w_f^0\right\} . \end{aligned}$$

Clearly $\mu(r \mid \|w_f\|) = 1$ on $[n_2, n_3]$.

We now argue that $\mu(r \mid \|w_f\|)$ is continuous.

Let $r_n \to r$ with $r_n \in \|w_f\|^0$. There is a subsequence $\mu(r_{n_k} \mid \|w_f\|) \to \rho$ in $[0,1]$.

Now $\left\|w_f^\alpha\right\|$ is open for $0 \leq \alpha \leq 1$, so $\|w_f\|^\alpha$ is open and $\{r \mid \mu(r \mid \|w_f\|) \leq t\}$ is closed for all t.

Hence $\mu(r \mid \|w_f\|)$ is lower semicontinuous and

$$\liminf \mu(r_n \mid \|w_f\|) \geq \mu(r \mid \|w_f\|).$$

As r_n in $\|w_f\|^0$ converging to r and $\mu(r_n \mid \|w_f\|)$ converging to ρ in $[0,1]$ implies $\mu(r \mid \|w_f\|) \geq \rho$.

This implies $\liminf \mu(r_n \mid \|w_f\|) \geq \rho$ and therefore we obtain that

$$\liminf \mu(r_n \mid \|w_f\|) = \rho = \mu(r \mid \|w_f\|).$$

We also have a subsequence converging to lim sup and from above we get that

$$\mu(r \mid \|w_f\|) \geq \limsup \mu(r_n \mid \|w_f\|).$$

Therefore lim inf is equal to lim sup which equals $\mu(r \mid \|w_f\|)$ and this function is continuous.

Finally, we show that $\mu(r \mid \|w_f\|)$ is increasing on $[n_1, n_2]$ or $[0, n_2]$ and decreasing on $[n_3, n_4]$.

We first argue that

$$\mu(r \mid \|w_f\|) = \sup\{(w \mid w_f) \mid \|w_f\| \leq r\} \tag{3.15}$$

for $n_1 \leq r \leq n_2$ or $0 \leq r \leq n_2$ and

$$\mu(r \mid \|w_f\|) = \sup\{(w \mid w_f) \mid \|w_f\| \geq r\} \tag{3.16}$$

for $n_3 \leq r \leq n_4$.

The proof of equations (3.15) and (3.16) are similar, so we will only prove equation (3.15).

Suppose for some fixed value of r, there is a w_0 so that $\|w_0\| < r$ and $\mu(w_0 \mid w_f)$ exceeds $\mu(w \mid \|w_f\|)$.

Now we know that

$$\{w \mid \|w\| = r\} \cap w_f^\alpha = \phi \text{ for } \alpha > \mu(r \mid \|w_f\|).$$

Also $z_0 \in w_f^{\alpha 0}$ for some $\alpha_0 > \mu(r \mid \|w_f\|)$. So w_f^α is a subset of $\{w \mid \|w\| < r\}$, for $\alpha \geq \alpha_0$, since the w_f^α are connected.

This implies that $n_2 < n_1$, a contradiction.

Now let $n_1 \leq x_1 < x_2 \leq n_2$ or $0 \leq x_1 < x_2 \leq n_2$.

We see

$$\mu(x_1 \mid \|w_f\|) \leq \mu(x_2 \mid \|w_f\|),$$

since

$$\{w \mid \|w\| \leq x_1\} \subset \{w \mid \|w\| \leq x_2\}.$$

If $n_3 \leq x_1 < x_2 \leq n_4$ then

$$\mu(x_1 \mid \|w_f\|) \geq \mu(x_2 \mid \|w_f\|),$$

because

$$\{w \mid \|w\| \leq x_1\} \supset \{w \mid \|w\| \leq x_2\}.$$

This completes the proof that $\|w_f\|$ is a truncated real fuzzy number. □

Theorem 3.12. *Let w_f be a fuzzy bicomplex number. Then for any α, $0 \leq \alpha \leq 1$*

$$\|w_f\|^\alpha = \left\|w_f^\alpha\right\| ,$$

where $\|w_f\|$ is a truncated real fuzzy number.

Proof. **Case I.** Suppose $0 \leq \alpha < 1$.

Also let $r \in \|w_f\|^\alpha$, then there exist a bicomplex number w such that $\|w\| = r$ and $\mu(w \mid w_f) > \alpha$.

Hence

$$r \in \|w_f\|^\alpha \Rightarrow r \in \left\|w_f^\alpha\right\| . \tag{3.17}$$

Further let $r \in \left\|w_f^\alpha\right\|$, then there exist a bicomplex number w such that $r = \|w\|$ and $\mu(w \mid w_f) > \alpha$.

This implies

$$\sup\{\mu(w \mid w_f) \mid \|w\| = r\} > \alpha.$$

So

$$r \in \left\|w_f^\alpha\right\| \Rightarrow r \in \|w_f\|^\alpha . \tag{3.18}$$

Thus from (3.17) and (3.18) we get that

$$\|w_f\|^\alpha = \left\|w_f^\alpha\right\| \text{ for } 0 \leq \alpha < 1 . \tag{3.19}$$

Case II. Let $\alpha = 1$.

Suppose $r \in \left\|w_f^1\right\|$, then there is a bicomplex number w so that $r = \|w\|$ and $\mu(w \mid w_f) = 1$.

Therefore the supremum of all $\mu(w \mid w_f)$ over all w such that $\|w\| = r$ is also one and $r \in \|w_f\|^1$.

Hence

$$r \in \left\|w_f^1\right\| \Rightarrow r \in \left\|w_f\right\|^1 \ . \tag{3.20}$$

Also let $r \in \left\|w_f\right\|^1$.

For each $m = 2, 3, ...$ there is a w_m in w_f^0 so that $\|w_m\| = r$ and

$$\mu\left(w_m \mid w_f\right) > 1 - \frac{1}{m},$$

where w_m belong to compact $supp(w_f)$ so there is a subsequence $w_{m_k} \to w$ with $\|w\| = r$ and $\mu\left(w \mid w_f\right) \geq 1$ and thus $r \in \left\|w_f^1\right\|$

Hence

$$r \in \left\|w_f\right\|^1 \Rightarrow r \in \left\|w_f^1\right\| \ . \tag{3.21}$$

Therefore from (3.20) and (3.21) we get that

$$\left\|w_f\right\|^1 = \left\|w_f^1\right\| \ . \tag{3.22}$$

Thus the theorem follows from (3.19) and (3.22) . □

Theorem 3.13. *Let w_f be any fuzzy bicomplex numbers, then*

$$\begin{aligned} (i)\ \left\|-w_f\right\| &= \left\|w_f\right\| \ \textit{and} \\ (ii)\ \left\|a.w_f\right\| &= |a| \,.\, \left\|w_f\right\| \quad \textit{where } a \in \mathbb{R}. \end{aligned}$$

Proof. The meaning of the equality is that the interval $\left\|-w_f\right\|^\alpha$ is equal to the interval $\left\|w_f\right\|^\alpha$ for $0 \leq \alpha \leq 1$.

From Theorem 3.12 we get that

$$\left\|-w_f\right\|^\alpha = \left\|-w_f^\alpha\right\| = \left\{\|-w\| \mid w \in w_f^\alpha\right\}. \tag{3.23}$$

Again in view of Theorem 3.12 we have

$$\left\|w_f\right\|^\alpha = \left\|w_f^\alpha\right\| = \left\{\|w\| \mid w \in w_f^\alpha\right\}. \tag{3.24}$$

Hence the result follows from (3.23) and (3.24) as

$$\|-w\| = \|w\| \ .$$

This proves the first part of the theorem.

For the second part of the theorem we have to prove the α-cuts of $\left\|a.w_f\right\|$ equal the corresponding α-cuts of $|a| \,.\, \left\|w_f\right\|$.

Now in view of Lemma 2.1 it follows from Theorem 3.12 that

$$\begin{aligned} \left\|a.w_f\right\|^\alpha &= \left\|(a.w_f)^\alpha\right\| = \left\|a.w_f^\alpha\right\| \\ &= \left\{\|a.w\| \mid w \in w_f^\alpha\right\}. \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} \left(|a| \,.\, \left\|w_f\right\|\right)^\alpha &= |a| \,.\, \left\|w_f\right\|^\alpha = |a| \,.\, \left\|w_f^\alpha\right\| \\ &= \left\{|a| \,.\, \left\|w_f\right\| \mid w \in w_f^\alpha\right\}. \end{aligned} \tag{3.26}$$

Thus the second part of the theorem follows from (3.25) and (3.26) . □

Theorem 3.14. *Let w_{f1} and w_{f_2} be two fuzzy bicomplex numbers, then*

$$\begin{aligned} (i)\ \ |\|w_{f_1}\| - \|w_{f_2}\|| &\le \|w_{f_1} + w_{f_2}\| \le \|w_{f_1}\| + \|w_{f_2}\| \ \ \textit{and} \\ (ii)\ \ |\|w_{f_1}\| - \|w_{f_2}\|| &\le \|w_{f_1} - w_{f_2}\| \le \|w_{f_1}\| + \|w_{f_2}\| \ . \end{aligned}$$

Proof. For $0 \le \alpha \le 1$, we have

$$\begin{aligned} |\|w_{f_1}\| - \|w_{f_2}\||^{\alpha} &= |(\|w_{f_1}\| - \|w_{f_2}\|)^{\alpha}| \\ &= |\|w_{f_1}\|^{\alpha} - \|w_{f_2}\|^{\alpha}| \\ &= |\|w^{\alpha}_{f_1}\| - \|w^{\alpha}_{f_2}\|| \\ &= \left\{\|w_1\| - \|w_2\| \mid w_i \in w^{\alpha}_{f_i}, i = 1,2\right\}, \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} \|w_{f_1} + w_{f_2}\|^{\alpha} &= \|(w_{f_1} + w_{f_2})^{\alpha}\| \\ &= \|w^{\alpha}_{f_1} + w^{\alpha}_{f_2}\| \\ &= \left\{\|w_1 + w_2\| \mid w_i \in w^{\alpha}_{f_i}, i = 1,2\right\} \ . \end{aligned} \tag{3.28}$$

We also deduce that

$$\begin{aligned} (\|w_{f_1}\| + \|w_{f_2}\|)^{\alpha} &= \|w_{f_1}\|^{\alpha} + \|w_{f_2}\|^{\alpha} \\ &= \|w^{\alpha}_{f_1}\| + \|w^{\alpha}_{f_2}\| \\ &= \left\{\|w_1\| + \|w_2\| \mid w_i \in w^{\alpha}_{f_i}, i = 1,2\right\}. \end{aligned} \tag{3.29}$$

Hence the result follows from (3.27) , (3.28) and (3.29) because

$$|\|w_1\| - \|w_2\|| \le \|w_1 + w_2\| \le \|w_1\| + \|w_2\| \ .$$

Similarly with the help of Theorem 3.12 one can easily established the second part of the theorem. □

Theorem 3.15. *Let w_{f1} and w_{f_2} be two fuzzy bicomplex numbers, then*

$$\|w_{f_1} \cdot w_{f_2}\| \le 2^{\frac{1}{2}} \|w_{f_1}\| \cdot \|w_{f_2}\| \ .$$

Proof. In order to prove this theorem, we wish to show that the interval $\|w_{f_1} \cdot w_{f_2}\|^{\alpha}$ is less than or equal to the interval $\left(2^{\frac{1}{2}} \|w_{f_1}\| \cdot \|w_{f_2}\|\right)^{\alpha}$ for $0 \le \alpha \le 1$.

From Theorem 3.12 and in view of Lemma 2.2 we get that

$$\begin{aligned} \|w_{f_1} \cdot w_{f_2}\|^{\alpha} &= \|(w_{f_1} \cdot w_{f_2})^{\alpha}\| \\ &= \|w^{\alpha}_{f_1} \cdot w^{\alpha}_{f_2}\| \\ &= \left\{\|w_1 \cdot w_2\| \mid w_i \in w^{\alpha}_{f_i}, i = 1,2\right\} \ . \end{aligned} \tag{3.30}$$

Also in view of Theorem 3.12 Lemma 2.2 we have

$$\begin{aligned} (\|w_{f_1}\| \cdot \|w_{f_2}\|)^{\alpha} &= \|w_{f_1}\|^{\alpha} \cdot \|w_{f_2}\|^{\alpha} \\ &= \|w^{\alpha}_{f_1}\| \cdot \|w^{\alpha}_{f_2}\| \\ &= \left\{\|w_1\| \cdot \|w_2\| \mid w_i \in w^{\alpha}_{f_i}, i = 1,2\right\}. \end{aligned} \tag{3.31}$$

Since

$$\|w_1 \cdot w_2\| \le 2^{\frac{1}{2}} \|w_1\| \cdot \|w_2\| \ ,$$

the theorem follows from (3.30) and (3.31) . □

In the next theorem we have established a few properties of fuzzy bicomplex conjugate number depending on the concept of it.

Theorem 3.16. *Suppose $w_f^{t_k} \mid k = 1, 2, 3$ be a fuzzy bicomplex conjugate number of a fuzzy bicomplex number w_f then*

$$(1) \quad w_f^{t_k^{t_k}} = w_f \ , \ (2) \ \left(w_{f_1} \pm w_{f_2}\right)^{t_k} = w_{f_1}^{t_k} \pm w_{f_2}^{t_k} \ ,$$
$$(3) \ \left(w_{f_1} \cdot w_{f_2}\right)^{t_k} = w_{f_1}^{t_k} \cdot w_{f_2}^{t_k}, \ (4) \ \left(\frac{w_{f_1}}{w_{f_2}}\right)^{t_k} = \frac{w_{f_1}^{t_k}}{w_{f_2}^{t_k}},$$
$$(5) \ |w_f|_{t_k} = \left|w_f^{t_k}\right|_{t_k} \ \textit{and} \ (6) \ \|w_f\| = \left\|w_f^{t_k}\right\|.$$

Proof. In view of Remark 3.3 we obtain for $0 \le \alpha \le 1$.

$$\begin{aligned}\left(w_f^{t_k^{t_k}}\right)^{\alpha} &= \left(w_f^{t_k^{\alpha}}\right)^{t_k} = \left(\left(w_f^{\alpha}\right)^{t_k}\right)^{t_k} \\ &= \left\{w^{t_k^{t_k}} \mid \text{ for all } w \in w_f^{\alpha}\right\}.\end{aligned}$$

Again

$$\begin{aligned} w_f^{\alpha} &= \{w \mid \mu\left(w \mid w_f\right) > \alpha\} \\ &= \left\{w \mid \text{ for all } w \in w_f^{\alpha}\right\}.\end{aligned}$$

As $w^{t_k^{t_k}} = w$. Hence the first part of the theorem follows from the above.

For the second part of the theorem we have to prove the α-cuts of $\left(w_{f_1} \pm w_{f_2}\right)^{t_k}$ equal the corresponding α-cuts of $w_{f_1}^{t_k} \pm w_{f_2}^{t_k}$.

Now it follows from Theorem 3.1 and Remark 3.3 that

$$\begin{aligned}\left(\left(w_{f_1} \pm w_{f_2}\right)^{t_k}\right)^{\alpha} &= \left(\left(w_{f_1} \pm w_{f_2}\right)^{\alpha}\right)^{t_k} = \left(w_{f_1}^{\alpha} \pm w_{f_2}^{\alpha}\right)^{t_k} \\ &= \left\{\left(w_1 \pm w_2\right)^{t_k} \mid w_i \in w_{f_i}^{\alpha}, i = 1, 2\right\}.\end{aligned}$$

and

$$\begin{aligned}\left(w_{f_1}^{t_k} \pm w_{f_2}^{t_k}\right)^{\alpha} &= \left(w_{f_1}^{t_k}\right)^{\alpha} \pm \left(w_{f_2}^{t_k}\right)^{\alpha} = \left(w_{f_1}^{\alpha}\right)^{t_k} \pm \left(w_{f_2}^{\alpha}\right)^{t_k} \\ &= \left\{w_1^{t_k} \pm w_2^{t_k} \mid w_i \in w_{f_i}^{\alpha}, i = 1, 2\right\}.\end{aligned}$$

Thus the second part of the theorem established as $\left(w_1 \pm w_2\right)^{t_k} = w_1^{t_k} \pm w_2^{t_k}$.

We also observe that

$$\begin{aligned}\left(\left(w_{f_1} \cdot w_{f_2}\right)^{t_k}\right)^{\alpha} &= \left(\left(w_{f_1} \cdot w_{f_2}\right)^{\alpha}\right)^{t_k} = \left(w_{f_1}^{\alpha} \cdot w_{f_2}^{\alpha}\right)^{t_k} \\ &= \left\{\left(w_1 \cdot w_2\right)^{t_k} \mid w_i \in w_{f_i}^{\alpha}, i = 1, 2\right\}. \end{aligned} \tag{3.32}$$

We can also see

$$\begin{aligned}\left(w_{f_1}^{t_k} \cdot w_{f_2}^{t_k}\right)^{\alpha} &= \left(w_{f_1}^{t_k}\right)^{\alpha} . \left(w_{f_2}^{t_k}\right)^{\alpha} = \left(w_{f_1}^{\alpha}\right)^{t_k} . \left(w_{f_2}^{\alpha}\right)^{t_k} \\ &= \left\{w_1^{t_k} \cdot w_2^{t_k} \mid w_i \in w_{f_i}^{\alpha}, i = 1, 2\right\}. \end{aligned} \tag{3.33}$$

From (3.32)and (3.33) we obtain that the corresponding α-cuts are equal. Hence the third part of the theorem established

For the fourth part of the theorem we deduce that

$$\begin{aligned}\left(\left(\frac{w_{f_1}}{w_{f_2}}\right)^{t_k}\right)^{\alpha} &= \left(\left(\frac{w_{f_1}}{w_{f_2}}\right)^{\alpha}\right)^{t_k} = \left(\left(w_{f_1}.w_{f_2}^{-1}\right)^{\alpha}\right)^{t_k} \\ &= \left(w_{f_1}^{\alpha}.\left(w_{f_2}^{-1}\right)^{\alpha}\right)^{t_k} = \left(w_{f_1}^{\alpha}.\left(w_{f_2}^{\alpha}\right)^{-1}\right)^{t_k} \\ &= \left\{\left(\frac{w_1}{w_2}\right)^{t_k} \mid w_i \in w_{f_i}^{\alpha}, i = 1, 2\right\}.\end{aligned}$$

and

$$\begin{aligned}\left(\frac{w_{f_1}^{t_k}}{w_{f_2}^{t_k}}\right)^{\alpha} &= \left(w_{f_1}^{t_k}.w_{f_2}^{t_k-1}\right)^{\alpha} = \left(w_{f_1}^{t_k}\right)^{\alpha}.\left(w_{f_2}^{t_k-1}\right)^{\alpha} = \left(w_{f_1}^{\alpha}\right)^{t_k}.\left(\left(w_{f_2}^{t_k}\right)^{\alpha}\right)^{-1} \\ &= \left(w_{f_1}^{\alpha}\right)^{t_k}.\left(\left(w_{f_2}^{\alpha}\right)^{t_k}\right)^{-1} \\ &= \left\{\frac{w_1^{t_k}}{w_2^{tk}} \mid w_i \in w_{f_i}^{\alpha}, i = 1, 2\right\}.\end{aligned}$$

Hence the α-cuts of $\left(\frac{w_{f_1}}{w_{f_2}}\right)^{t_k}$ equal the corresponding α-cuts of $\frac{w_{f_1}^{t_k}}{w_{f_2}^{t_k}}$ implying two fuzzy bicomplex numbers are equal. Thus the fourth part of the theorem follows.

Again we have from the Remark 3.3 that

$$|w_f|_{t_k}^{\alpha} = \left|w_f^{\alpha}\right|_{t_k} = \left\{|w|_{t_k} \mid \text{ for all } w \in w_f\right\}.$$

and

$$\left(\left|w_f^{t_k}\right|_{t_k}\right)^{\alpha} = \left|\left(w_f^{t_k}\right)^{\alpha}\right|_{t_k} = \left|\left(w_f^{\alpha}\right)^{t_k}\right|_{t_k} = \left\{\left|w^{t_k}\right|_{t_k} \mid \text{ for all } w \in w_f\right\}.$$

Hence the forth part of the theorem follows as $|w|_{t_k} = |w^{t_k}|_{t_k}$.

Again we have in view of Theorem 3.12 that

$$\|w_f\|^{\alpha} = \left\|w_f^{\alpha}\right\| = \{\|w\| \mid \text{ for all } w \in w_f\}.$$

We can also see

$$\left\|w_f^{t_k}\right\|^{\alpha} = \left\|\left(w_f^{t_k}\right)^{\alpha}\right\| = \left\|\left(w_f^{\alpha}\right)^{t_k}\right\| = \left\{\left\|w^{t_k}\right\| \mid \text{ for all } w \in w_f\right\}.$$

Hence the last part of the theorem follows as $\|w\| = \|w^{t_k}\|$. □

Theorem 3.17. *Let z_{f_1} and z_{f_2} are any two fuzzy complex numbers with membership functions $\mu(z_1 \mid z_{f_1})$ and $\mu(z_2 \mid z_{f_2})$ respectively. Also let w_f be fuzzy bicomplex number such that $w_f = z_{f_1} + i_2 z_{f_2}$. Then for any α, $0 \leq \alpha \leq 1$,*

$$w_f^{\alpha} = z_{f_1}^{\alpha} \times z_{f_2}^{\alpha}.$$

Proof. **Case I.** Suppose $0 \leq \alpha < 1$.

Also let $w \in w_f^{\alpha}$, then

$$\min\left(\mu(z_1 \mid z_{f_1}),\ \mu(z_2 \mid z_{f_2})\right) > \alpha$$

where $w = z_1 + i_2 z_2$ implying that both the membership functions $\mu(z_1 \mid z_{f_1})$ and $\mu(z_2 \mid z_{f_2})$ exceed α and therefore $(z_1\ ,\ z_2) \in z_{f_1}^{\alpha} \times z_{f_2}^{\alpha}$.

Thus

$$w \in w_f^\alpha \Rightarrow (z_1\ ,\ z_2) \in z_{f_1}^\alpha \times z_{f_2}^\alpha \text{ where } w = z_1 + i_2 z_2\ . \tag{3.34}$$

Again suppose $(z_1\ ,\ z_2) \in z_{f_1}^\alpha \times z_{f_2}^\alpha$.

Then the minimum of the membership functions at z_1 and z_2 respectively exceeds α so that $\mu(w \mid w_f) > \alpha$.

So $w \in w_f^\alpha$ where $w = z_1 + i_2 z_2$.

Therefore

$$(z_1\ ,\ z_2) \in z_{f_1}^\alpha \times z_{f_2}^\alpha \Rightarrow w \in w_f^\alpha \text{ where } w = z_1 + i_2 z_2\ . \tag{3.35}$$

Thus from (3.34) and (3.35) we get that

$$w_f^\alpha = z_{f_1}^\alpha \times z_{f_2}^\alpha \quad \text{for } 0 \leq \alpha < 1\ . \tag{3.36}$$

Case II. Let $\alpha = 1$.

If $(z_1\ ,\ z_2) \in z_{f_1}^1 \times z_{f_2}^1$ then we easily see that $\mu\left(w \mid w_f\right) = 1$ where $w = z_1 + i_2 z_2$.

Therefore $w \in w_f^1$.

Hence

$$(z_1\ ,\ z_2) \in z_{f_1}^1 \times z_{f_2}^1 \Rightarrow w \in w_f^1\ . \tag{3.37}$$

Next let $w \in w_f^1$.

Then for any z_1 and z_2 we have $w = z_1 + i_2 z_2$ and $\mu(z_1 \mid z_{f_1}) = \mu(z_2 \mid z_{f_2}) = 1$.

Hence $(z_1\ ,\ z_2) \in z_{f_1}^1 \times z_{f_2}^1$.

Thus

$$w \in w_f^1 \Rightarrow (z_1\ ,\ z_2) \in z_{f_1}^1 \times z_{f_2}^1\ . \tag{3.38}$$

Therefore from (3.37) and (3.38) we get that

$$w_f^1 = z_{f_1}^1 \times z_{f_2}^1\ . \tag{3.39}$$

Thus the theorem follows from (3.36) and (3.39) . □

Theorem 3.18. *Let w_{f_1} and w_{f_2} are any two fuzzy bicomplex numbers such that $w_{f_1} = z_{f_1} + i_2 z_{f_2}$ and $w_{f_2} = z_{f_3} + i_2 z_{f_4}$ where z_{f_1}, z_{f_2}, z_{f_3} and z_{f_4} are any four fuzzy complex numbers Then*

$$w_{f_1} \pm w_{f_2} = \left(z_{f_1} \pm z_{f_3}\right) + i_2\left(z_{f_2} \pm z_{f_4}\right)\ .$$

Proof. We show that the addition formula is true.

Let $W = w_{f_1} + w_{f_2}$, then

$$\mu\left(w \mid W\right) = \sup\left\{\Lambda\left(w_1, w_2\right) \mid w_1 + w_2 = w\right\},$$

where

$$\Lambda\left(w_1, w_2\right) = \min\left\{\mu(w_1 \mid w_{f_1}),\ \mu(w_2 \mid w_{f_2})\right\}\ .$$

Now we define $\Gamma\left(z_1, z_2, z_3, z_4\right)$ to be the minimum of

$$\mu\left(z_i \mid z_{f_i}\right) \mid i = 1, 2, 3, 4.$$

So we can write that

$$\Gamma\left(z_1, z_2, z_3, z_4\right) = \Lambda\left(w_1, w_2\right)\ ,$$

where

$$w_i = z_i + i_2 z_i \mid i = 1, 2 \ .$$

Also let

$$P = z_{f_1} + z_{f_3} \text{ and } Q = z_{f_2} + z_{f_4} \ \text{ so that}$$
$$\mu(p \mid P) = \sup\{\Lambda(z_1, z_3) \mid z_1 + z_3 = p\}$$

and

$$\mu(q \mid Q) = \sup\{\Lambda(z_2, z_4) \mid z_2 + z_4 = q\} \ ,$$

where $\Lambda(z_1, z_3)$ and $\Lambda(z_2, z_4)$ are the minimum of $\mu\left(z_i \mid z_{f_i}\right) \mid i = 1, 3$ and $\mu\left(z_j \mid z_{f_j}\right) \mid j = 2, 4$.

If $G = P + i_2 Q$, then $\mu(g \mid G)$ is the minimum of $\mu(p \mid P)$, $\mu(q \mid Q)$ where $g = p + i_2 q$.

We first argue that

$$\mu(w \mid W) \leq \mu(w \mid G) \ ,$$

where $w = p + i_2 q$, $z_1 + z_3 = p$ and $z_2 + z_4 = q$.

Now $\Gamma(z_1, z_2, z_3, z_4)$ is less than or equal to $\Lambda(z_1, z_3)$ and $\Lambda(z_2, z_4)$ implying it is also less than or equal to $\mu(p \mid P)$ and $\mu(q \mid Q)$.

Hence

$$\Gamma(z_1, z_2, z_3, z_4) \leq \mu(w \mid G),$$

which implies that

$$\mu(w \mid W) \leq \mu(w \mid G) \ . \tag{3.40}$$

Next we show that

$$\mu(w \mid G) \leq \mu(w \mid W) \ ,$$

where $w = p + i_2 q$.

For any $\epsilon > 0$ there exists $z_i^* \, (i = 1, 2, 3, 4)$ so that $p = z_1^* + z_3^*$ and $q = z_2^* + z_4^*$ for which

$$\Lambda(z_1^*, z_3^*) > \mu(p \mid P) - \epsilon$$

and

$$\Lambda(z_2^* + z_4^*) > \mu(q \mid Q) - \epsilon$$

holds.

Therefore it follows from above that

$$\Gamma(z_1^*, z_2^*, z_3^*, z_4^*) > \mu(w \mid G),$$

which implies that

$$\mu(w \mid W) > \mu(w \mid G) \ .$$

Since $\epsilon > 0$ is arbitrary, we get from above that

$$\mu(w \mid W) \geq \mu(w \mid G) \ . \tag{3.41}$$

Therefore from (3.40) and (3.41) we get that

$$\mu(w \mid W) = \mu(w \mid G) \ .$$

Thus the result follows from above.

Similarly one can easily verify that

$$w_{f_1} - w_{f_2} = \left(z_{f_1} - z_{f_3}\right) + i_2\left(z_{f_2} - z_{f_4}\right) \ .$$

□

In view of Theorem 3.1 , Theorem 3.17 and Theorem 3.18 one can easily verified the following corollary:

Corollary 3.19. *Suppose w_{f_1} and w_{f_2} are any two fuzzy bicomplex numbers such that $w_{f_1} = z_{f_1} + i_2 z_{f_2}$ and $w_{f_2} = z_{f_3} + i_2 z_{f_4}$ where z_{f_1}, z_{f_2}, z_{f_3} and z_{f_4} are any four fuzzy complex numbers Then for any α, $0 \leq \alpha \leq 1$,*

$$(w_{f_1} \pm w_{f_2})^{\alpha} = \left(z^{\alpha}_{f_1} \pm z^{\alpha}_{f_3}\right) + i_2 \left(z^{\alpha}_{f_2} \pm z^{\alpha}_{f_4}\right) .$$

The proof is omitted.

Theorem 3.20. *Let z_{f_1} and z_{f_2} are any two fuzzy complex numbers . Also let w_f be fuzzy bicomplex number such that $w_f = z_{f_1} + i_2 z_{f_2}$. Then for any α, $0 \leq \alpha \leq 1$,*

$$\|w_f\|^{\alpha} = \left(\left|z^{\alpha}_{f_1}\right|^2 + \left|z^{\alpha}_{f_2}\right|^2\right)^{\frac{1}{2}} .$$

Proof. In view of Theorem 3.12 we have

$$\|w_f\|^{\alpha} = \left\|w^{\alpha}_f\right\| = \left\{\left(z_1^2 + z_2^2\right)^{\frac{1}{2}} \mid z_1 \in z^{\alpha}_{f_1},\ z_2 \in z^{\alpha}_{f_2}\right\} = \left(\left|z^{\alpha}_{f_1}\right|^2 + \left|z^{\alpha}_{f_2}\right|^2\right)^{\frac{1}{2}} .$$

Hence the result follows. □

Theorem 3.21. *Let w_f be fuzzy bicomplex number such that $w_f = z_{f_1} + i_2 z_{f_2}$ where z_{f_1} and z_{f_2} are any two fuzzy complex numbers. Then*

$$w_f = (z_{f_1} - i_1 z_{f_2}) \left(\frac{1 + i_1.i_2}{2}\right) + (z_{f_1} + i_1 z_{f_2}) \left(\frac{1 - i_1.i_2}{2}\right) .$$

Proof. The meaning of the equality is that the interval $\left(z_{f_1} + i_2 z_{f_2}\right)^{\alpha}$ is equal to the interval $\left[\left(z_{f_1} - i_1 z_{f_2}\right)\left(\frac{1+i_1.i_2}{2}\right) + \left(z_{f_1} + i_1 z_{f_2}\right)\left(\frac{1-i_1.i_2}{2}\right)\right]^{\alpha}$ for $0 \leq \alpha \leq 1$.

From Corollary 3.19 we get that

$$w^{\alpha}_f = \left(z_{f_1} + i_2 z_{f_2}\right)^{\alpha} = z^{\alpha}_{f_1} + i_2 z^{\alpha}_{f_2} = \left\{z_1 + z_2 \mid z_i \in z^{\alpha}_{f_i},\ i = 1, 2\right\} . \tag{3.42}$$

Also in view of Corollary 3.19 we have

$$\left(z_{f_1} \pm i_1 z_{f_2}\right)^{\alpha} = z^{\alpha}_{f_1} \pm i_1 z^{\alpha}_{f_2} = \left\{z_1 \pm i_1 z_2 \mid z_i \in z^{\alpha}_{f_i},\ i = 1, 2\right\} . \tag{3.43}$$

Since

$$z_1 + i_2 z_2 = (z_1 - i_1 z_2) \left(\frac{1 + i_1.i_2}{2}\right) + (z_1 + i_1 z_2) \left(\frac{1 - i_1.i_2}{2}\right) ,$$

the theorem follows from (3.42) and (3.43) . □

Theorem 3.22. *Suppose w_{f_1} and w_{f_2} are any two fuzzy bicomplex numbers such that $w_{f_1} = z_{f_1} + i_2 z_{f_2}$ and $w_{f_2} = z_{f_3} + i_2 z_{f_4}$ where z_{f_1}, z_{f_2}, z_{f_3} and z_{f_4} are any four fuzzy complex numbers Then*

$$\begin{aligned} w_{f_1} + w_{f_2} &= \left[\left(z_{f_1} - i_1 z_{f_2}\right) + \left(z_{f_3} - i_1 z_{f_4}\right)\right] \left(\frac{1 + i_1.i_2}{2}\right) \\ &\quad + \left[\left(z_{f_1} + i_1 z_{f_2}\right) + \left(z_{f_3} + i_1 z_{f_4}\right)\right] \left(\frac{1 - i_1.i_2}{2}\right) . \end{aligned}$$

Proof. For $0 \leq \alpha \leq 1$, we have

$$\begin{aligned}
(w_{f_1} + w_{f_2})^{\alpha} &= \left(z_{f_1}^{\alpha} + z_{f_3}^{\alpha}\right) + i_2\left(z_{f_2}^{\alpha} + z_{f_4}^{\alpha}\right) \\
&= \left(z_{f_1}^{\alpha} + i_2 z_{f_3}^{\alpha}\right) + \left(z_{f_2}^{\alpha} + i_2 z_{f_4}^{\alpha}\right) \\
&= \left\{(z_1 + i_2 z_3) + (z_2 + i_2 z_4) \mid z_i \in z_{f_i}^{\alpha},\ i = 1,2,3,4\right\}, \qquad (3.44)
\end{aligned}$$

and

$$\begin{aligned}
&\left[\left(z_{f_1} \pm i_1 z_{f_2}\right) + \left(z_{f_3} \pm i_1 z_{f_4}\right)\right]^{\alpha} \\
= &\left(z_{f_1} \pm i_1 z_{f_2}\right)^{\alpha} + \left(z_{f_3} \pm i_1 z_{f_4}\right)^{\alpha} \\
= &\left(z_{f_1}^{\alpha} \pm i_1 z_{f_2}^{\alpha}\right) + \left(z_{f_3}^{\alpha} \pm i_1 z_{f_4}^{\alpha}\right) \\
= &\left\{(z_1 \pm i_1 z_2) + (z_3 \pm i_1 z_4) \mid z_i \in z_{f_i}^{\alpha},\ i = 1,2,3,4\right\}. \qquad (3.45)
\end{aligned}$$

Hence the theorem follows from (3.44) and (3.45) because

$$\begin{aligned}
(z_1 + i_2 z_3) + (z_2 + i_2 z_4) = &\left[(z_1 - i_1 z_2) + (z_3 - i_1 z_4)\right]\left(\frac{1 + i_1.i_2}{2}\right) \\
&+ \left[(z_1 + i_1 z_2) + (z_3 + i_1 z_4)\right]\left(\frac{1 - i_1.i_2}{2}\right).
\end{aligned}$$

□

References

[1] J. J. Buckley, Fuzzy complex numbers, Fuzzy Sets and Systems 33 (1989) 333–345.
[2] J. J. Buckley and Y. Qu, Fuzzy complex analysis I :Differentiation, Fuzzy Sets and Systems 41 (1991) 269–284.
[3] J. J. Buckley, Fuzzy complex analysis II : Integration, Fuzzy Sets and Systems 49 (1992) 171–179.
[4] S. K. Datta, T. Biswas and S. Tamang, Some inequalities involving fuzzy complex numbers, Theory and Applications of Mathematics and Computer Science 4 (1) (2014) 106–113.
[5] A. Kaufmann and M. Gupta, Introduction to Fuzzy Arithmetic: Theory and Applications, Van Nostrand Reinhold, New York 1985.
[6] A. E. Taylor, General Theory of Functions and Integration, Blaisdell Waltham, MA 1965.
[7] A. Wilansky, Functional Analysis, Blaisdell, New York 1964.
[8] C. Wu and J. Qiu, Some remarks for fuzzy complex analysis, Fuzzy Sets and Systems 106 (1999) 231–238.
[9] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

Sanjib Kumar Datta (sanjib_kr_datta@yahoo.co.in)
Department of Mathematics,University of Kalyani, P.O. Kalyani, Dist-Nadia,PIN-741235, West Bengal, India

Tanmay Biswas (tanmaybiswas_math@rediffmail.com)
Rajbari, Rabindrapalli, R. N. Tagore Road, P.O. Krishnagar, Dist-Nadia,PIN-741101, West Bengal, India