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T-fuzzy subsemigroups and T-fuzzy ideals of regular Γ-semigroups

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Abstract. In this paper, we introduce the notion of T-fuzzy ideal, T-fuzzy bi-ideal, T-fuzzy quasi-ideal, T-fuzzy interior-ideal and generalized T-fuzzy bi-ideals of a gamma-semigroup. We also introduce the concept of regular, left regular and strongly regular gamma semigroup and establish the relationship between these type of regularities. We provide an example to establish that strong regularity implies regularity whereas the converse is not true. We characterize regularity of gamma semigroup through T-fuzzy subset, T-fuzzy ideal, T-fuzzy bi-ideal, T-fuzzy quasi-ideal, T-fuzzy interior-ideal and generalized T-fuzzy bi-ideal. Finally we obtain some equivalent condition for gamma semigroup to be regular in terms of these ideals.

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1. Introduction

The fundamental concepts of fuzzy set was introduced by Zadeh[12]. Fuzzy semigroups have been first considered by Kurki. In [2, 3, 4, 5, 6], Kurki has given some properties of fuzzy ideals and fuzzy bi-ideals in semigroups. Rosenfeld[7] introduced the concept of fuzzy subgroups and give some of its properties. The notion of quasi-ideals was introduced by Steinfeld[11]. The idea of gamma semigroup was introduced by Sen[10]. Many fundamental results in semigroup theory have been extended to gamma semigroup by Sen and Saha[9]. In 1960, Schweizer and Sklar[8] introduced the definition of t-norm to generalize the triangular inequality in metric spaces. T-fuzzy concept is another generalization of fuzzy set theory. In this paper, we introduce the notion of T-fuzzy ideal, T-fuzzy bi-ideal, T-fuzzy quasi-ideal, T-fuzzy interior-ideal and generalized T-fuzzy bi-ideal of a gamma-semigroup. We study the concepts regularity in gamma-semigroup and obtain some characterization of regular gamma-semigroups using different T-fuzzy ideals. Equivalent conditions on T-fuzzy ideals, T-fuzzy quasi-ideals and T-fuzzy bi-ideals of regular gamma semigroup are also obtained.

2. Preliminaries

In this section, we recall some definitions that are necessary for the development of the concepts studied in this paper.

Definition 2.1. Let S be a semigroup. Let A and B be subsets of S, the product of A and B is defined as $AB = \{ ab \in S \mid a \in A \text{ and } b \in B \}$. A nonempty subset A of S is called a subsemigroup of S if $AA \subseteq A$. A nonempty subset A of S is called a left (resp. right) ideal of S if $SA \subseteq A$ (resp.$AS \subseteq A$). A is called a two-sided ideal (simply, ideal) of S if it is both a left and a right ideal of S. A nonempty subset A of S is called an interior ideal of S if $SAS \subseteq A$, and a *quasi-ideal* of S if $AS \cap SA \subseteq A$. A subsemigroup A of S is called a bi-ideal of S if $ASA \subseteq A$. A semigroup S is called regular if for each element $a \in S$ there exists $x \in S$ such that $a = axa$. A function μ from a nonempty set A into the unit interval $[0, 1]$ is called a *fuzzy subset* of A.

Definition 2.2 ([1])**.** Let M and Γ be any two nonempty sets. M is called a Γ-semigroup if for all $a, b, c \in M$ and $x, y \in \Gamma$,
(1) $M\Gamma M \subseteq M$ and $\Gamma M\Gamma \subseteq \Gamma$,
(2) $(axb)yc = a(xby)c = ax(byc)$.

Notation 2.3 ([1])**.** For subsets A and B of M, let $A\Gamma B = \{a\gamma b \mid a \in A, b \in B, \gamma \in \Gamma\}$.

Definition 2.4 ([1])**.** Let M be a Γ-semigroup and A a nonempty subset of M. A is called a left (resp. right) ideal of M if
(1) $A\Gamma A \subseteq A$,
(2) $M\Gamma A \subseteq A$ (resp. $A\Gamma M \subseteq A$).

A is a two-sided ideal (simply, ideal) of a Γ-semigroup M if it is both a left ideal and a right ideal of M.
Throughout the discussion, M denotes Γ-semigroup unless otherwise specified.

Definition 2.5 ([1])**.** A subsemigroup Q of M is called a quasi-ideal of M if $Q\Gamma M \cap M\Gamma Q \subseteq Q$.

Definition 2.6 ([1])**.** A subsemigroup B of M is called a bi-ideal of M if $B\Gamma M\Gamma B \subseteq B$.

Definition 2.7. A subset A of a Γ-semigroup M is called an interior ideal if
(1) $A\Gamma A \subseteq A$,
(2) $M\Gamma A\Gamma M \subseteq A$.

Definition 2.8 ([1])**.** Let μ and λ be any two fuzzy subsets of M. Then $\mu \wedge \lambda$ and $\mu \circ \lambda$ are fuzzy subsets of M defined by

$(\mu \wedge \lambda)(x) = min\{\mu(x),\ \lambda(x)\},$

$$(\mu \circ \lambda)(z) = \begin{cases} \sup\limits_{z=x\gamma y} \{min\{\mu(x),\ \lambda(y)\}\}, \text{if } z \text{ can be expressed as } z = x\gamma y, \\ \qquad\qquad \text{where } x,\ y,\ z \in M \text{ and } \gamma \in \Gamma, \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

Definition 2.9. A fuzzy subset μ of M is called a fuzzy subsemigroup if for all $a,\ b \in M$ and $\gamma \in \Gamma$, $\mu(a\gamma b) \geq min\{\mu(a),\ \mu(b)\}$.

Definition 2.10. A fuzzy subset μ of M is called a fuzzy left (resp. right) ideal of M if for all $a,\ b \in M$ and $\gamma \in \Gamma$,
(1) $\mu(a\gamma b) \geq min\{\mu(a),\ \mu(b)\}$,
(2) $\mu(a\gamma b) \geq \mu(b)$ (resp. $\mu(a\gamma b) \geq \mu(a)$).

μ is called a fuzzy ideal of M if it is both a fuzzy left ideal and a fuzzy right ideal of M.

The characteristic function of M is denoted by $\mathbf{M}$.

Definition 2.11. A fuzzy subset μ of M is called a *fuzzy quasi-ideal* of M if for all $x,\ y \in M$ and $\gamma \in \Gamma$,
(1) $\mu(x\gamma y) \geq min(\mu(x),\ \mu(y))$,
(2) $(\mu \circ \mathbf{M}) \wedge (\mathbf{M} \circ \mu) \leq \mu$.

Definition 2.12. A fuzzy subset μ of M is called a fuzzy bi-ideal of M if for all $x,\ y \in M$ and $\gamma \in \Gamma$,
(1) $\mu(x\gamma y) \geq min\{\mu(x),\ \mu(y)\}$,
(2) $\mu \circ \mathbf{M} \circ \mu \leq \mu$.

Definition 2.13. A fuzzy subset μ of M is called a fuzzy interior ideal of M if for all $a, b, x \in M$ and $\nu_1, \nu_2, \nu \in \Gamma$,
(1) $\mu(a\,\nu\,b) \geq min\{\ \mu(a), \mu(b)\}$,
(2) $\mu(a\,\nu_1\,x\,\nu_2\,b) \geq \mu(x)$.

Definition 2.14 ([8])**.** A t-norm is a function $T : [0,\ 1] \times [0,\ 1] \to [0,\ 1]$ that satisfies the following conditions for all $x,\ y,\ z \in [0,\ 1]$,
(T1) $T(x,\ 1) = x$,
(T2) $T(x',\ y') \leq T(x,\ y)$ if $x' \leq x$ and $y' \leq y$ (monotonicity),
(T3) $T(x,\ y) = T(y,\ x)$ (commutative),
(T4) $T(x,\ T(y,\ z)) = T(T(x,\ y),\ z)$ (associativity).

Replacing 1 by 0 in condition (T1) we obtain the concept of triangular conorm (t-conorm). In general for any t-norm T, $T(x,\ y) \leq min\{x,\ y\}$, $T(x,\ 0) = 0$, $T(0,\ 0) = 0$, and $T(1,\ 1) = 1$ are always true. Now we generalize Definition 2.8 through t-norm T.

Definition 2.15. Let λ and μ be any two fuzzy subsets of M and T be any t-norm. Then $\mu \cap \lambda$ and $\mu * \lambda$ are fuzzy subsets of M defined by

$$(\mu \cap \lambda)(x) = T(\mu(x),\ \lambda(x))$$

$$(\mu * \lambda)(x) = \begin{cases} \sup\limits_{x=y\gamma z} \{T(\mu(y),\ \lambda(z))\}, \text{if } x \text{ can be expressed as } x = y\gamma z, \\ \qquad\qquad \text{where } x,\ y,\ z \in M \text{ and } \gamma \in \Gamma, \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

Definition 2.16. A fuzzy subset λ of a gamma semigroup M is said to satisfy the idempotent property if $(\lambda * \lambda)(x) = \lambda(x)$ for all $x \in M$ and $\gamma \in \Gamma$.

3. T-fuzzy subsemigroups and T-fuzzy ideals of regular Γ-semigroups

In this section, we introduce the concepts of T-fuzzy subsemigroups and different types of T-fuzzy ideals of a Γ-semigroup and regular Γ-semigroup and study some basic properties of these ideals.

Definition 3.1. A fuzzy subset μ of M is called a fuzzy subsemigroup of M with respect to a t-norm T [briefly, T-fuzzy subsemigroup of M] if for all $x,\ y \in M$ and $\gamma \in \Gamma$, $\mu(x\gamma y) \geq T(\mu(x),\ \mu(y))$.

Definition 3.2. A fuzzy subset μ of M is called a T-fuzzy quasi-ideal of M if, for all $x,\ y \in M$ and $\gamma \in \Gamma$,
(1) $\mu(x\gamma y) \geq T(\mu(x),\ \mu(y))$,
(2) $(\mu * \mathbf{M}) \cap (\mathbf{M} * \mu) \leq \mu$.

Definition 3.3. A fuzzy subset μ of M is called a T-fuzzy bi-ideal of M if for all $x,\ y \in M$ and $\gamma \in \Gamma$,
(1) $\mu(x\gamma y) \geq T(\mu(x),\ \mu(y))$,
(2) $\mu * \mathbf{M} * \mu \leq \mu$.

Definition 3.4. A fuzzy subset μ of M is called a fuzzy interior ideal of M with respect to t-norm T (in short, T-fuzzy interior ideal) if
(1) $\mu(a\,\nu\, b) \geq T(\mu(a), \mu(b))$,
(2) $\mu(a\,\nu_1\, x\,\nu_2\, b) \geq \mu(x)$.

Definition 3.5 ([1])**.** M is said to be regular if for each element $a \in M$, there exist $m \in M$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $a = a\gamma_1 m\gamma_2 a$. M is said to be strongly regular if for each element $a \in M$ and $\gamma_1, \gamma_2 \in \Gamma$, there exists $m \in M$ such that $a = x\gamma_1 m\gamma_2 y$, where $x \in R(a)$ and $y \in L(a)$. An element $a \in M$ is said to be regular if there exist x such that $axa = a$. Here, $R(a)$ and $L(a)$ denote the right and left ideals generated by the element a respectively.

Clearly, M is regular if and only if $a \in a\Gamma a$, for all $a \in M$.

Definition 3.6 ([1])**.** M is said to be left (right) regular if for each element $a \in M$, there exist $x \in M$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $a = x\gamma_1 a\gamma_2 a(a = a\gamma_1 a\gamma_2 x)$.

Example 3.7. Let M be the set of all integers of the form $4n + 1$ where n is an integer and Γ denote the set of all integers of the form $4n+3$. If $a\gamma b$ is $a+\gamma+b, \gamma a\mu$ is $\gamma + a + \mu$ (usual sum of the integers) for all $a, b \in M$ and $\gamma, \mu \in \Gamma$, then M is strongly regular. Set $a = 4n + 1, \gamma_1 = 4n_1 + 3$ and $\gamma_2 = 4n_2 + 3$. let $x = a, y = b$

and $m = 4(-n - n_1 - n_2 - 2) + 1$. Then $a = a\gamma_1 m\gamma_2 a$. Thus M is strongly regular, regular and left (right) regular.

Example 3.8 ([1]). Let $M = \{0, 1, 2, 3, 4, 5\}$ and $\Gamma = M$. If the composition of any two elements is taken as multiplication modulo six, then M is regular but not strongly regular. For $2 \in M$ and $3 \in \Gamma$ there is no $x \in R(2)$ and $y \in L(2)$ such that $2 = x3m3y$.

If $\Gamma = \{0, 3\}$, then 2 is not regular and hence M is not regular.

For the sake of continuity of the subject matter we quote the following results, the proof of which are available in [1].

Theorem 3.9 ([1]). *Let I be a nonempty subset of M. I is left (resp. right) ideal of M if and only if χ_I is a T-fuzzy left (resp. right) ideal of M.*

Theorem 3.10 ([1]). *Let I be any nonempty subset of M and T be any t-norm. Then*

(1) *I is a quasi-ideal of M if and only if χ_I is a T-fuzzy quasi-ideal of M.*
(2) *I is a bi-ideal of M if and only if χ_I is a T-fuzzy bi-ideal of M.*

Lemma 3.11 ([1]). *For any nonempty subset A and B of M,*

(1) $\chi_A \cap \chi_B = \chi_{A\cap B}$.
(2) $\chi_A * \chi_B = \chi_{A\Gamma B}$.

Lemma 3.12 ([1]). *Any T-fuzzy quasi-ideal of M is a T-fuzzy bi-ideal of M.*

Theorem 3.13 ([1]). (1) *M is strongly regular if and only if $A \cap B = A\gamma B$, where A and B respectively right and left ideals of M and $\gamma \in \Gamma$.* (2) *M is regular if and only if $A \cap B = A\gamma B$, where A and B respectively right and left ideals of M.*

Lemma 3.14 ([1]). *M is regular if and only if $Q\Gamma M\Gamma Q = Q$ for every quasi-ideal Q of M.*

Lemma 3.15 ([1]). *M is regular if and only if $Q\Gamma A\Gamma Q = A\cap Q$ for every quasi-ideal Q and every ideal A of M.*

Lemma 3.16 ([1]). *M is left regular if and only if $I \cap B \subseteq I\Gamma B$ for every ideal I and for every bi-ideal B of M.*

Lemma 3.17. *If λ and μ are any two T-fuzzy subsets of M having idempotent properties, then $(\mu * \lambda) \cap (\mu * \lambda) = \mu * \lambda$.*

Proof. Let μ and λ be any two T-fuzzy subsets of M having idempotent properties. Let $a, b, c \in M$ and $\nu \in \Gamma$. Consider

$$\begin{aligned}
((\mu * \lambda) \cap (\mu * \lambda))(a) &= T[(\mu * \lambda)(a), (\mu * \lambda)(a)] \\
&= T[\sup_{a=b\,\nu\,c} \{T(\mu(b), \lambda(c)\}, \sup_{a=b\,\nu\,c} \{T(\mu(b), \lambda(c)\}\,] \\
&= \sup_{a=b\,\nu\,c} \{T[T(\mu(b), \lambda(c)), T(\mu(b), \lambda(c))]\} \\
&= \sup_{a=b\,\nu\,c} \{T[T(\mu(b), T(\lambda(c), T(\mu(b), \lambda(c))))]\} \\
&= \sup_{a=b\,\nu\,c} \{T[T(\mu(b), T(T(\mu(b), \lambda(c)), \lambda(c)))]\} \\
&= \sup_{a=b\,\nu\,c} \{T[T(\mu(b), T(\mu(b), T(\lambda(c), \lambda(c))))]\}
\end{aligned}$$

$$= \sup_{a=b\,\nu\,c} \{T[T(\mu(b), T(\mu(b), \lambda(c)))]\},$$
since λ has idempotent property.
$$= \sup_{a=b\,\nu\,c} \{T[T(\mu(b), \mu(b)), \lambda(c)]\}$$
$$= \sup_{a=b\,\nu\,c} \{T[\mu(b), \lambda(c)]\}, \text{ by the idempotent property of } \mu.$$
$$= (\mu * \lambda)(a).$$

In case a cannot be expressed as $a = b\,\nu\,c$, $(\mu * \lambda)(a) = 0 = T[(\mu * \lambda)(a), (\mu * \lambda)(a)]$. Then $(\mu * \lambda) \cap (\mu * \lambda) = \mu * \lambda$. □

Note 3.18. Using a similar argument, one can show that
$(\mu * \theta * \lambda) \cap (\mu * \theta * \lambda) = \mu * \theta * \lambda$ for any three T-fuzzy subsets μ, θ, λ of M having idempotent properties.

Now we characterize regularity, strongly regularity and left regularity of M through T-fuzzy subsets.

Theorem 3.19. *The following conditions are equivalent:*

(1) *M is left regular.*
(2) *$\lambda \cap \mu \leq \lambda * \mu$ for every T-fuzzy ideal λ and T-fuzzy bi-ideal μ of M.*

Proof. (1)⇒(2). Let M be left regular and $a \in M$. Then, there exist $x \in M$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $a = x\gamma_1 a\gamma_2 a$. Let λ be a T-fuzzy ideal and μ be a T-fuzzy bi-ideal of M. Consider,

$$\begin{aligned}(\lambda * \mu) &= \sup_{a=x\,\gamma\,y} \{T(\lambda(x), \mu(y))\}\\ &\geq T(\lambda(x\gamma_1 a), \mu(a)), \text{ as } x = x\gamma_1 a\gamma_2 a\\ &\geq T(\lambda(a), \mu(a)),\ \lambda(x\gamma, a) \geq \lambda(a)\\ &= (\lambda \cap \mu)(a).\end{aligned}$$

Hence $\lambda \cap \mu \leq \lambda * \mu$.

(2)⇒(1). Suppose λ and μ are respectively any T-fuzzy ideal and any T-fuzzy bi-ideal of M such that $\lambda \cap \mu \leq \lambda * \mu$. Let I be any ideal and B be any bi-ideal of M and $x \in I \cap B$. By Theorem 3.9, χ_I is a T-fuzzy ideal of M and by Theorem 3.10, χ_B is a T-fuzzy bi-ideal of M. Now by Lemma 3.11, we have $\chi_{I\cap B} = \chi_I \cap \chi_B$ and hence,

$$\begin{aligned}1 &= (\chi_{I\cap B})(x)\\ &= (\chi_I \cap \chi_B)(x) \leq (\chi_I * \chi_B)(x)\\ &= (\chi_{I\Gamma B})(x) \text{ (by Lemma 3.11)}.\end{aligned}$$

Thus it follows that $x \in I\Gamma B$ and hence $I \cap B \subseteq I\Gamma B$. Hence by Lemma 3.16, M is left regular. □

Theorem 3.20. *M is regular if and only if $\lambda * \mu = \lambda \cap \mu$, for every T-fuzzy right ideal λ and every T-fuzzy left ideal μ of M.*

Proof. Let M be regular and let λ be a T-fuzzy right ideal, μ be a T-fuzzy left ideal of M and $z \in M$. Consider

$$\begin{aligned}(\lambda * \mu)(z) &= \sup_{z=x\,\gamma\,y} \{T(\lambda(x), \mu(y))\}\\ &\leq \sup_{z=x\,\gamma\,y} \{T(\lambda(x\gamma y), \mu(x\gamma y))\}\\ &= \sup_{z=x\,\gamma\,y} \{T(\lambda(z), \mu(z))\}\\ &= \sup_{z=x\,\gamma\,y} \{(\lambda\cap\mu)(z)\}\\ &= (\lambda\cap\mu)(z),\end{aligned}$$

and so, we have $\lambda * \mu \leq \lambda \cap \mu$. Next, let $y \in M$. Since M is regular, there exist $\gamma_1, \gamma_2 \in \Gamma$ and $x \in M$ such that $y = y\gamma_1 x\gamma_2 y$. Now,

$$\begin{aligned}(\lambda * \mu)(y) &= \sup_{y=a\,\gamma\,b} \{T(\lambda(a), \mu(b))\}\\ &\geq T(\lambda(y), \mu(y)), \text{ since } y = y(\gamma_1 x\gamma_2)y,\ \gamma = \gamma_1 x\gamma_2 \in \Gamma,\\ &= (\lambda\cap\mu)(y),\end{aligned}$$

so $\lambda * \mu \geq \lambda \cap \mu$. This shows that $\lambda * \mu = \lambda \cap \mu$, for every T-fuzzy right ideal λ and every T-fuzzy left ideal μ of M.

Conversely, suppose that $\lambda * \mu = \lambda \cap \mu$, for any T-fuzzy right ideal λ and any T-fuzzy left ideal μ of M. Let A be a right ideal and B be a left ideal of M. By Theorem 3.9, χ_A is a T-fuzzy right ideal and χ_B is a T-fuzzy left ideal of M. By Lemma 3.11, we have

$\chi_{A\cap B} = \chi_A \cap \chi_B = \chi_A * \chi_B = \chi_{A\Gamma B}$.

Thus $A \cap B = A\Gamma B$, for any right ideal A and any left ideal B. Hence by Theorem 3.13(2), M is regular. □

Theorem 3.21. *The following conditions are equivalent:*

(1) *M is regular.*

(2) *$\lambda \cap \lambda \leq \lambda * \mathbf{M} * \lambda \leq \lambda$ for every T-fuzzy bi-ideal λ of M.*

(3) *$\lambda \cap \lambda \leq \lambda * \mathbf{M} * \lambda \leq \lambda$ for every T-fuzzy quasi-ideal λ of M.*

Proof. (1) ⇒(2). Let M be a regular Γ-semigroup and λ be any T-fuzzy bi-ideal of M and $a \in M$. Since M is regular, there exist $\gamma_1, \gamma_2 \in \Gamma$ and $m \in M$ such that $a = a\gamma_1 m\gamma_2 a$. Then

$$\begin{aligned}(\lambda * \mathbf{M} * \lambda)(a) &= \sup_{a=x\,\gamma_3\,m_1\,\gamma_4\,y} \{T[\lambda(x), T(\mathbf{M}(m_1), \lambda(y))]\}\\ &\geq T[\lambda(a), T(\mathbf{M}(m), \lambda(a))]\\ &= T[\lambda(a), \lambda(a)]\\ &= (\lambda\cap\lambda)(a),\end{aligned}$$

and so $\lambda \cap \lambda \leq \lambda * \mathbf{M} * \lambda$ but since λ is a T-fuzzy bi-ideal, we have $\lambda \cap \lambda \leq \lambda * \mathbf{M} * \lambda \leq \lambda$. Thus (1) implies (2).

(2) ⇒ (3). Since every T-fuzzy quasi-ideal is a T-fuzzy bi-ideal, (2) ⇒ (3) is trivial.

(3) ⇒ (1). Let Q be any quasi-ideal of M and let $a \in Q$. Then $\chi_Q(a) = 1$ and by Theorem 3.10(1), χ_Q is a T-fuzzy quasi-ideal of M. By (3), we have

$$\begin{aligned}(\chi_Q \cap \chi_Q)(a) &\leq (\chi_Q * \mathbf{M} * \chi_Q)(a)\\ &\leq \chi_Q(a).\end{aligned}$$

Thus, by Lemma 3.11,

$$(\chi_{Q\cap Q})(a) \leq (\chi_{Q\Gamma M\Gamma Q})(a) \leq \chi_Q(a).$$

So $(\chi_Q)(a) \leq (\chi_{(Q\Gamma M\Gamma Q))}(a) \leq \chi_Q(a)$. Hence $(\chi_Q)(a) = (\chi_{(Q\Gamma M\Gamma Q))}(a)$. It follows that $Q = Q\Gamma M\Gamma Q$, for any quasi-ideal Q of M. Therefore, by Lemma 3.14, M is regular. □

Theorem 3.22. *Let μ be a fuzzy subset of M having idempotent property. The following conditions are equivalent:*
(1) *M is regular.*
(2) *$\lambda = \lambda * \mathbf{M} * \lambda$ for every T-fuzzy bi-ideal λ of M.*
(3) *$\lambda = \lambda * \mathbf{M} * \lambda$ for every T-fuzzy quasi-ideal λ of M.*

Proof. If λ is a T-fuzzy bi-ideal of M having idempotent property, $\lambda \cap \lambda = \lambda$. Then the result follows from Theorem 3.21. □

Corollary 3.23. *The following conditions are equivalent:*
(1) *M is regular.*
(2) *$\lambda = \lambda \circ \mathbf{M} \circ \lambda$ for every fuzzy bi-ideal λ of M.*
(3) *$\lambda = \lambda \circ \mathbf{M} \circ \lambda$ for every fuzzy quasi-ideal λ of M.*

Taking $T = min$ in Theorem 3.21, the result is obtained.

Theorem 3.24. *The following conditions are equivalent:*
(1) *M is regular.*
(2) *$\lambda \cap \mu = \mu * \lambda * \mu$ for every T-fuzzy ideal λ and every T-fuzzy bi-ideal μ of M having idempotent property.*
(3) *$\lambda \cap \mu = \mu * \lambda * \mu$ for every T-fuzzy ideal λ and every T-fuzzy quasi-ideal μ of M having idempotent property.*

Proof. (1) ⇒ (2). Let λ be a T-fuzzy ideal and μ be a T-fuzzy bi-ideal of M having idempotent property, respectively. Then

$$\mu * \lambda * \mu \leq \mu * \mathbf{M} * \mu \leq \mu$$

and

$$\mu * \lambda * \mu \leq \mathbf{M} * \lambda * \mathbf{M} * \lambda.$$

Thus $\mu * \lambda * \mu \leq \lambda \cap \mu$.
Assume that $a \in M$. Since M is regular, there exist $\gamma_1, \gamma_2 \in \Gamma$ and $m \in M$ such that $a = a\gamma_1 m\gamma_2 a = a\gamma_1 m\gamma_2(a\gamma_1 m\gamma_2 a) = a\gamma a\gamma a$, where $\gamma = \gamma_1 m\gamma_2 \in \Gamma$.
Since λ is a T-fuzzy ideal of M, $\lambda(a\gamma a\gamma a) \geq T(\lambda(a), \lambda(a)) = (\lambda \cap \lambda)(a)$.Then,

$$\begin{aligned}(\mu * \lambda * \mu)(a) &= \sup_{a=x\,\gamma_1\,y\,\gamma_2\,z} \{T(\mu(x), T(\lambda(y), \mu(z)))\} \\ &\geq T(\mu(a), T(\lambda(a), \mu(a))) \\ &= T(T(\mu(a), \mu(a)), \lambda(a)) \\ &= T(\mu(a), \lambda(a)), \text{ by idempotent property of } \mu \\ &\geq (\mu \cap \lambda)(a),\end{aligned}$$

and hence $(\mu * \lambda * \mu) \geq \lambda \cap \mu$. Thus $\lambda \cap \mu = \mu * \lambda * \mu$.

(2) $\Rightarrow$ (3). It is clear from Lemma 3.12.

(3) $\Rightarrow$ (1). Assume that (3) holds. Let A and Q be any ideal and any quasi-ideal of M respectively. By Theorem 3.9 and Theorem 3.10(1) we have χ_A is a $T-$fuzzy ideal of M and χ_Q is a T-fuzzy quasi-ideal of M. By Lemma 3.11, we have

$$\chi_{A\cap Q} = \chi_A \cap \chi_Q = \chi_Q * \chi_A * \chi_Q = \chi_{(Q\Gamma A\Gamma Q)}.$$

Thus $A \cap Q = Q\Gamma A\Gamma Q$. By Lemma 3.15, it follows that M is regular. □

Lemma 3.25. *Let μ be a fuzzy subset of a regular Γ-semigroup M. Then the following conditions are equivalent:*

(1) *μ is a T-fuzzy ideal of M.*

(2) *μ is a T-fuzzy interior ideal of M.*

Proof. (1) $\Rightarrow$ (2) follows directly.

(2) $\Rightarrow$ (1). Let μ be a T-fuzzy interior ideal of M. Let $a, b \in M$. Then, since M is regular, there exist $\nu_1, \nu_2, \nu_3, \nu_4 \in \Gamma$ and $x, y \in M$ such that $a = a\,\nu_1 x\,\nu_2\,a$ and $b = b\,\nu_3 y\,\nu_4 b$. Consider,

$$\begin{aligned}\mu(a\nu b) &= \mu(a\,\nu_1\,x\,\nu_2\,a\,\nu\,b\,\nu_3\,y\,\nu_4\,b)\\ &= \mu((a\,\nu_1\,x\,\nu_2\,a)\,\nu\,b\,\nu_3\,(y\,\nu_4\,b))\\ &\geq \mu(b).\end{aligned}$$

In a similar way, we have

$$\begin{aligned}\mu(a\nu b) &= \mu((a\,\nu_1\,x)\,\nu_2\,a\,(\nu\,b\,\nu_3\,y\,\nu_4\,b))\\ &\geq \mu(a).\end{aligned}$$

This implies that μ is a T-fuzzy ideal of M. □

Theorem 3.26. *For a Γ-semigroup M, the following conditions are equivalent:*

(1) *M is regular.*

(2) *$\lambda \cap \lambda \leq \lambda * \mathbf{M} * \lambda \leq \lambda$ for every generalized T-fuzzy bi-ideal λ of M.*

(3) *$\lambda \cap \lambda \leq \lambda * \mathbf{M} * \lambda \leq \lambda$ for every T-fuzzy bi-ideal λ of M.*

(4) *$\lambda \cap \lambda \leq \lambda * \mathbf{M} * \lambda \leq \lambda$ for every T-fuzzy quasi-ideal λ of M.*

Proof. (1)$\Rightarrow$(2). Let M be a regular Γ-semigroup and λ be any T-fuzzy bi-ideal of M and $a \in M$. Since M is regular, there exists $\nu_1, \nu_2 \in \Gamma$ and $m \in M$ such that $a = a\,\nu_1\,m\,\nu_2\,a$. Then

$$\begin{aligned}(\lambda * \mathbf{M} * \lambda)(a) &= \sup_{a=x\,\nu_3\,m\,\nu_4\,y} \{T(\lambda(x), T(\mathbf{M}(m), \lambda(y)))\}\\ &\geq T(\lambda(a), T(\mathbf{M}(m), \lambda(a)))\\ &= T(\lambda(a), \lambda(a))\\ &= (\lambda \cap \lambda)(a)\end{aligned}$$

and so $\lambda \cap \lambda \leq \lambda * \mathbf{M} * \lambda$. Since λ is a generalized T-fuzzy bi-ideal, we have $\lambda \cap \lambda \leq \lambda * \mathbf{M} * \lambda \leq \lambda$. Thus (1) implies (2).

(2)$\Rightarrow$(3). Since every T-fuzzy bi-ideal is a generalized bi-ideal, (2)$\Rightarrow$(3) is trivial.

(3)$\Rightarrow$(4). As every T-fuzzy quasi-ideal is a T-fuzzy bi-ideal, (3)$\Rightarrow$(4) is trivial.

(4)$\Rightarrow$(1). Let Q be any quasi-ideal of M and let $a \in Q$. Then $\chi_Q(a) = 1$ and by

Theorem 3.10(1), χ_Q is a T-fuzzy quasi-ideal of M.
By (4), we have

$$\begin{aligned}(\chi_Q \cap \chi_Q)(a) &\leq (\chi_Q * \mathbf{M} * \chi_Q)(a)\\ &\leq \chi_Q(a).\end{aligned}$$

Thus

$$\begin{aligned}(\chi_{Q \cap Q})(a) &\leq \chi_{Q\Gamma M\Gamma Q}(a)\\ &\leq \chi_Q(a)\end{aligned}$$

and so we have $\chi_Q(a) \leq \chi_{Q\Gamma M\Gamma Q}(a) \leq \chi_Q(a)$.
This shows that $\chi_Q(a) = \chi_{Q\Gamma M\Gamma Q}(a)$. It follows that $Q = Q\Gamma M\Gamma Q$, for any quasi-ideal Q of M. By Lemma 3.14, M is regular. □

Theorem 3.27. *Let λ be any fuzzy subset of M having idempotent property. Then the following conditions are equivalent:*
(1) *M is regular.*
(2) *$\lambda = \lambda * \mathbf{M} * \lambda$ for every generalized T-fuzzy bi-ideal λ of M.*
(3) *$\lambda = \lambda * \mathbf{M} * \lambda$ for every T-fuzzy bi-ideal λ of M.*
(4) *$\lambda = \lambda * \mathbf{M} * \lambda$ for every T-fuzzy quasi-ideal λ of M.*

Proof. If λ is a generalized T-fuzzy bi-ideal having idempotent property, then $\lambda \cap \lambda = \lambda$. Thus the result follows from Theorem 3.26. □

Corollary 3.28. *Let λ be any fuzzy subset of M having idempotent property. The following conditions are equivalent:*
(1) *M is regular.*
(2) *$\lambda = \lambda \circ \mathbf{M} \circ \lambda$ for every generalized bi-ideal λ of M.*
(3) *$\lambda = \lambda \circ \mathbf{M} \circ \lambda$ for every bi-ideal λ of M.*
(4) *$\lambda = \lambda * \mathbf{M} * \lambda$ for every fuzzy quasi-ideal λ of M.*

The results follow once we take T =*min* in Theorem 3.27.

Theorem 3.29. *Let λ and μ be any T-fuzzy subsets of M having idempotent property. Then the following conditions are equivalent:*
(1) *M is regular.*
(2) *$\lambda \cap \mu = \mu * \lambda * \mu$ for every T-fuzzy ideal λ and every generalized T-fuzzy bi-ideal μ of M.*
(3) *$\lambda \cap \mu = \mu * \lambda * \mu$ for every T-fuzzy ideal λ and every T-fuzzy bi-ideal μ of M.*
(4) *$\lambda \cap \mu = \mu * \lambda * \mu$ for every T-fuzzy ideal λ and every T-fuzzy quasi-ideal μ of M.*

Proof. (1)⇒(2). Let λ and μ be a T-fuzzy ideal and a generalized T-fuzzy bi-ideal of M respectively both having the idempotent property. Then

$$\mu * \lambda * \mu \leq \mu * \mathbf{M} * \mu \leq \mu$$

and

$$\mu * \lambda * \mu \leq \mathbf{M} * \lambda * \mathbf{M} \leq \lambda.$$

Thus $\mu*\lambda*\mu \leq \lambda\cap\mu$. Assume that $a \in M$. Since M is regular, there exist $\nu_1, \nu_2 \in \Gamma$ and $m \in M$ such that $a = a\,\nu_1\,m\,\nu_2\,a = a\,\nu_1\,m\,\nu_2\,(a\,\nu_1\,m\,\nu_2\,a) = a\,\nu\,m\,\nu\,a$, where $\nu = \nu_1\,m\,\nu_2 \in \Gamma$. Then

$$\begin{aligned}(\mu*\lambda*\mu)(a) &= \sup_{a=x\,\nu_1\,y\,\nu_2\,z}\{T(\mu(x), T(\lambda(y), \mu(z)))\}\\ &\geq T(\mu(a), T(\lambda(a), \mu(a)))\\ &= T(T(\mu(a), \mu(a)), \lambda(a))\\ &= T((\mu\cap\mu)(a), \lambda(a))\\ &= T(\mu(a), \lambda(a))\\ &= (\mu\cap\lambda)(a).\end{aligned}$$

Thus we obtain that $\lambda\cap\mu = \mu*\lambda*\mu$.
Hence (1)⇒(2).

(2)⇒(3) is clear.

(3)⇒(4) is trivial by Lemma 3.12.

(4)⇒(1). Assume that (4) holds. Let A and Q be any ideal and any quasi-ideal of M respectively. By Theorem 3.9 and Theorem 3.10(1) we have χ_A is a T-fuzzy ideal of M and χ_Q is a T-fuzzy quasi-ideal of M. By Lemma 3.11, we have

$$\begin{aligned}\chi_{A\cap Q} &= \chi_A\cap\chi_Q\\ &= \chi_Q*\chi_A*\chi_Q\\ &= \chi_{(Q\Gamma A\Gamma Q)}.\end{aligned}$$

Thus $A\cap Q = Q\Gamma A\Gamma Q$. By Lemma 3.15, it follows that M is regular. □

Theorem 3.30. *Let λ and μ be any fuzzy subsets of M having idempotent property. Then, the following conditions are equivalent:*

(1) *M is regular.*

(2) *$\lambda\cap\mu = \mu*\lambda*\mu$ for every T-fuzzy ideal λ and every T-fuzzy quasi-ideal μ of M.*

(3) *$\lambda\cap\mu = \mu*\lambda*\mu$ for every interior ideal λ and every T-fuzzy quasi-ideal μ of M.*

(4) *$\lambda\cap\mu = \mu*\lambda*\mu$ for every T-fuzzy ideal λ and every T-fuzzy bi-ideal μ of M.*

(5) *$\lambda\cap\mu = \mu*\lambda*\mu$ for every T-fuzzy interior ideal λ and every T-fuzzy bi-ideal μ of M.*

(6) *$\lambda\cap\mu = \mu*\lambda*\mu$ for every T-fuzzy ideal λ and every generalized bi-ideal μ of M.*

(7) *$\lambda\cap\mu = \mu*\lambda*\mu$ for every T-fuzzy interior ideal λ and every generalized bi-ideal μ of M.*

Proof. (1)⇒(7). Let λ and μ be any T-fuzzy interior ideal and generalized T-fuzzy bi-ideal of M having idempotent properties respectively. Then

$$\mu*\lambda*\mu \leq \mu*\mathbf{M}*\mu \leq \mu$$

and

$$\mu*\lambda*\mu \leq \mathbf{M}*\mu*\mathbf{M} \leq \lambda.$$

This means that $(\mu * \lambda * \mu) \cap (\mu * \lambda * \mu) \leq \mu \cap \lambda$.
By Lemma 3.17, $(\mu * \lambda * \mu) \leq \mu \cap \lambda$.
Let $a \in M$. Then, since M is regular, there exist $x \in M$, $\nu_1, \nu_2 \in \Gamma$ such that $a = a\,\nu_1\,x\,\nu_2\,a (= a\,\nu_1\,x\,\nu_2\,a\,\nu_1\,x\,\nu_2\,a)$. Now

$$\begin{aligned}
(\mu * \lambda * \mu)(a) &= \sup_{a = s\,\nu_3\,t\,\nu_4\,u} \{T(\mu(s), T(\lambda(t), \mu(u)))\} \\
&\geq T(\mu(a), T(\lambda(x\,\nu_2\,a\,\nu_1\,x), \mu(a))) \\
&\geq T(\mu(a), T(\lambda(a), \mu(a))), \\
&\quad [\text{Since } \lambda(x\,\nu_2\,a\,\nu_1\,x) \geq \lambda(a)] \\
&= T(\mu(a), T(\mu(a), \lambda(a))) \\
&= T(T(\mu(a), \mu(a), \lambda(a))) \\
&= T(\mu(a), \lambda(a)) \\
&= (\mu \cap \lambda)(a).
\end{aligned}$$

Thus $\mu * \lambda * \mu = \mu \cap \lambda$ and so (1)⇒(7).

It is clear that (7)⇒(5)⇒(3)⇒(2) and (7)⇒(6)⇒(4)⇒(2).

(2)⇒(1). Let μ be any idempotent T-fuzzy quasi-ideal of M. We have $\mu = \mu \cap \mathbf{M} = \mu * \mathbf{M} * \mu$. It follows from the fact $\mathbf{M}$ itself is an ideal and from Theorem 3.26 that M is regular and hence (2)⇒(1). □

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