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Fuzzy generalized continuity

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Abstract. In this paper some different types of fuzzy generalized closed sets are introduced and studied here. Also their mutual relationships have been established. Afterwards, some different kinds of fuzzy generalized continuous functions have been introduced and studied and made some mutual relationships among themselves. In the last section, it has been shown that under these types of fuzzy generalized continuous functions, fuzzy normality remains invariant.

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1. Introduction

In [2], Balasubramanian and Sundaram have introduced fuzzy generalized closed set in 1997. Also in [8], Murugesan and Thangavelu have investigated the relationships between some types of fuzzy generalized closed sets. Afterwards, in 2014, Bhattacharyya [4] has introduced and studied some different types of fuzzy generalized closed sets and found the mutual relationships between them. In this paper another kinds of fuzzy generalized closed sets are introduced and studied. Also some different types of fuzzy generalized continuity are introduced and studied under which fuzzy normality remains invariant.

2. Preliminaries

In 1965, L.A. Zadeh introduced fuzzy set [12] A which is a mapping from a non-empty set X into the closed interval $I = [0, 1]$, i.e., $A \in I^X$. The support [12] of a fuzzy set A, denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement [12] of a fuzzy set A in a fuzzy topological space (fts, for short) X in the sense of Chang [6] is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [12] while AqB means A is quasi-coincident (q-coincident, for short) [10] with B, i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. For a fuzzy set A, clA and $intA$ will stand for fuzzy closure [6] and fuzzy interior [6] respectively.

A fuzzy set A in an fts (X, τ) is called fuzzy semiopen [1] (resp., fuzzy α-open [5], fuzzy preopen [9]) if $A \leq clintA$ (resp., $A \leq intclintA$, $A \leq intclA$). The complement of these three fuzzy sets in an fts (X, τ) are respectively called fuzzy semiclosed [1], fuzzy α-closed [5], fuzzy preclosed [9], i.e., a fuzzy set A in X is called fuzzy semiclosed (resp., fuzzy α-closed, fuzzy preclosed) if $intclA \leq A$ (resp., $clintclA \leq A$, $clintA \leq A$). For a fuzzy set A, $sclA$ [1] (resp., αclA [5], $pclA$ [9]) is the smallest fuzzy semiclosed (resp., fuzzy α-closed, fuzzy preclosed) set containing A. The collection of all fuzzy semiopen (resp., fuzzy α-open) sets in X is denoted by $FSO(X)$ (resp., $F\alpha O(X)$).

3. Fuzzy generalized closed sets and fuzzy generalized continuity

In [8] the relationships between different types of fuzzy generalized closed sets are studied. In this section some different types of fuzzy generalized closed sets and fuzzy generalized continuities have been introduced and found some mutual relationships among themselves.

Definition 3.1. A fuzzy set A in an fts (X, τ) is said to be fuzzy

(i) generalized closed (fg-closed, for short) [4, 2] if $clA \leq U$ whenever $A \leq U$ and $U \in \tau$,

(ii) generalized semiclosed (fgs-closed, for short) [3] if $sclA \leq U$ whenever $A \leq U$ and $U \in \tau$,

(iii) semi-generalized closed (fsg-closed, for short) [3] if $sclA \leq U$ whenever $A \leq U$ and $U \in FSO(X)$,

(iv) weakly closed (fw-closed, for short) if $clA \leq U$ whenever $A \leq U$ and $U \in FSO(X)$,

(v) weakly generalized closed (fwg-closed, for short) if $clintA \leq U$ whenever $A \leq U$ and $U \in \tau$,

(vi) generalized α-closed ($fg\alpha$-closed, for short) [3] if $\alpha clA \leq U$ whenever $A \leq U$ and $U \in F\alpha O(X)$,

(vii) α-generalized closed ($f\alpha g$-closed, for short) [3] if $\alpha clA \leq U$ whenever $A \leq U$ and $U \in \tau$,

(viii) strongly g-closed (fs^*g-closed, for short) if $clA \leq U$ whenever $A \leq U$ and U is fg-open in X.

The complements of the above mentioned sets are called their respective open sets.

Definition 3.2. A function $f : (X, \tau_X) \to (Y, \tau_Y)$ is called fuzzy

(i) generalized continuous (fg-continuous, for short) [3] if $f^{-1}(V)$ is fg open in X for each $V \in \tau_Y$,

(ii) strongly g-continuous (fs^*g-continuous, for short) if $f^{-1}(V) \in \tau_X$ for each fg-open set V in Y,

(iii) semi-generalized continuous (fsg-continuous, for short) [3] if $f^{-1}(V)$ is fsg-open in X for each $V \in \tau_Y$,

(iv) generalized semi-continuous (fgs-continuous, for short) [3] if $f^{-1}(V)$ is fgs-open in X for each $V \in \tau_Y$.

Remark 3.3. It is clear from Definition 3.1 that every fuzzy closed set is fg-closed. But the converse need not be true as seen form the following example.

Example 3.4. fg-closed $\not\Rightarrow$ fuzzy closed.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.4, A(b) = 0.55$, $B(a) = 0.5, B(b) = 0.6$. Then (X, τ) is an fts. Consider the fuzzy set D in (X, τ) defined by $D(a) = 0.5, D(b) = 0.8$. Clearly D is not fuzzy closed in (X, τ). But 1_X is the only fuzzy open set in X containing D and so $clD = 1_X \leq 1_X \Rightarrow D$ is fg-closed in (X, τ).

Remark 3.5. From Definition 3.1(i) and (ii), we can conclude that fg-closed set is fgs-closed, but not conversely as it seen from the following example.

Example 3.6. fgs-closed $\not\Rightarrow$ fg-closed.
Let $X = \{a, b\}, \tau = \{0_X, 1_X, A\}$ where $A(a) = 0.4, A(b) = 0.5$. Then (X, τ) is an fts. The collection of fuzzy semiopen sets in X is $\{0_X, 1_X, A, U\}$ where $A \leq U \leq 1_X \setminus A$ and that of fuzzy semiclosed sets is $\{0_X, 1_X, 1_X \setminus A, 1_X \setminus U\}$ where $A \leq 1_X \setminus U \leq 1_X \setminus A$. Let B be a fuzzy set in X defined by $B(a) = B(b) = 0.4$. It is clear that B is not fg-closed. Indeed, $B \leq A \in \tau$, but $clB = 1_X \setminus A \not\leq A$. But $B \leq A \in \tau$ and $sclB = A \leq A \Rightarrow B$ is fgs-closed in X.

Remark 3.7. It is clear from Definition 3.1(ii) and (iii) that fsg-closed set is fgs-closed. But the converse need not be true as seen from the following example.

Example 3.8. fgs-closed $\not\Rightarrow$ fsg-closed.
Consider Example 3.4. We first prove that D is fgs-closed in (X, τ). Now the collection of all fuzzy semiopen sets in (X, τ) is $\{0_X, 1_X, A, B, U\}$ where $U > A$ so that that of fuzzy semiclosed sets in (X, τ) is $\{0_X, 1_X, 1_X \setminus A, 1_X \setminus B, 1_X \setminus U\}$ where $1_X \setminus U < 1_X \setminus A$. Now 1_X is the only fuzzy open set in X containing D and so $sclD \leq 1_X \Rightarrow D$ is fgs-closed in X. Now as $D > A$, $D \in FSO(X)$ containing D, but $sclD = 1_X \not\leq D \Rightarrow D$ is not fsg-closed in X.

Remark 3.9. fg-closedness and fsg-closedness are independent notions follow from the next two examples.

Example 3.10. fg-closed $\not\Rightarrow$ fsg-closed.
Consider Example 3.4. Here D is not fsg-closed as shown in Example 3.8, but D is fg-closed as shown in Example 3.4.

Example 3.11. fsg-closed $\not\Rightarrow$ fg-closed.
Consider Example 3.6. Here B is not fg-closed. But $B \leq A(\in FSO(X))$ and so $sclB = A \leq A \Rightarrow B$ is fsg-closed.

Remark 3.12. From Definition 3.1(i) and (iv), it is clear that fw-closed set is fg-closed. But the converse need not be true as shown in the following example.

Example 3.13. fg-closed $\not\Rightarrow$ fw-closed.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.4, A(b) = 0.5$. Then (X, τ) is an fts. The collection of fuzzy semiopen sets in X is $\{0_X, 1_X, A, U\}$ where $A \leq U \leq 1_X \setminus A$ and that of fuzzy semiclosed sets is $\{0_X, 1_X, 1_X \setminus A, 1_X \setminus U\}$ where $A \leq 1_X \setminus U \leq 1_X \setminus A$. Consider the fuzzy set B defined by $B(a) = B(b) = 0.5$. Then 1_X is the only fuzzy open set in X containing B and so B is fg-closed. Now $B \in FSO(X)$. Therefore, $B \leq B$ and $clB = 1_X \setminus A \not\leq B \Rightarrow B$ is not fw-closed.

Remark 3.14. It is clear from Definition 3.1(i) and (v) that fg-closed set is fwg-closed, but the converse may not be true as it seen from the following example.

Example 3.15. fwg-closed $\not\Rightarrow$ fg-closed.
Consider Example 3.6. Here B is not fg-closed. Now $B \leq A \in \tau \Rightarrow clintB = cl0_X = 0_X \leq A \Rightarrow B$ is fwg-closed.

Remark 3.16. fg-closedness and $fg\alpha$-closedness are independent notions as it seen from the following examples.

Example 3.17. fg-closed $\not\Rightarrow$ $fg\alpha$-closed.
Consider Example 3.4 and the fuzzy set C defined by $C(a) = 0.5, C(b) = 0.7$. We claim that C is fg-closed. Infact, 1_X is the only fuzzy open set in X containing C and so $clC = 1_X \leq 1_X \Rightarrow C$ is fg-closed. Again $C \in F\alpha O(X)$ as $intclintC = intclB = int1_X = 1_X > C$. Then $C \leq C \in F\alpha O(X)$ and $\alpha clC = 1_X \not\leq C \Rightarrow C$ is not $fg\alpha$-closed.

Example 3.18. $fg\alpha$-closed $\not\Rightarrow$ fg-closed.
Consider Example 3.4 and the fuzzy set E defined by $E(a) = E(b) = 0.4$. Then $E \leq A \in \tau$, but $clE = 1_X \setminus B \not\leq A \Rightarrow E$ is not fg-closed. Again $A \in F\alpha O(X)$ such that $E \leq A, \alpha clE = E \leq A \Rightarrow E$ is $fg\alpha$-closed.

Remark 3.19. From Definition 3.1(i) and (vii), we can conclude that fg-closed $\Rightarrow f\alpha g$-closed, but not conversely follows from the next example.

Example 3.20. $f\alpha g$-closed $\not\Rightarrow$ fg-closed.
Consider Example 3.18. Here E is not fg-closed. Again $E \leq A \in \tau$ and $\alpha clE = E \leq A \Rightarrow E$ is $f\alpha g$-closed.

Remark 3.21. Since fuzzy open set is fg-open, fs^*g-closed set is fg-closed, but not conversely as it seen from the following example.

Example 3.22. fg-closed $\not\Rightarrow$ fs^*g-closed.
Consider Example 3.6 and the fuzzy set B defined by $B(a) = B(b) = 0.5$. Here B is fg-closed. Indeed, 1_X is the only fuzzy open set in X containing B and so $clB \leq 1_X \Rightarrow B$ is fg-closed. Also B is fg-open and $B \leq B$ and $clB = 1_X \setminus A \not\leq B \Rightarrow B$ is not fs^*g-closed.

Remark 3.23. It is clear from Definition 3.1(vi) and (vii) that $fg\alpha$-closed $\Rightarrow f\alpha g$-closed, but not conversely as it seen from the following example.

Example 3.24. $f\alpha g$-closed $\not\Rightarrow$ $fg\alpha$-closed.
Consider Example 3.17. Here C is not $fg\alpha$-closed. Now 1_X is the only fuzzy open set in X containing C and so C is $f\alpha g$-closed.

Remark 3.25. It is clear from Definition 3.1(v) and (viii) that fs^*g-closed $\Rightarrow fwg$-closed, but not conversely as it seen from the following example.

Example 3.26. fwg-closed $\not\Rightarrow fs^*g$-closed.
Consider Example 3.22. Here B is not fs^*g-closed. Now 1_X is the only fuzzy open set in X containing B and so B is fwg-closed.

Remark 3.27. From Definition 3.1(vii) and (viii) that fs^*g-closed $\Rightarrow f\alpha g$-closed, but the converse need not be true as it seen from the following example.

Example 3.28. $f\alpha g$-closed $\not\Rightarrow fs^*g$-closed.
Consider Example 3.22. Here B is not fs^*g-closed. But 1_X is the only fuzzy open set in X containing B and so B is $f\alpha g$-closed.

Remark 3.29. From Definition 3.1(ii) and (viii), it is clear that fs^*g-closed set is fgs-closed, but not conversely as it seen from the following example.

Example 3.30. fgs-closed $\not\Rightarrow fs^*g$-closed.
Consider Example 3.22. Here B is not fs^*g-closed. But 1_X is the only fuzzy open set in X containing B and so B is fgs-closed.

Remark 3.31. Definition 3.1 (iv) and (v) together imply that fw-closed $\Rightarrow fwg$-closed, but the converse may not be true as seen from the following example.

Example 3.32. fwg-closed $\not\Rightarrow fw$-closed.
Consider Example 3.22. Here $B \in FSO(X)$ so that $B \leq B$, but $clB = 1_X \setminus A \not\leq B \Rightarrow B$ is not fw-closed. But 1_X is the only fuzzy open set in X containing B and so B is fwg-closed.

Remark 3.33. Definition 3.1 (v) and (viii) together imply that fs^*g-closed $\Rightarrow fwg$-closed, but the converse may not be true as it seen from the following example.

Example 3.34. fwg-closed $\not\Rightarrow fs^*g$-closed.
Consider Example 3.32. Here B is fwg-closed. Now B is fg-open in X so that $B \leq B$, but $clB = 1_X \setminus A \not\leq B \Rightarrow B$ is not fs^*g-closed.

Remark 3.35. It is clear from Definition 3.1(ii) that fuzzy closed set is fgs-closed, but not conversely as it seen from the following example.

Example 3.36. fgs-closed $\not\Rightarrow$ fuzzy closed.
Consider Example 3.6. Here B is fgs-closed. But B is not fuzzy closed in (X, τ).

Example 3.37. $fg\alpha$-closed $\not\Rightarrow fs^*g$-closed
Consider Example 3.22. Here B is not fs^*g-closed. Now 1_X is the only fuzzy α-open set in X containing B and so $\alpha clB \leq 1_X \Rightarrow B$ is $fg\alpha$-closed.

Example 3.38. fsg-closed $\not\Rightarrow fs^*g$-closed.
Consider Example 3.11. Here B is fsg-closed. Now 1_X is the only fuzzy open set in X containing $1_X \setminus B$ and so $cl(1_X \setminus B) = 1_X \leq 1_X \Rightarrow 1_X \setminus B$ is fg-closed $\Rightarrow B$ is fg-open in X. Now $B \leq B$, $clB = 1_X \setminus A \not\leq B \Rightarrow B$ is not fs^*g-closed.

Remark 3.39. It is clear from Definition 3.1(iii) and (iv) that fw-closed set is fsg-closed, but the converse may not be true as seen from the following example.

Example 3.40. fsg-closed $\not\Rightarrow$ fw-closed.
Consider Example 3.6. Here $B \leq A \in FSO(X)$ and $sclB = A \leq A \Rightarrow B$ is fsg-closed. Now $B \leq A \in FSO(X)$, but $clB = 1_X \setminus A \not\leq A \Rightarrow B$ is not fw-closed.

Remark 3.41. From Definition 3.2 (iii) and (iv) that fsg-continuity $\Rightarrow$ fgs-continuity, but the converse need not be true as seen from the following example.

Example 3.42. fgs-continuity $\not\Rightarrow$ fsg-continuity.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$, $\tau_1 = \{0_X, 1_X, E\}$ where $A(a) = 0.4, A(b) = 0.55$, $B(a) = 0.5, B(b) = 0.6$ and $E(a) = 0.5, E(b) = 0.2$. Then (X, τ) and (X, τ_1) are fts's. Consider the identity function $i : (X, \tau) \to (X, \tau_1)$. Then $1_X \setminus E \in \tau_1^c$ and $i^{-1}(1_X \setminus E) = 1_X \setminus E$ where $(1_X \setminus E)(a) = 0.5, (1_X \setminus E)(b) = 0.8$. By Example 3.8, $1_X \setminus E$ is fgs-closed but not fsg-closed and hence i is fgs-continuous but not fsg-continuous.

Remark 3.43. It is clear from Definition 3.2 and Remark 3.35 that fs^*g-continuity implies fgs-continuity and fsg-continuity, but not conversely as it seen from the next two examples.

Example 3.44. fgs-continuity $\not\Rightarrow$ fs^*g-continuity.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_1 = \{0_X, 1_X, C\}$ where $A(a) = 0.4, A(b) = 0.5$ and $C(a) = C(b) = 0.6$. Then (X, τ) and (X, τ_1) are fts's. Consider the identity function $i : (X, \tau) \to (X, \tau_1)$. Now $1_X \setminus C \in \tau_1^c$ and $i^{-1}(1_X \setminus C) = 1_X \setminus C$ which is fgs-closed in (X, τ) by Example 3.6. But $1_X \setminus C \notin \tau^c$ ($1_X \setminus C \in \tau_1^c \Rightarrow 1_X \setminus C$ is fg-closed in (X, τ_1)). Therefore i is fgs-continuous but not fs^*g-continuous.

Example 3.45. fsg-continuity $\not\Rightarrow$ fs^*g-continuity.
Consider Example 3.44. Here also $1_X \setminus C$ is fsg-closed and so i is fsg-continuous but not fs^*g-continuous.

Remark 3.46. From Definition 3.2, it is clear that fs^*g-continuity $\Rightarrow$ fg-continuity, but the converse need not be true as it seen from the following example.

Example 3.47. fg-continuity $\not\Rightarrow$ fs^*g-continuity.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_1 = \{0_X, 1_X, B\}$ where $A(a) = 0.4, A(b) = 0.5$, $B(a) = B(b) = 0.4$. Then (X, τ) and (X, τ_1) are fts's. Consider the identity function $i : (X, \tau) \to (X, \tau_1)$. Now $1_X \setminus B \in \tau_1^c$, $i^{-1}(1_X \setminus B) = 1_X \setminus B$. Now 1_X is the only fuzzy open set in X containing $1_X \setminus B$ and so $1_X \setminus B$ is fg-closed in (X, τ) and so i is fg-continuous.
Again we know that every fuzzy closed set is fg-closed, $1_X \setminus B$ is fg-closed in (X, τ_1), but $1_X \setminus B \notin \tau^c$ and so i is not fs^*g-continuous.

4. fg^*s-closed sets in a fuzzy topological space

In this section we first introduce a new type of fuzzy generalized closed set and then find mutual relationship of this newly defined set with the sets defined in Section 3.

Definition 4.1. A fuzzy set A in a fts (X, τ) is called fuzzy g^*s-closed (fg^*s-closed, for short) if $sclA \leq U$ whenever $A \leq U$ and U is fgs-open in (X, τ).
The complement of an fg^*s-closed set is called fg^*s-open.

Theorem 4.2. *Every fuzzy closed set in a fts (X, τ) is fg^*s-closed.*

Proof. Let $A \in \tau^c$ and U be an fgs-open set in X such that $A \leq U$. Then $sclA \leq clA = A \leq U \Rightarrow A$ is fg^*s-closed in X. □

The converse of the above theorem need not be true as seen from the following example.

Example 4.3. fg^*s-closed $\not\Rightarrow$ fuzzy closed.
Consider Example 3.4. Here the collection of fuzzy semiopen sets in X is $\{0_X, 1_X, A, B, U\}$ where $U > A$ and that of fuzzy semiclosed sets is $\{0_X, 1_X, 1_X \setminus A, 1_X \setminus B, 1_X \setminus U\}$ where $1_X \setminus U < 1_X \setminus A$. Here 1_X is the only fuzzy open set in X containing $D \Rightarrow sclD = 1_X \Rightarrow D$ is fgs-closed in $(X, \tau) \Rightarrow 1_X \setminus D$ is fgs-open in (X, τ). Again, $1_X \setminus D$ is fgs-open in X containing $1_X \setminus D \Rightarrow scl(1_X \setminus D) = 1_X \setminus D \leq 1_X \setminus D \Rightarrow 1_X \setminus D$ is fg^*s-closed, but $1_X \setminus D \notin \tau^c \Rightarrow 1_X \setminus D$ is not fuzzy closed in (X, τ).

Theorem 4.4. *Union of two fg^*s-closed sets is fg^*s-closed.*

Proof. Let A and B be two fg^*s-closed sets in a fts (X, τ). Let U be an fgs-open set in X such that $A \bigvee B \leq U$. Then $A \leq U$ and $B \leq U$. By hypothesis, $sclA \leq U$, $sclB \leq U$. Now $scl(A \bigvee B) = sclA \bigvee sclB \leq U$ (Clearly, $sclA \bigvee sclB \leq scl(A \bigvee B)$. To prove the converse, let $x_\alpha \in scl(A \bigvee B)$. Then for any fuzzy semiopen set U in X with $x_\alpha qU$, $Uq(A \bigvee B)$. Then there exists $y \in X$ such that $U(y) + (A \bigvee B)(y) > 1 \Rightarrow U(y) + max\{A(y), B(y)\} > 1 \Rightarrow U(y) + A(y) > 1$ or $U(y) + B(y) > 1 \Rightarrow UqA$ or $UqB \Rightarrow x_\alpha \in sclA$ or $x_\alpha \in sclB \Rightarrow x_\alpha \in sclA \bigvee sclB$). Hence the proof. □

Theorem 4.5. *Every fg^*s-closed set in a fts (X, τ) is fgs-closed.*

Proof. Let A be fg^*s-closed in X. Let $U \in \tau$ be such that $A \leq U$. Since every fuzzy open set is fgs-open (by Remark 3.35), by assumption, $sclA \leq U$. Hence A is fgs-closed. □

The converse of the above theorem need not be true as seen from the following example.

Example 4.6. fgs-closed $\not\Rightarrow$ fg^*s-closed.
Consider Example 3.6. Here B is fgs-closed in X. Again $(1_X \setminus B)(a) = 1 - B(a) = 1 - B(b) = (1_X \setminus B)(b) = 0.6$. And so 1_X is the only fuzzy open set in X containing $1_X \setminus B \Rightarrow scl(1_X \setminus B) \leq 1_X \Rightarrow 1_X \setminus B$ is also fgs-closed in $X \Rightarrow B$ is fgs-open in X. Now $B \leq B$ where B is fgs-open in $X \Rightarrow sclB = A \not\leq B \Rightarrow B$ is not fg^*s-closed in X.

Remark 4.7. It is clear from Definition 3.1 that A is fuzzy semiclosed $\Rightarrow A$ is fgs-closed and so $A \in FSO(X) \Rightarrow A$ is fgs-open in X.

Theorem 4.8. *Every fg^*s-closed set in a fts X is fsg-closed in X.*

Proof. Let A be fg^*s-closed set in X. Let $U \in FSO(X)$ and $A \leq U$. By Remark 4.7, U is fgs-open in X. By assumption, $sclA \leq U \Rightarrow A$ is fsg-closed in X. □

The converse of the above theorem need not be true as seen from the following example.

Example 4.9. fsg-closed $\not\Rightarrow$ fg^*s-closed.
Consider Example 4.6. Here $B \leq A$ and $A \in FSO(X)$ and so $sclB = A \leq A \Rightarrow A$ is fsg-closed, but not fg^*s-closed in X.

Remark 4.10. fg^*s-closedness is independent of the following classes of fuzzy closedness, viz., fg-closedness, fw-closedness, $fg\alpha$-closedness, $f\alpha g$-closedness, fuzzy preclosedness.

Example 4.11. fuzzy preclosed $\not\Rightarrow$ fg^*s-closed.
Consider Example 4.6. Here $clintB = cl0_X = 0_X \leq B \Rightarrow B$ is fuzzy preclosed, but B is not fg^*s-closed as shown in Example 4.6.

Example 4.12. fg^*s-closed $\not\Leftarrow$ fuzzy preclosed.
Consider Example 3.22. Here B is not fuzzy preclosed. Indeed, $clintB = clA = 1_X \setminus A \not\leq B$. Now we show that B is fg^*s-closed. Here 1_X is the only fuzzy open set in X containing B and so $clB \leq 1_X \Rightarrow B$ is fg-closed in $X \Rightarrow 1_X \setminus B$ is fg-open in X. But $1_X \setminus B = B$. Now $B \leq 1_X \setminus B(= B)$ where $1_X \setminus B$ is fg-open in $X \Rightarrow sclB = B \leq B \Rightarrow B$ is fg^*s-closed in X.

Example 4.13. fg^*s-closed $\not\Rightarrow$ fw-closed.
Consider Example 4.12. Now $B \in FSO(X)$, $B \leq B$, but $clB = 1_X \setminus A \not\leq B \Rightarrow B$ is not fw-closed, though B is fg^*s-closed in X.

Example 4.14. fw-closed $\not\Rightarrow$ fg^*s-closed.
Consider Example 3.6 and the fuzzy set B defined by $B(a) = B(b) = 0.6$. Then 1_X is the only fuzzy semiopen set in X containing B and so $clB \leq 1_X \Rightarrow B$ is fw-closed in X. Again from Example 4.6, it is clear that B is fgs-open. So $B \leq B$, but $sclB = 1_X \not\leq B \Rightarrow B$ is not fg^*s-closed in X.

Example 4.15. fg-closed $\not\Rightarrow$ fg^*s-closed.
Consider Example 4.3 and consider the fuzzy set C defined by $C(a) = 0.5, C(b) = 0.7$. We claim that C is fg-closed. Infact, 1_X is the only fuzzy open set in X containing C and so $clC \leq 1_X \Rightarrow C$ is fg-closed. Again, C is fgs-open in X. Indeed, $1_X \setminus C \leq B \Rightarrow scl(1_X \setminus C) = 1_X \setminus C \leq B \Rightarrow 1_X \setminus C$ is fgs-closed in $X \Rightarrow C$ is fgs-open in X. Now $C \leq C$ and $sclC = 1_X \not\leq C \Rightarrow C$ is not fg^*s-closed in X.

Example 4.16. fg^*s-closed $\not\Rightarrow$ fg-closed.
Let $X = \{a, b\}, \tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.4, A(b) = 0.5, B(a) = 0.5, B(b) = 0.6$. Then (X, τ) is an fts. Then the collection of fuzzy semiopen sets in X is $\{0_X, 1_X, A, B, U\}$ where $U > A$ and that of fuzzy semiclosed sets is $\{0_X, 1_X, 1_X \setminus A, 1_X \setminus B, 1_X \setminus U\}$ where $1_X \setminus U < 1_X \setminus A$. Consider the fuzzy set E defined by $E(a) = E(b) = 0.5$. We claim that E is fg^*s-closed in X. Let F be a fuzzy set defined by $F(a) > 0.5, F(b) = 0.5$. We first show that $1_X \setminus F$ is fgs-closed in X. Now $1_X \setminus F < B \in \tau$ and $scl(1_X \setminus F) = 1_X \setminus F < B \Rightarrow 1_X \setminus F$ is fgs-closed in $X \Rightarrow F$ is fgs-open in X. Now $E < F$ and so $sclE = E < F \Rightarrow E$ is fg^*s-closed in X. Now B is a fuzzy open set in X containing E, but $clE = 1_X \setminus A \not\leq B \Rightarrow E$ is not fg-closed in X.

Example 4.17. fg^*s-closed $\not\Rightarrow$ $fg\alpha$-closed.
Consider Example 4.16. Here E is fg^*s-closed. We claim that E is not $fg\alpha$-closed

in X. Now $E < B(\in F\alpha O(X))$, but $\alpha clE = 1_X \setminus A \not\leq B \Rightarrow E$ is not $fg\alpha$-closed in X.

Example 4.18. $fg\alpha$-closed $\not\Rightarrow fg^*s$-closed.
Consider Example 4.14. Here B is not fg^*s-closed in X. Now 1_X is the only fuzzy open set in X containing B and so $\alpha clB = 1_X \leq 1_X \Rightarrow B$ is $fg\alpha$-closed in X.

Example 4.19. $f\alpha g$-closed $\not\Rightarrow fg^*s$-closed.
Consider Example 4.15. Now 1_X is the only fuzzy open set in X containing C and so $\alpha clC \leq 1_X \Rightarrow C$ is $f\alpha g$-closed in X though C is not fg^*s-closed in X.

Example 4.20. fg^*s-closed $\not\Rightarrow f\alpha g$-closed.
Consider Example 4.16. Here E is fg^*s-closed in X. Now $E < B \in \tau$, $\alpha clE = 1_X \setminus A \not\leq B \Rightarrow E$ is not $f\alpha g$-closed in X.

Example 4.21. fg^*s-closed $\not\Rightarrow fs^*g$-closed.
Consider Example 4.12. Here B is fg^*s-closed in X. Now B is fg-open in X and so $B \leq B$, but $clB = 1_X \setminus A \not\leq B \Rightarrow B$ is not fs^*g-closed in X.

Example 4.22. fs^*g-closed $\not\Rightarrow fg^*s$-closed.
Consider Example 4.14. Here B is not fg^*s-closed in X. Now $B \leq 1_X$ where 1_X is the only fg-open set in X. Now $clB \leq 1_X \Rightarrow B$ is fs^*g-closed in X.

5. fg^*s-continuous function in a fuzzy topological space

In this section a new type of fuzzy generalized continuity has been introduced and studied and found mutual relationships of this newly defined function with other fuzzy generalized functions defined in Section 3.

Definition 5.1. A function $f : (X, \tau_X) \to (Y, \tau_Y)$ is called fg^*s-continuous if $f^{-1}(V)$ is fg^*s-closed in X whenever $V \in \tau_Y^c$.

Definition 5.2 ([6])**.** A function $f : (X, \tau_X) \to (Y, \tau_Y)$ is called fuzzy continuous if $f^{-1}(V) \in \tau_X^c$ for all $V \in \tau_Y^c$.

Theorem 5.3. *If $f : (X, \tau_X) \to (Y, \tau_Y)$ is fuzzy continuous, then it is fg^*s-continuous.*

Proof. Let $V \in \tau_Y^c$. Then $f^{-1}(V) \in \tau_X^c$. By Theorem 4.2, $f^{-1}(V)$ is fg^*s-closed in X and hence the proof. □

The converse of the above theorem is not necessarily true, as seen from the following example.

Example 5.4. fg^*s-continuity $\not\Rightarrow$ fuzzy continuity.
Let $X = \{a, b\}$, $\tau_X = \{0_X, 1_X, A, B\}$, where $A(a) = 0.4, A(b) = 0.55$, $B(a) = 0.5, B(b) = 0.6$, $Y = \{c, d\}$, $\tau_Y = \{0_Y, 1_Y, C\}$, where $C(c) = 0.8, C(d) = 0.5$. Then (X, τ_X) and (Y, τ_Y) are fts's. Let us consider the fuzzy function $f : (X, \tau_X) \to (Y, \tau_Y)$ defined by $f(a) = d, f(b) = c$. We claim that f is fg^*s-continuous, but not fuzzy continuous. Consider the fuzzy set D in X defined by $D(a) = 0.5, D(b) = 0.8$. Now $1_Y \setminus C \in \tau_Y^c$. $[f^{-1}(1_Y \setminus C)](a) = (1_Y \setminus C)f(a) = (1_Y \setminus C)(d) = 1 - C(d) = 1 - 0.5 = 0.5$, $[f^{-1}(1_Y \setminus C)](b) = (1_Y \setminus C)f(b) = (1_Y \setminus C)(c) = 1 - C(c) = 1 - 0.8 = 0.2$. Therefore, $f^{-1}(1_Y \setminus C) = 1_X \setminus D$ which is fg^*s-closed in X (as shown in Example 4.3), but not fuzzy closed in X.

Theorem 5.5. *If a function $f : (X, \tau_X) \to (Y, \tau_Y)$ is fg^*s-continuous, then it is fgs-continuous.*

Proof. Let $V \in \tau_Y^c$. As f is fg^*s-continuous, $f^{-1}(V)$ is fg^*s-closed in X. By Theorem 4.5, $f^{-1}(V)$ is fgs-closed in X and hence by Definition 3.2(iv), f is fgs-continuous. □

The converse of the above theorem may not be true, in general, as shown in the following example.

Example 5.6. fgs-continuity $\not\Rightarrow$ fg^*s-continuity.
Let $X = \{a, b\}$, $Y = \{c, d\}$, $\tau_X = \{0_X, 1_X, A\}$, $\tau_Y = \{0_Y, 1_Y, C\}$ where $A(a) = 0.4, A(b) = 0.5$, $C(c) = C(d) = 0.6$. Then (X, τ_X) and (Y, τ_Y) are fts's. Consider the function $f : (X, \tau_X) \to (Y, \tau_Y)$ defined by $f(a) = c, f(b) = d$. Let B be a fuzzy set in X defined by $B(a) = B(b) = 0.4$. Now $1_Y \setminus C \in \tau_Y^c$. $[f^{-1}(1_Y \setminus C)](a) = (1_Y \setminus C)f(a) = (1_Y \setminus C)(c) = 1 - C(c) = 1 - 0.6 = 0.4 = B(a)$, $[f^{-1}(1_Y \setminus C)](b) = (1_Y \setminus C)f(b) = (1_Y \setminus C)(d) = 1 - C(d) = 1 - 0.6 = 0.4 = B(b)$. Therefore, $f^{-1}(1_Y \setminus C) = B$ which is fgs-closed in X, but not fg^*s-closed in X as shown in Example 4.6.

Remark 5.7. It is clear from Definition 5.1 that $f : (X, \tau_X) \to (Y, \tau_Y)$ is fg^*s-continuous iff $f^{-1}(V)$ is fg^*s-open in X for every $V \in \tau_Y$. Also composition of two fg^*s-continuous functions may not be so as fg^*s-closed set need not be fuzzy closed as seen from Example 4.3. Also inverse image of an fg^*s-closed set under fg^*s-continuous function need not be fg^*s-closed follows from Example 5.8.

Example 5.8. Let $X = \{a, b\}$, $\tau_X = \{0_X, 1_X, C\}$ where $C(a) = 0.8, C(b) = 0.5$. Then (X, τ_X) is a fts. Let us consider the function $f : (X, \tau_X) \to (X, \tau_X)$ defined by $f(a) = b, f(b) = a$. Now $1_X \setminus C$ being fuzzy closed is fg^*s-closed in (X, τ_X). Now $[f^{-1}(1_X \setminus C)](a) = (1_X \setminus C)f(a) = (1_X \setminus C)(b) = 1 - C(b) = 0.5$, $[f^{-1}(1_X \setminus C)](b) = (1_X \setminus C)f(b) = (1_X \setminus C)(a) = 1 - C(a) = 0.2$. Let D be a fuzzy set in X defined by $D(a) = 0.5, D(b) = 0.2$. Then $f^{-1}(1_X \setminus C) = D$. We claim that D is not fg^*s-closed. We first show that D is fgs-open in X, i.e., $1_X \setminus D$ is fgs-closed in X. Now 1_X is the only fuzzy open set in X containing $1_X \setminus D$ and so $scl(1_X \setminus D) \leq 1_X \Rightarrow 1_X \setminus D$ is fgs-closed and so D is fgs-open set in X. Again, $D \leq D$, but $sclD = 1_X \not\leq D \Rightarrow D$ is not fg^*s-closed in X.

To achieve the desire result that the composition of two fg^*s-continuous functions is fg^*s-continuous, we need to define some sort of space.

Definition 5.9. A fts (X, τ) is called
(i) fT_{g^*s}-space if every fg^*s-closed set in (X, τ) is fuzzy closed in (X, τ),
(ii) fT_b-space [3] if every fgs-closed set in (X, τ) is fuzzy closed in (X, τ),
(iii) fT_{sg}-space if every fsg-closed set in (X, τ) is fuzzy closed in (X, τ),
(iv) fT_g-space if every fg-closed set in (X, τ) is fuzzy closed in (X, τ).

Theorem 5.10. *Let (X, τ_X), (Y, τ_Y) and (Z, τ_Z) be fts's where (Y, τ_Y) be an fT_{g^*s}-space. Let $f : (X, \tau_X) \to (Y, \tau_Y)$ and $g : (Y, \tau_Y) \to (Z, \tau_Z)$ be fg^*s-continuous functions. Then $g \circ f : (X, \tau_X) \to (Z, \tau_Z)$ is fg^*s-continuous.*

Proof. Let $V \in \tau_Z^c$. As g is fg^*s-continuous, $g^{-1}(V)$ is fg^*s-closed in Y. As Y is fT_{g^*s}-space, $g^{-1}(V) \in \tau_Y^c$. Again f is fg^*s-continuous, so $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is fg^*s-closed in X. As a result, $g \circ f$ is fg^*s-continuous. □

Remark 5.11. According to Definition 5.9(iv), in a fT_g-space, fg-closed set is fg^*s-closed.

Theorem 5.12. *If a function $f : (X, \tau_X) \to (Y, \tau_Y)$ is fg^*s-continuous, then it is fsg-continuous.*

Proof. Let $V \in \tau_Y^c$. Then $f^{-1}(V)$ is fg^*s-closed in X. By Theorem 4.8, $f^{-1}(V)$ is fsg-closed in X and hence the result. □

The converse of the above theorem need not be true, as seen from the following example.

Example 5.13. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_1 = \{0_X, 1_X, C\}$, where $A(a) = 0.4, A(b) = 0.5, C(a) = C(b) = 0.6$. Then (X, τ) and (X, τ_1) are a fts's. Consider the identity function $i : (X, \tau) \to (X, \tau_1)$. Let B be a fuzzy set in X defined by $B(a) = B(b) = 0.4$. Then $B = 1_X \setminus C \in \tau_1^c$. Thus $i^{-1}(B) = B$ is fsg-closed, but not fg^*s-closed (as shown in Example 4.9).

Remark 5.14. fg-continuity and fg^*s-continuity are independent notions as follow from the next two examples.

Example 5.15. fg-continuity $\not\Rightarrow$ fg^*s-continuity.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_1 = \{0_X, 1_X, B\}$, where $A(a) = 0.4, A(b) = 0.5, B(a) = 0.5, B(b) = 0.4$. The collection of fuzzy semiopen sets in X is $\{0_X, 1_X, A, U\}$ where $A \le U \le 1_X \setminus A$ and that of fuzzy semiclosed sets is $\{0_X, 1_X, 1_X \setminus A, 1_X \setminus U\}$ where $A \le 1_X \setminus U \le 1_X \setminus A$. Consider the identity function $i : (X, \tau) \to (X, \tau_1)$.

Let D be a fuzzy set in X defined by $D(a) = 0.5, D(b) = 0.6$. Then $D = 1_X \setminus B \in \tau_1^c$. We claim that D is fg-closed but not fg^*s-closed in X. Now 1_X is the only fuzzy open set in X containing D and so D is fg-closed. Now D is fgs-open in X also. Indeed, $(1_X \setminus D)(a) = 0.5, (1_X \setminus D)(b) = 0.4$ and 1_X is the only fuzzy open set in X containing $1_X \setminus D$ and so $scl(1_X \setminus D) \le 1_X \Rightarrow 1_X \setminus D$ is fgs-closed in X and so D is fgs-open in X. Again $D \le D$ and $sclD = 1_X \not\le D \Rightarrow D$ is not fg^*s-closed in X.

Example 5.16. fg^*s-continuity $\not\Rightarrow$ fg-continuity.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$, $\tau_1 = \{0_X, 1_X, B\}$ where $B(a) = 0.5, B(b) = 0.6$. Then (X, τ) and (X, τ_1) are fts's. Consider the identity function $i : (X, \tau) \to (X, \tau_1)$. Now $1_X \setminus B \in \tau_1^c$. $i^{-1}(1_X \setminus B) = 1_X \setminus B \le A \in \tau$, but $cl(1_X \setminus B) = 1_X \setminus A \not\le A \Rightarrow i$ is not fg-continuous. Now clearly B is fgs-closed in (X, τ) (as 1_X is the only fuzzy open set in X containing B) and so $1_X \setminus B$ is fgs-open in (X, τ). Now $1_X \setminus B \le 1_X \setminus B$ and so $scl(1_X \setminus B) = 1_X \setminus B \le 1_X \setminus B \Rightarrow 1_X \setminus B$ is fg^*s-closed in (X, τ).

Remark 5.17. It is clear from Definition 3.2(iv) and Definition 5.1 that fs^*g-continuous function is fg^*s-continuous. But the converse may not be true as seen from the following example.

Example 5.18. fg^*s-continuity $\not\Rightarrow$ fs^*g-continuity.
Consider Example 5.16. Here $1_X \setminus B$ is fg^*s-closed in (X, τ) where $1_X \setminus B \in \tau_1^c$ which shows that i is fg^*s-continuous. But $1_X \setminus B \notin \tau^c$ whereas $1_X \setminus B$ is fg-closed in (X, τ_1). Indeed, $1_X \setminus B \in \tau_1$ and $cl(1_X \setminus B) = 1_X \setminus B \leq B$ and so i is not fs^*g-continuous.

6. fg^*s-closed and fg^*s-open functions : Some applications

In this section we have introduced and studied fg^*s-closed and fg^*s-open functions and found mutual relations with fuzzy closed [11] and fuzzy open [11] functions. Also it has been shown that fuzzy normal space remains invariant under different types of fuzzy generalized continuity defined in Section 3.

Definition 6.1. A function $f : (X, \tau_X) \to (Y, \tau_Y)$ is called fg^*s-closed (resp., fg^*s-open) function if for each fuzzy closed (resp., fuzzy open) set U in X, $f(U)$ is fg^*s-closed (resp., fg^*s-open) in Y.

Definition 6.2 ([11])**.** A function $f : (X, \tau_X) \to (Y, \tau_Y)$ is called fuzzy closed (resp., fuzzy open) function if $f(U) \in \tau_Y^c$ (resp., $f(U) \in \tau_Y$) for every $U \in \tau_X^c$ (resp., $U \in \tau_X$).

Theorem 6.3. *If $f : (X, \tau_X) \to (Y, \tau_Y)$ is fuzzy closed (resp., fuzzy open) function, then it is fg^*s-closed (resp., fg^*s-open) function.*

Proof. It follows from the fact that every fuzzy closed (resp., fuzzy open) set is fg^*s-closed (resp., fg^*s-open) set. $\square$

The converse of the above theorem need not be true, in general, as seen from the following example.

Example 6.4. fg^*s-closed (fg^*s-open) function $\not\Rightarrow$ fuzzy closed (resp., fuzzy open) function.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, C\}$, $\tau_1 = \{0_X, 1_X, A, B\}$ where $C(a) = 0.5, C(b) = 0.8$, $A(a) = 0.4, A(b) = 0.55$, $B(a) = 0.5, B(b) = 0.6$. Then (X, τ) and (X, τ_1) are fts's. Consider the identity function $i : (X, \tau) \to (X, \tau_1)$. Here $1_X \setminus C \in \tau^c$. Now $i(1_X \setminus C) = 1_X \setminus C$ is fg^*s-closed in (X, τ_1) as seen in Example 5.4, but $1_X \setminus C \notin \tau_1^c \Rightarrow i$ is not fuzzy closed function though it is fg^*s-closed function.
Similarly $C \in \tau$ and $i(C) = C$ is fg^*s-open in (X, τ_1) as $1_X \setminus C$ is fg^*s-closed in (X, τ_1) and so i is fg^*s-open function. But $i(C) = C \notin \tau_1$ and so i is not fuzzy open function.

Theorem 6.5. *A bijective function $f : (X, \tau_X) \to (Y, \tau_Y)$ is fg^*s-closed iff for each fuzzy set S in Y and for each $U \in \tau_X$ such that $f^{-1}(S) \leq U$, then there is an fg^*s-open set V in Y such that $S \leq V$ and $f^{-1}(V) \leq U$.*

Proof. Let S be a fuzzy set in Y and $U \in \tau_X$ be such that $f^{-1}(S) \leq U$. Then $1_X \setminus f^{-1}(S) \geq 1_X \setminus U \in \tau_X^c$. As f is fg^*s-closed function, $f(1_X \setminus U)$ is fg^*s-closed set in Y. Now $f(1_X \setminus U) \leq f(1_X \setminus f^{-1}(S)) = 1_Y \setminus ff^{-1}(S) = 1_Y \setminus S$ (as f is bijective) $\Rightarrow S \leq 1_Y \setminus f(1_X \setminus U)$ which is fg^*s-open in Y. Let $V = 1_Y \setminus f(1_X \setminus U)$. Now $f^{-1}(V) = f^{-1}(1_Y \setminus f(1_X \setminus U)) = 1_X \setminus f^{-1}(f(1_X \setminus U)) = U$ (as f is bijective).

Conversely, let $F \in \tau_X^c$. Then $f(F)$ is a fuzzy set in Y. Let $S = 1_Y \setminus f(F)$. Then $f^{-1}(S) = f^{-1}(1_Y \setminus f(F)) = 1_X \setminus f^{-1}f(F) = 1_X \setminus F$ (as f is bijective) $\in \tau_X$. By hypothesis, there is an fg^*s-open set V in Y such that $S \leq V$, $f^{-1}(V) \leq 1_X \setminus F$. Since $1_Y \setminus f(F) \leq V$,

$$1_Y \setminus V \leq f(F). \tag{6.1}$$

Since $F \leq 1_X \setminus f^{-1}(V)$ and f is bijective,

$$f(F) \leq f(1_X \setminus f^{-1}(V)) = 1_Y \setminus ff^{-1}(V) = 1_Y \setminus V. \tag{6.2}$$

Combining (6.1) ad (6.2), we get $f(F) = 1_Y \setminus V$ which is fg^*s-closed in $Y \Rightarrow f$ is fg^*s-closed function. □

Theorem 6.6. *If $f : (X, \tau_X) \to (Y, \tau_Y)$ is fuzzy closed function and $g : (Y, \tau_Y) \to (Z, \tau_Z)$ is fg^*s-closed function, then $g \circ f : (X, \tau_X) \to (Z, \tau_Z)$ is fg^*s-closed function.*

Proof. Let $V \in \tau_X^c$. As f is fuzzy closed function, $f(V) \in \tau_Y^c$. Again, g being fg^*s-closed function, $g(f(V))$ is fg^*s-closed set in Z. Consequently, $g \circ f$ is fg^*s-closed function. □

Definition 6.7 ([7])**.** An fts (X, τ) is called fuzzy normal space if for any two fuzzy closed sets A, B in X with $A \not{q} B$, there exist two fuzzy open sets U, V in X such that $A \leq U$, $B \leq V$ and $U \not{q} V$.

Theorem 6.8. *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a bijective, fg^*s-continuous, fuzzy open function. If X is fuzzy normal and fT_{g^*s}-space, then Y is fuzzy normal.*

Proof. Let A, B be two fuzzy closed sets in Y with $A \not{q} B$. Then $f^{-1}(A)$, $f^{-1}(B)$ are fg^*s-closed in X as f is fg^*s-continuous function. As X is fT_{g^*s}-space, $f^{-1}(A)$, $f^{-1}(B) \in \tau_X^c$. Now we claim that $f^{-1}(A) \not{q} f^{-1}(B)$. Indeed, $f^{-1}(A) q f^{-1}(B) \Rightarrow$ there is $x \in X$ such that $[f^{-1}(A)](x) + [f^{-1}(B)](x) > 1 \Rightarrow A(f(x)) + B(f(x)) > 1 \Rightarrow AqB$, a contradiction. as $f(x) \in Y$. As X is fuzzy normal, there are $U, V \in \tau_X$ such that $f^{-1}(A) \leq U, f^{-1}(B) \leq V$ and $U \not{q} V$. As f is bijective, $A = ff^{-1}(A) \leq f(U), B = ff^{-1}(B) \leq f(V)$. Since f is fuzzy open function, $f(U), f(V) \in \tau_Y$. We claim that $f(U) \not{q} f(V)$. If, $f(U)qf(V)$ then there exists $y \in Y$ such that $[f(U)](y) + [f(V)](y) > 1 \Rightarrow U(f^{-1}(y)) + V(f^{-1}(y) > 1$ as f is bijective. Let $z = f^{-1}(y)$. Then $U(z) + V(z) > 1$ where $z \in X \Rightarrow UqV$, a contradiction. Hence $f(U) \not{q} f(V) \Rightarrow Y$ is fuzzy normal space. □

Now we can state the following four theorems the proofs of which are followed from Theorem 6.8 as follows.

Theorem 6.9. *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a bijective, fgs-continuous, fuzzy open function. If X is fuzzy normal and fT_b-space, then Y is fuzzy normal.*

Theorem 6.10. *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a bijective, fsg-continuous, fuzzy open function. If X is fuzzy normal and fT_{sg}-space, then Y is fuzzy normal.*

Theorem 6.11. *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a bijective, fg-continuous, fuzzy open function. If X is fuzzy normal and fT_g-space, Then Y is fuzzy normal.*

Theorem 6.12. *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a bijective, fs^*g-continuous, fuzzy open function. If X is fuzzy normal space, then Y is fuzzy normal space.*

Proof. Let A, B be two fuzzy closed sets in Y with $A \not{q} B$. A and B are fg-closed in Y. As f is fs^*g-continuous function, $f^{-1}(A), f^{-1}(B) \in \tau_X^c$. The rest follows from Theorem 6.8. □

According to referee's comment we now site examples of Definition 5.9.

Example 6.13. fT_g-space.
Let $X = \{a\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) \geq 0.7$. Then (X, τ) is a fts. We claim that fg-closed sets are $0_X, 1_X, 1_X \setminus A$ only. Let us take a fuzzy set U defined by $U(a) > 0.3$. Then $U \leq A \in \tau$. But $clU = 1_X \not\leq A \Rightarrow U$ is not fg-closed in X.

Example 6.14. fT_b-space.
Consider Example 6.13. Here the collection of fuzzy semiopen sets in X is $\{0_X, 1_X, A\}$ only and so the collection of fuzzy semiclosed sets is the collection of fuzzy closed sets in X and so (X, τ) is fT_b -space.

Example 6.15. fT_{g^*s}-space.
Consider Example 6.14. Here fuzzy semiclosed sets are fuzzy closed sets only and so fgs-closed and fg-closed sets are same. Then fgs-open set is fg-open and hence fuzzy open (shown in Example 6.13) set in X. Therefore, every fg^*s-closed set in X is fg-closed and hence fuzzy closed in X (shown in Example 6.13). Consequently, (X, τ) is a fT_{g^*s}-space.

Example 6.16. fT_{sg}-space.
Consider Example 6.14. Since here fuzzy semiopen (resp., fuzzy semiclosed) set is fuzzy open (resp., fuzzy closed), every fsg-closed set is fg-closed and hence fuzzy closed (shown in Example 6.13) in X. Consequently, (X, τ) is a fT_{sg}-space.

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