Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 11, No. 4, (April 2016), pp. 633–644
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

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Generalized fuzzy Volterra spaces

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Received 3 August 2014; Revised 18 September 2015; Accepted 19 October 2015

Abstract. In this paper, fuzzy ε_r-Volterra spaces and fuzzy ε_p-Volterra spaces are defined and several characterizations of fuzzy ε_r-Volterra spaces and fuzzy ε_p-Volterra spaces are studied. The conditions for fuzzy ε_r-Volterra spaces and fuzzy ε_p-Volterra spaces, to be fuzzy Volterra spaces are also investigated in this paper.

2010 AMS Classification: 54 A 40, 03 E 72

Keywords: Fuzzy dense set, Fuzzy nowhere dense set, Fuzzy first category set, fuzzy residual set, fuzzy pre-open set, fuzzy submaximal space, Fuzzy Baire space, Fuzzy Volterra space.

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1. Introduction

In order to deal with uncertainties, the idea of fuzzy sets and fuzzy set operations was introduced by L.A.Zadeh in his classical paper [22] in the year 1965. This inspired mathematicians to fuzzify Mathematical Structures. The first notion of fuzzy topological space had been defined by C.L.Chang [5] in 1968. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. The concepts of Volterra spaces have been studied extensively in classical topology in [6, 7, 8, 9, 10]. The concepts of fuzzy Volterra spaces and fuzzy weakly Volterra spaces in fuzzy topological spaces are introduced and studied by the authors in [21].

In classical topology, the concept of generalized Volterra spaces was introduced and studied by Milan Matejdes in [11] and [12]. Motivated on these lines, the concept of generalized Volterra spaces in fuzzy setting is introduced and studied in this paper. A general concept of classification of fuzzy sets of a fuzzy topological space (X, T) with respect to a given non-empty system ε consisting of fuzzy residual sets, fuzzy second category sets and fuzzy pre-open sets, is considered for defining generalized fuzzy Volterra spaces.

2. Preliminaries

Now we introduce some basic notions and results used in the sequel. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang.

Definition 2.1. A fuzzy set λ in a set X is a function from X to $[0, 1]$, that is, $\lambda : X \longrightarrow [0, 1]$.

Definition 2.2. Let λ and μ be fuzzy sets in X. Then for all $x \in X$,

(i). $\lambda = \mu \Leftrightarrow \lambda(x) = \mu(x)$.
(ii). $\lambda \leq \mu \Leftrightarrow \lambda(x) \leq \mu(x)$.
(iii). $\psi = \lambda \vee \mu \Leftrightarrow \psi(x) = max\{\lambda(x), \mu(x)\}$.
(iv). $\delta = \lambda \wedge \mu \Leftrightarrow \delta(x) = min\{\lambda(x), \mu(x)\}$.
(v). $\eta = \lambda^c \Leftrightarrow \eta(x) = 1 - \lambda(x)$.

For a family $\{\lambda_i / i \in I\}$ of fuzzy sets in (X, T), the union $\psi = \vee_i \lambda_i$ and intersection $\delta = \wedge_i \lambda_i$ are defined respectively as

(vi). $\psi(x) = sup_i\{\lambda_i(x) \ / \ x \in X\}$.
(vii). $\delta(x) = inf_i\{\lambda_i(x) \ / \ x \in X\}$.

Definition 2.3 ([1])**.** Let (X, T) be a fuzzy topological space. For a fuzzy set λ of X, the interior $int(\lambda)$ and the closure $cl(\lambda)$ are defined respectively as $int(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$ and $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$.

Lemma 2.4 ([1])**.** *Let λ be any fuzzy set in a fuzzy topological space (X, T). Then $1 - cl(\lambda) = int(1 - \lambda)$ and $1 - int(\lambda) = cl(1 - \lambda)$.*

Lemma 2.5 ([1])**.** *For a family $\mathcal{A} = \{\lambda_\alpha\}$ of fuzzy sets of a fuzzy space X, $\vee\big(cl(\lambda_\alpha)\big) \leq cl\big(\vee(\lambda_\alpha)\big)$. In case $\mathcal{A}$ is a finite set, $\vee\big(cl(\lambda_\alpha)\big) = cl\big(\vee(\lambda_\alpha)\big)$. Also $\vee\big(int(\lambda_\alpha)\big) \leq int\big(\vee(\lambda_\alpha)\big)$.*

Definition 2.6 ([19])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy dense set if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$.

Definition 2.7 ([19])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy nowhere dense set if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is, $intcl(\lambda) = 0$.

Definition 2.8 ([2])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy G_δ-set in (X, T) if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$ where $\lambda_i \in T$, for $i \in I$.

Definition 2.9 ([2])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy F_σ-set in (X, T) if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where $1 - \lambda_i \in T$, for $i \in I$.

Definition 2.10 ([19])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy first category set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where λ_i's are fuzzy nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy second category.

Definition 2.11 ([19])**.** Let λ be a fuzzy first category set in a fuzzy topological space (X, T). Then $1 - \lambda$ is called a fuzzy residual set in (X, T).

Definition 2.12 ([20])**.** Let (X, T) be fuzzy topological space. A fuzzy set λ in (X, T) is called a fuzzy σ-nowhere dense set if λ is a fuzzy F_σ-set in (X, T) such that $int(\lambda) = 0$.

Definition 2.13 ([21])**.** Let (X,T) be a fuzzy topological space. Then (X,T) is called a fuzzy Volterra space if $cl\Big(\wedge_{k=1}^{N}(\lambda_k)\Big) = 1$, where (λ_k)'s are fuzzy dense and fuzzy G_δ-sets in (X,T).

Definition 2.14 ([16])**.** A fuzzy topological space (X,T) is called a fuzzy nodec space if every non-zero fuzzy nowhere dense set λ is fuzzy closed in (X,T). That is, if λ is a fuzzy nowhere dense set in (X,T), then $1-\lambda \in T$.

Definition 2.15 ([2])**.** A fuzzy topological space (X,T) is called a fuzzy submaximal space if for each fuzzy set λ in (X,T) such that $cl(\lambda) = 1$, then $\lambda \in T$ in (X,T).

Definition 2.16 ([14])**.** Let (X,T) be a fuzzy topological space. Then (X,T) is called a fuzzy Baire space if $int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T).

Definition 2.17 ([18])**.** A fuzzy topological space (X,T) is called a fuzzy D-Baire space if every fuzzy first category set in (X,T) is a fuzzy nowhere dense set in (X,T). That is, (X,T) is a fuzzy D-Baire space if $int\ cl(\lambda) = 0$, for each fuzzy first category set λ in (X,T).

Definition 2.18. A fuzzy set λ in a fuzzy topological space X is called a fuzzy pre-open if $\lambda \leq intcl(\lambda)$ and fuzzy pre-closed if $clint(\lambda) \leq \lambda$ [4].

Lemma 2.19. *Let (X,T) be any fuzzy topological space and λ be any fuzzy set in (X,T). The fuzzy pre-closure and the fuzzy pre-interior of λ, are defined as follows:*

(1) $pcl(\lambda) = \wedge\{\mu/\lambda \leq \mu, \mu \text{ is a fuzzy pre-closed set of } X\}$ [13].
(2) $pint(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \text{ is a fuzzy pre-open set of } X\}$ [13].

Definition 2.20 ([17])**.** Let (X,T) be a fuzzy topological space. A fuzzy set λ in (X,T) is called a fuzzy pre-nowhere dense set if there exists no non-zero fuzzy pre-open set μ in (X,T) such that $\mu < pcl(\lambda)$. That is, $pint\ pcl(\lambda) = 0$.

3. Fuzzy ε_r-Volterra spaces

Definition 3.1. A fuzzy topological space (X,T) is said to be a fuzzy ε_r-Volterra space if $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 1$, where (λ_i)'s are fuzzy dense and fuzzy residual sets in (X,T).

Example 3.2. Let $X = \{a,b,c\}$. The fuzzy sets λ,μ and ν are defined on X as follows :

$\lambda : X \to [0,1]$ is defined as $\lambda(a) = 0.8;\ \ \lambda(b) = 0.6;\ \ \lambda(c) = 0.7,$
$\mu : X \to [0,1]$ is defined as $\mu(a) = 0.6;\ \ \mu(b) = 0.9;\ \ \mu(c) = 0.8,$
$\nu : X \to [0,1]$ is defined as $\nu(a) = 0.7;\ \ \nu(b) = 0.5;\ \ \nu(c) = 0.9.$

Clearly $T = \{0, \lambda, \mu, \nu, \lambda\vee\mu, \lambda\vee\nu, \mu\vee\nu, \lambda\wedge\mu, \lambda\wedge\nu, \mu\wedge\nu, \lambda\vee(\mu\wedge\nu), \lambda\wedge(\mu\vee\nu), \mu\vee(\lambda\wedge\nu), \mu\wedge(\lambda\vee\nu), \nu\vee(\lambda\wedge\mu), \nu\wedge(\lambda\vee\mu), \lambda\vee\mu\vee\nu, \lambda\wedge\mu\wedge\nu, 1\}$ is a fuzzy topology on X.

Now the fuzzy sets $1-[\lambda\wedge(\mu\vee\nu)] = (1-\lambda)\vee[1-(\lambda\vee\mu)]\vee(1-[\lambda\vee(\mu\wedge\nu)])\vee[1-(\mu\vee\nu)]\vee[1-(\lambda\vee\mu\vee\nu)]$, $1-(\lambda\wedge\mu\wedge\nu) = (1-\mu)\vee[1-(\lambda\wedge\mu)]\vee[1-(\mu\wedge\nu)]\vee(1-[\mu\wedge(\lambda\vee\nu)])$ and $1-(\lambda\wedge\nu) = (1-\nu)\vee[1-(\mu\vee\nu)]\vee(1-[\nu\wedge(\lambda\vee\mu)])\vee(1-[\mu\vee(\lambda\wedge\nu)])\vee(1-[\lambda\wedge(\mu\vee\nu)])$ are fuzzy first category sets in (X,T) and hence $[\lambda \wedge (\mu \vee \nu)]$, $(\lambda \wedge \mu \wedge \nu)$ and $(\lambda \wedge \nu)$ are fuzzy residual sets in (X, T). Also, $cl[\lambda \wedge (\mu \vee \nu)] = 1$, $cl(\lambda \wedge \mu \wedge \nu) = 1$ and $cl(\lambda \wedge \nu) = 1$. Thus, $[\lambda \wedge (\mu \vee \nu)]$, $(\lambda \wedge \mu \wedge \nu)$ and $(\lambda \wedge \nu)$ are fuzzy dense and fuzzy residual sets in (X, T). Then $cl\left([\lambda \wedge (\mu \vee \nu)] \wedge (\lambda \wedge \mu \wedge \nu) \wedge (\lambda \wedge \nu)\right) = 1$. Hence (X, T) is a fuzzy ε_r-Volterra space.

Proposition 3.3. *If (X, T) is a fuzzy ε_r-Volterra space, then $int\left[\vee_{i=1}^{N}(\mu_i)\right] = 0$, where (μ_i)'s are fuzzy first category sets such that $int(\mu_i) = 0$ in (X, T).*

Proof. Let (μ_i)'s ($i = 1$ to N) be fuzzy first category sets such that $int(\mu_i) = 0$ in (X, T). Then $(1 - \mu_i)$'s are fuzzy residual sets such that $cl(1 - \mu_i) = 1$ in (X, T). That is, $(1 - \mu_i)$'s are fuzzy residual and fuzzy dense sets in (X, T). Since (X, T) is a fuzzy ε_r-Volterra space, $cl\left[\wedge_{i=1}^{N}(1 - \mu_i)\right] = 1$. Then $cl\left[1 - \vee_{i=1}^{N}(\mu_i)\right] = 1$ and hence $1 - int\left[\vee_{i=1}^{N}(\mu_i)\right] = 1$. Therefore, we have $int\left[\vee_{i=1}^{N}(\mu_i)\right] = 0$, where (μ_i)'s are fuzzy first category sets in (X, T) such that $int(\mu_i) = 0$. □

Theorem 3.4 ([15])**.** *If λ is a fuzzy dense and fuzzy G_δ-set in a fuzzy topological space (X, T), then λ is a fuzzy residual set.*

Proposition 3.5. *Let (X, T) be a fuzzy ε_r-Volterra space. Then (X, T) is a fuzzy Volterra space.*

Proof. Let (λ_i)'s ($i = 1$ to N) be fuzzy dense and fuzzy G_δ-sets in (X, T). Then, by theorem 3.4, (λ_i)'s are fuzzy residual sets in (X, T). This implies that (λ_i)'s are fuzzy dense and fuzzy residual sets in (X, T). Since (X, T) is a fuzzy ε_r-Volterra space, $cl\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1$. Hence $cl\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1$, where (λ_i)'s are fuzzy dense and fuzzy G_δ-sets in (X, T). Therefore (X, T) a fuzzy Volterra space. □

Remark 3.6. The converse of the above proposition need not be true. That is, a fuzzy Volterra space need not be a fuzzy ε_r-Volterra space. For, consider the following example:

Example 3.7. Let $X = \{a, b, c\}$. The fuzzy sets λ,μ and ν are defined on X as follows :

$\lambda : X \to [0, 1]$ is defined as $\lambda(a) = 0.8$; $\lambda(b) = 0.5$; $\lambda(c) = 0.7$,
$\mu : X \to [0, 1]$ is defined as $\mu(a) = 0.6$; $\mu(b) = 0.9$; $\mu(c) = 0.4$,
$\nu : X \to [0, 1]$ is defined as $\nu(a) = 0.4$; $\nu(b) = 0.7$; $\nu(c) = 0.8$.

Clearly $T = \{0, \lambda, \mu, \nu, \lambda \vee \mu, \lambda \vee \nu, \mu \vee \nu, \lambda \wedge \mu, \lambda \wedge \nu, \mu \wedge \nu, \lambda \vee (\mu \wedge \nu), \lambda \wedge (\mu \vee \nu), \mu \vee (\lambda \wedge \nu), \mu \wedge (\lambda \vee \nu), \nu \vee (\lambda \wedge \mu), \nu \wedge (\lambda \vee \mu), \lambda \vee \mu \vee \nu, \lambda \wedge \mu \wedge \nu, 1\}$ is a fuzzy topology on X.

Now the fuzzy sets $\alpha = \lambda \wedge (\lambda \vee \mu) \wedge (\mu \wedge \nu) \wedge [\mu \vee (\lambda \wedge \nu)] \wedge [\lambda \wedge (\mu \vee \nu)]$ and $\beta = \nu \wedge (\lambda \vee \nu) \wedge (\mu \vee \nu) \wedge [\nu \vee (\lambda \wedge \mu)] \wedge (\lambda \vee \mu \vee \nu)$ are fuzzy dense and fuzzy $G_\delta-$sets in (X, T). Also, $cl(\alpha \wedge \beta) = 1$. Hence the fuzzy topological space (X, T) is a fuzzy Volterra space.

Now the fuzzy sets $1 - \lambda$, $1 - \mu$, $1 - \nu$, $1 - (\lambda \vee \mu)$, $1 - (\lambda \vee \nu)$, $1 - (\mu \vee \nu)$, $1 - (\lambda \wedge \nu)$, $1 - (\mu \wedge \nu)$, $1 - [\lambda \vee (\mu \wedge \nu)]$, $1 - [\nu \vee (\lambda \wedge \mu)]$, $1 - [\nu \wedge (\lambda \vee \mu)]$, $1 - [\mu \vee (\lambda \wedge \nu)]$, $1 - [\lambda \wedge (\mu \vee \nu)]$, $1 - (\lambda \vee \mu \vee \nu)$, $1 - [\mu \wedge (\lambda \vee \nu)]$ are fuzzy nowhere dense sets in (X, T).

Now the fuzzy sets $1 - [\nu \wedge (\lambda \vee \mu)] = (1 - \nu) \vee [1 - (\lambda \vee \nu)] \vee (1 - [\lambda \vee (\mu \wedge \nu)]) \vee [1 - (\mu \vee \nu)] \vee (1 - [\mu \vee (\lambda \wedge \nu)])$, $1 - (\lambda \wedge \nu) = [1 - (\lambda \vee \mu)] \vee (1 - [\lambda \vee (\mu \wedge \nu)]) \vee$

$(1-[\lambda\wedge(\mu\vee\nu)])\vee[1-(\lambda\vee\mu\vee\nu)]\vee(1-[\nu\wedge(\lambda\vee\mu)])$ and $1-(\mu\wedge\nu)=(1-[\nu\wedge(\lambda\vee\mu)])\vee[1-(\mu\wedge\nu)]\vee(1-\mu)\vee(1-[\mu\vee(\lambda\wedge\nu)])\vee(1-[\mu\wedge(\lambda\vee\nu)])$ are fuzzy first category sets in (X,T) and hence $[\nu\wedge(\lambda\vee\mu)]$, $(\lambda\wedge\nu)$ and $(\mu\wedge\nu)$ are fuzzy residual sets in (X,T). Also, $cl[\nu\wedge(\lambda\vee\mu)]=1$, $cl(\lambda\wedge\nu)=1$ and $cl(\mu\wedge\nu)=1$. Thus, $[\nu\wedge(\lambda\vee\mu)]$, $(\lambda\wedge\nu)$ and $(\mu\wedge\nu)$ are fuzzy dense and fuzzy residual sets in (X,T). But $cl\,([\nu\wedge(\lambda\vee\mu)]\wedge(\lambda\wedge\nu)\wedge(\mu\wedge\nu))=1-(\lambda\wedge\mu)\neq 1$. Hence (X,T) is not a fuzzy ε_r-Volterra space.

The following propositions give conditions for fuzzy Volterra spaces to be fuzzy ε_r-Volterra spaces.

Proposition 3.8. *If each fuzzy nowhere dense set is a fuzzy closed set in a fuzzy Volterra space (X,T), then (X,T) is a fuzzy ε_r-Volterra space.*

Proof. Let (λ_i)'s $(i=1$ to $N)$ be fuzzy dense and fuzzy residual sets in (X,T). Since (λ_i)'s are fuzzy residual sets, $(1-\lambda_i)$'s are fuzzy first category sets in (X,T). Now $1-\lambda_i=\vee_{j=1}^{\infty}(\mu_{ij})$, where (μ_{ij})'s are fuzzy nowhere dense sets in (X,T). By hypothesis, the fuzzy nowhere dense sets (μ_{ij})'s are fuzzy closed sets and hence $(1-\lambda_i)$'s are fuzzy F_σ-sets in (X,T). This implies that (λ_i)'s are fuzzy G_δ-sets in (X,T). Hence (λ_i)'s are fuzzy dense and fuzzy G_δ-sets in (X,T). Since (X,T) is a fuzzy Volterra space, $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big)=1$. Hence $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big)=1$, where (λ_i)'s are fuzzy dense and fuzzy residual sets in (X,T) implies that (X,T) is a fuzzy ε_r-Volterra space. □

Proposition 3.9. *If a fuzzy topological space (X,T) is a fuzzy Volterra space and fuzzy nodec space, then (X,T) is a fuzzy ε_r-Volterra space.*

Proof. Let (X,T) be a fuzzy Volterra and fuzzy nodec space and (λ_i)'s $(i=1$ to $N)$ be fuzzy dense and fuzzy residual sets in (X,T). Since (λ_i)'s are fuzzy residual sets, $(1-\lambda_i)$'s are fuzzy first category sets in (X,T). Now $1-\lambda_i=\vee_{j=1}^{\infty}(\mu_{ij})$, where (μ_{ij})'s are fuzzy nowhere dense sets in (X,T). Since (X,T) is a fuzzy nodec space, [by definition 2.14] fuzzy nowhere dense sets (μ_{ij})'s are fuzzy closed sets in (X,T). Then by proposition 3.8, (X,T) is a fuzzy ε_r-Volterra space. □

Theorem 3.10 ([14])**.** *If λ is a fuzzy dense and fuzzy open set in a fuzzy topological space, then $1-\lambda$ is a fuzzy nowhere dense set in (X,T).*

Proposition 3.11. *If a fuzzy topological space (X,T) is a fuzzy Volterra and fuzzy submaximal space, then (X,T) is a fuzzy ε_r-Volterra space.*

Proof. Let (X,T) be a fuzzy Volterra and fuzzy submaximal space and (λ_i)'s $(i=1$ to $N)$ be fuzzy dense and fuzzy residual sets in (X,T). Since (X,T) is a fuzzy submaximal space, the fuzzy dense sets (λ_i)'s are fuzzy open sets in (X,T). Since (λ_i)'s are fuzzy dense and fuzzy open sets in (X,T), by theorem 3.10, $(1-\lambda_i)$'s are fuzzy nowhere dense sets in (X,T). Since (λ_i)'s are fuzzy open sets in (X,T), $(1-\lambda_i)$'s are fuzzy closed sets in (X,T). Hence the fuzzy nowhere dense sets $(1-\lambda_i)$'s are fuzzy closed sets in (X,T). Since (X,T) is a fuzzy Volterra space and the fuzzy nowhere dense sets $(1-\lambda_i)$'s are fuzzy closed sets in (X,T), by proposition 3.8, (X,T) is a fuzzy ε_r-Volterra space. □

Theorem 3.12 ([14]). *If λ is a fuzzy nowhere dense set in a fuzzy topological space (X, T), then $1 - \lambda$ is a fuzzy dense set in (X, T).*

Proposition 3.13. *If a fuzzy topological space (X, T) is a fuzzy ε_r-Volterra and fuzzy D-Baire space, then $int\big(\vee_{i=1}^{N} (\lambda_i)\big) = 0$, where (λ_i)'s are fuzzy first category sets in (X, T).*

Proof. Let (X, T) be a fuzzy ε_r-Volterra and fuzzy D-Baire space and (λ_i)'s ($i =$ 1 to N) be fuzzy first category sets in (X, T). Since (X, T) is a fuzzy D-Baire space, the fuzzy first category sets (λ_i)'s are fuzzy nowhere dense sets in (X, T) and hence, by theorem 3.12, $(1 - \lambda_i)$'s are fuzzy dense sets in (X, T). Then $(1 - \lambda_i)$'s are fuzzy dense and fuzzy residual sets in (X, T). Since (X, T) is a fuzzy ε_r-Volterra space, $cl\big(\wedge_{i=1}^{N} (1 - \lambda_i)\big) = 1$. This implies that $cl\big(1 - \vee_{i=1}^{N}(\lambda_i)\big) = 1$ and hence $1 - int\big(\vee_{i=1}^{N} (\lambda_i)\big) = 1$. Therefore, $int\big(\vee_{i=1}^{N} (\lambda_i)\big) = 0$, where (λ_i)'s are fuzzy first category sets in (X, T). □

Proposition 3.14. *If $\vee_{i=1}^{\infty}(\mu_i)$, where (μ_i)'s are fuzzy nowhere dense sets, is a fuzzy nowhere dense set in a fuzzy Baire space (X, T), then (X, T) is a fuzzy ε_r-Volterra space.*

Proof. Let (X, T) be a fuzzy Baire space and (λ_i)'s ($i = 1$ to N) be fuzzy dense and fuzzy residual sets in (X, T). Since (λ_i)'s are fuzzy residual sets, $(1 - \lambda_i)$'s are fuzzy first category sets in (X, T) and hence $1 - \lambda_i = \vee_{i=1}^{\infty}(\mu_{ij})$, where (μ_{ij})'s are fuzzy nowhere dense sets in (X, T). By hypothesis, $(1 - \lambda_i)$'s are fuzzy nowhere dense sets in (X, T). Let (μ_α)'s be fuzzy nowhere dense sets in (X, T) in which the first N fuzzy nowhere dense sets be $1 - \lambda_i$. Since (X, T) is a fuzzy Baire space, $int\big(\vee_{\alpha=1}^{\infty}(\mu_\alpha)\big) = 0$. But $int\big(\vee_{i=1}^{N} (1 - \lambda_i)\big) \leq int\big(\vee_{\alpha=1}^{\infty} (\mu_\alpha)\big)$ and $int\big(\vee_{\alpha=1}^{\infty} (\mu_\alpha)\big) = 0$. Then $int\big(\vee_{i=1}^{N}(1 - \lambda_i)\big) \leq 0$. That is, $int\big(\vee_{i=1}^{N}(1 - \lambda_i)\big) = 0$. Then $int\big(1 - \wedge_{i=1}^{N}(\lambda_i)\big) = 0$ and hence $1 - cl\big(\wedge_{i=1}^{N} (\lambda_i)\big) = 0$. This implies that $cl\big(\wedge_{i=1}^{N} (\lambda_i)\big) = 1$, where (λ_i)'s are fuzzy dense and fuzzy residual sets in (X, T). Therefore (X, T) is a fuzzy ε_r-Volterra space. □

Proposition 3.15. *If each fuzzy first category set is a fuzzy closed set in a fuzzy Baire space (X, T), then (X, T) is a fuzzy ε_r-Volterra space.*

Proof. Let (X, T) be a fuzzy Baire space and (λ_i)'s ($i = 1$ to N) be fuzzy dense and fuzzy residual sets in (X, T). Since (λ_i)'s are fuzzy residual sets, $(1 - \lambda_i)$'s are fuzzy first category sets in (X, T). By hypothesis, the fuzzy first category sets $(1 - \lambda_i)$'s are fuzzy closed sets in (X, T) and hence (λ_i)'s are fuzzy open sets in (X, T). Since (λ_i)'s are fuzzy dense and fuzzy open sets in (X, T), by theorem 3.10, $(1 - \lambda_i)$'s are fuzzy nowhere dense sets in (X, T). Let (μ_α)'s be fuzzy nowhere dense sets in (X, T) in which the first N fuzzy nowhere dense sets be $1 - \lambda_i$. Since (X, T) is a fuzzy Baire space, $int\big(\vee_{\alpha=1}^{\infty} (\mu_\alpha)\big) = 0$. But $int\big(\vee_{i=1}^{N} (1 - \lambda_i)\big) \leq int\big(\vee_{\alpha=1}^{\infty} (\mu_\alpha)\big)$ and $int\big(\vee_{\alpha=1}^{\infty}(\mu_\alpha)\big) = 0$. Then $int\big(\vee_{i=1}^{N}(1 - \lambda_i)\big) \leq 0$. That is, $int\big(\vee_{i=1}^{N}(1 - \lambda_i)\big) = 0$. Then $int\big(1 - \wedge_{i=1}^{N}(\lambda_i)\big) = 0$ and hence $1 - cl\big(\wedge_{i=1}^{N} (\lambda_i)\big) = 0$. This implies that $cl\big(\wedge_{i=1}^{N} (\lambda_i)\big) = 1$, where (λ_i)'s are fuzzy dense and fuzzy residual sets in (X, T). Therefore (X, T) is a fuzzy ε_r-Volterra space. □

Theorem 3.16 ([14]). *Let (X, T) be a fuzzy topological space. Then the following are equivalent :*

(1) (X, T) *is a fuzzy Baire space.*
(2) $int(\lambda) = 0$ *for every fuzzy first category set* λ *in* (X, T).
(3) $cl(\mu) = 1$, *for every fuzzy residual set* μ *in* (X, T).

Proposition 3.17. *If a fuzzy* ε_r*-Volterra space is a fuzzy Baire space, then* $cl\big(\wedge_{i=1}^{N} (\lambda_i)\big) = 1$, *where* (λ_i)*'s are fuzzy residual sets in* (X, T).

Proof. Let (λ_i)'s ($i = 1$ to N) be fuzzy residual sets in (X, T). Since (X, T) is a fuzzy Baire space, by theorem 3.16, $cl(\lambda_i) = 1$, $\forall\ i$. Then (λ_i)'s are fuzzy dense and fuzzy residual sets in (X, T). Since (X, T) is a fuzzy ε_r-Volterra space, $cl\big(\wedge_{i=1}^{N} (\lambda_i)\big) = 1$. Therefore, $cl\big(\wedge_{i=1}^{N} (\lambda_i)\big) = 1$, where (λ_i)'s are fuzzy residual sets in (X, T). □

Proposition 3.18. *If* $\wedge_{i=1}^{N}(\lambda_i)$ *is a fuzzy residual set in a fuzzy Baire space* (X, T), *where* (λ_i)*'s are fuzzy residual sets, then* (X, T) *is a fuzzy* ε_r*-Volterra space.*

Proof. Let (λ_i)'s ($i = 1$ to N) be fuzzy dense and fuzzy residual sets in (X, T). Then, by hypothesis, $\wedge_{i=1}^{N}(\lambda_i)$ is a fuzzy residual set in (X, T). Since (X, T) is a fuzzy Baire space, by theorem 3.16, $cl\big(\wedge_{i=1}^{N} (\lambda_i)\big) = 1$. Hence $cl\big(\wedge_{i=1}^{N} (\lambda_i)\big) = 1$, where (λ_i)'s are fuzzy dense and fuzzy residual sets in (X, T). Therefore (X, T) is a fuzzy ε_r-Volterra space. □

Theorem 3.19 ([20])**.** *In a fuzzy topological space* (X, T), *a fuzzy set* λ *is fuzzy* σ*-nowhere dense if and only if* $1 - \lambda$ *is a fuzzy dense and fuzzy* G_δ*-set.*

Proposition 3.20. *Let* (X, T) *be a fuzzy* ε_r*-Volterra space. Then* $int(\vee_{i=1}^{N}(\lambda_i)) = 0$, *where* (λ_i)*'s are fuzzy* σ*-nowhere dense sets in* (X, T).

Proof. Let (X, T) be a fuzzy ε_r-Volterra space and (λ_i)'s ($i = 1$ to N) be fuzzy σ-nowhere dense sets in (X, T). Since (λ_i)'s are fuzzy σ-nowhere dense sets, by theorem 3.19, $(1 - \lambda_i)$'s are fuzzy dense and fuzzy G_δ-sets in (X, T). Then, by theorem 3.4, $(1 - \lambda_i)$'s are fuzzy residual sets in (X, T). Hence $(1 - \lambda_i)$'s are fuzzy dense and fuzzy residual sets in (X, T). Since (X, T) is a fuzzy ε_r-Volterra space, $cl\big(\wedge_{i=1}^{N} (1 - \lambda_i)\big) = 1$. Then $cl\big(1 - \vee_{i=1}^{N}(\lambda_i)\big) = 1$, implies that $1 - int\big(\vee_{i=1}^{N}(\lambda_i)\big) = 1$. Therefore $int(\vee_{i=1}^{N}(\lambda_i) = 0$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X, T). □

4. Fuzzy ε_p-Volterra spaces

Definition 4.1. A fuzzy topological space (X, T) is said to be a fuzzy ε_p-Volterra space if $cl(\wedge_{i=1}^{N}(\lambda_i)) = 1$, where (λ_i)'s are fuzzy pre-open and fuzzy G_δ-sets in (X, T).

Example 4.2. Let $X = \{a, b, c\}$. The fuzzy sets λ,μ and ν are defined on X as follows :

$\lambda : X \to [0, 1]$ is defined as $\lambda(a) = 0.8$; $\lambda(b) = 0.6$; $\lambda(c) = 0.7$,
$\mu : X \to [0, 1]$ is defined as $\mu(a) = 0.6$; $\mu(b) = 0.9$; $\mu(c) = 0.8$,
$\nu : X \to [0, 1]$ is defined as $\nu(a) = 0.7$; $\nu(b) = 0.5$; $\nu(c) = 0.9$.

Clearly $T = \{0, \lambda, \mu, \nu, \lambda \vee \mu, \lambda \vee \nu, \mu \vee \nu, \lambda \wedge \mu, \lambda \wedge \nu, \mu \wedge \nu, \lambda \vee (\mu \wedge \nu), \lambda \wedge (\mu \vee \nu), \mu \vee (\lambda \wedge \nu), \mu \wedge (\lambda \vee \nu), \nu \vee (\lambda \wedge \mu), \nu \wedge (\lambda \vee \mu), \lambda \vee \mu \vee \nu, \lambda \wedge \mu \wedge \nu, 1\}$ is a fuzzy topology on X.

Now the fuzzy sets $\lambda \wedge (\mu \vee \nu)$, $\lambda \wedge \mu \wedge \nu$ and $\lambda \wedge \nu$ are fuzzy pre-open and fuzzy G_δ-sets in (X, T) and $cl\,([\lambda \wedge (\mu \vee \nu)] \wedge (\lambda \wedge \mu \wedge \nu) \wedge (\lambda \wedge \nu)) = 1$. Hence the fuzzy topological space (X, T) is a fuzzy ε_p-Volterra space.

Example 4.3. Let $X = \{a, b, c\}$. The fuzzy sets λ,μ and ν are defined on X as follows :

$\lambda : X \to [0, 1]$ is defined as $\lambda(a) = 0.5$; $\lambda(b) = 0.4$; $\lambda(c) = 0.7$; $\lambda(d) = 0.8$,

$\mu : X \to [0, 1]$ is defined as $\mu(a) = 0.5$; $\mu(b) = 0.8$; $\mu(c) = 0.5$; $\mu(d) = 0.7$

$\nu : X \to [0, 1]$ is defined as $\nu(a) = 0.5$; $\nu(b) = 0.7$; $\nu(c) = 0.6$; $\nu(d) = 0.4$.

Clearly $T = \{0, \lambda, \mu, \nu, \lambda \vee \mu, \lambda \vee \nu, \mu \vee \nu, \lambda \wedge \mu, \lambda \wedge \nu, \mu \wedge \nu, \lambda \wedge (\mu \vee \nu), \mu \wedge (\lambda \vee \nu), \nu \vee (\lambda \wedge \mu), \nu \wedge (\lambda \vee \mu), \lambda \wedge \mu \wedge \nu, 1\}$ is a fuzzy topology on X.

Now the fuzzy sets $\lambda = \lambda \wedge (\lambda \vee \mu) \wedge (\lambda \vee \nu)$, $\lambda \wedge \mu = \mu \wedge (\mu \vee \nu) \wedge [\nu \vee (\lambda \wedge \mu)] \wedge [\lambda \wedge (\mu \vee \nu)] \wedge [\mu \wedge (\lambda \vee \nu)]$ and $\lambda \wedge \mu \wedge \nu = \nu \wedge (\lambda \wedge \nu) \wedge (\mu \wedge \nu) \wedge [\nu \wedge (\lambda \vee \mu)]$ are fuzzy pre-open and fuzzy G_δ-sets in (X, T). But $cl[\lambda \wedge (\lambda \wedge \mu) \wedge (\lambda \wedge \mu \wedge \nu)] = 1 - (\lambda \wedge \mu \wedge \nu) \neq 1$. Hence the fuzzy topological space (X, T) is not a fuzzy ε_p-Volterra space.

Also, λ and $\lambda \wedge \mu$ are fuzzy dense and fuzzy residual sets in (X, T) and $\lambda \wedge \mu \wedge \nu$ is a fuzzy residual set but not a fuzzy dense set in (X, T). Then $cl[\lambda \wedge (\lambda \wedge \mu)] = 1$. Hence the fuzzy topological space (X, T) is a fuzzy ε_r-Volterra space.

Proposition 4.4. *If (X, T) is a fuzzy ε_p-Volterra space, then $int\left[\vee_{i=1}^{N}(\mu_i)\right] = 0$, where (μ_i)'s are fuzzy pre-closed and fuzzy F_σ-sets in (X, T).*

Proof. Let (μ_i)'s $(i = 1$ to $N)$ be fuzzy pre-closed and fuzzy F_σ-sets in (X, T). Then $(1 - \mu_i)$'s are fuzzy pre-open and fuzzy G_δ-sets in (X, T). Since (X, T) is a fuzzy ε_p-Volterra space, $cl\left[\wedge_{i=1}^{N}(1 - \mu_i)\right] = 1$. Then $cl\left[1 - \vee_{i=1}^{N}(\mu_i)\right] = 1$ and hence $1 - int\left[\vee_{i=1}^{N}(\mu_i)\right] = 1$. Therefore, we have $int\left[\vee_{i=1}^{N}(\mu_i)\right] = 0$, where (μ_i)'s are fuzzy pre-closed and fuzzy F_σ-sets in (X, T). □

Proposition 4.5. *If (X, T) is a fuzzy ε_p-Volterra space, then (X, T) is a fuzzy Volterra space.*

Proof. Let (λ_i)'s $(i = 1 \text{ to } N)$ be fuzzy dense and fuzzy G_δ-sets in a fuzzy topological space (X, T). Since (λ_i)'s are fuzzy dense sets, $cl(\lambda_i) = 1$. Now $intcl(\lambda_i) = int(1) = 1$. Then $\lambda_i \leq intcl(\lambda_i)$. Hence (λ_i)'s are fuzzy pre-open sets in (X, T). Since (X, T) is a fuzzy ε_p-Volterra space and (λ_i)'s are fuzzy pre-open and fuzzy G_δ-sets in (X, T), $cl(\wedge_{i=1}^{N}(\lambda_i)) = 1$. Hence $cl(\wedge_{i=1}^{N}(\lambda_i)) = 1$, where (λ_i)'s are fuzzy dense and fuzzy G_δ-sets in (X, T). Therefore (X, T) is a fuzzy Volterra space. □

Remark 4.6. The converse of the above proposition need not be true. That is, a fuzzy Volterra space need not be a fuzzy ε_p-Volterra space. For, consider the following examples:

Example 4.7. Let $X = \{a, b, c\}$. The fuzzy sets λ,μ and ν are defined on X as follows :

$\lambda : X \to [0, 1]$ is defined as $\lambda(a) = 1$; $\lambda(b) = 0.2$; $\lambda(c) = 0.9$,

$\mu : X \to [0, 1]$ is defined as $\mu(a) = 0.3$; $\mu(b) = 1$; $\mu(c) = 0.2$,

$\nu : X \to [0, 1]$ is defined as $\nu(a) = 0.7$; $\nu(b) = 0.4$; $\nu(c) = 1$.

Clearly $T = \{0, \lambda, \mu, \nu, \lambda \vee \mu, \lambda \vee \nu, \mu \vee \nu, \lambda \wedge \mu, \lambda \wedge \nu, \mu \wedge \nu, \lambda \vee (\mu \wedge \nu), \mu \vee (\lambda \wedge \nu), \nu \wedge (\lambda \vee \mu), 1\}$ is a fuzzy topology on X.

Now the fuzzy sets $\lambda \wedge \nu = \lambda \wedge \nu \wedge (\mu \vee \nu) \wedge [\nu \wedge (\lambda \vee \mu)] \wedge [\mu \vee (\lambda \wedge \nu)]$ and $\nu \wedge (\lambda \vee \mu) = (\lambda \vee \mu) \wedge [\lambda \vee (\mu \wedge \nu)] \wedge (\lambda \vee \nu)$ are fuzzy dense and fuzzy G_δ-sets in (X, T). Then $cl(\lambda \wedge \nu) = 1$ and $cl[\nu \wedge (\lambda \vee \mu)] = 1$. Also, $cl(\lambda \wedge \nu) \wedge [\nu \wedge (\lambda \vee \mu)] = 1$. Hence (X, T) is a fuzzy Volterra space.

Also, the fuzzy sets $\lambda \wedge \nu = \lambda \wedge \nu \wedge (\mu \vee \nu) \wedge [\nu \wedge (\lambda \vee \mu)] \wedge [\mu \vee (\lambda \wedge \nu)]$, $\nu \wedge (\lambda \vee \mu) = (\lambda \vee \mu) \wedge [\lambda \vee (\mu \wedge \nu)] \wedge (\lambda \vee \nu)$ and $\lambda \wedge \mu = \lambda \wedge \mu \wedge \nu \wedge (\lambda \wedge \mu) \wedge (\lambda \wedge \nu) \wedge (\mu \wedge \nu)$ are fuzzy pre-open and fuzzy G_δ-sets in (X, T). But $cl[(\lambda \wedge \nu) \wedge [\nu \wedge (\lambda \vee \mu)] \wedge (\lambda \wedge \mu)] \neq 1$ Hence (X, T) is not a fuzzy ε_p-Volterra space.

Example 4.8. Let $X = \{a, b, c\}$. The fuzzy sets λ,μ and ν are defined on X as follows :

$\lambda : X \to [0, 1]$ is defined as $\lambda(a) = 0.8$; $\lambda(b) = 0.5$; $\lambda(c) = 0.7$,

$\mu : X \to [0, 1]$ is defined as $\mu(a) = 0.6$; $\mu(b) = 0.9$; $\mu(c) = 0.4$,

$\nu : X \to [0, 1]$ is defined as $\nu(a) = 0.4$; $\nu(b) = 0.7$; $\nu(c) = 0.8$.

Clearly $T = \{0, \lambda, \mu, \nu, \lambda \vee \mu, \lambda \vee \nu, \mu \vee \nu, \lambda \wedge \mu, \lambda \wedge \nu, \mu \wedge \nu, \lambda \vee (\mu \wedge \nu), \lambda \wedge (\mu \vee \nu), \mu \vee (\lambda \wedge \nu), \mu \wedge (\lambda \vee \nu), \nu \vee (\lambda \wedge \mu), \nu \wedge (\lambda \vee \mu), \lambda \vee \mu \vee \nu, \lambda \wedge \mu \wedge \nu, 1\}$ is a fuzzy topology on X.

Now the fuzzy sets $\alpha = \lambda \wedge (\lambda \vee \mu) \wedge (\mu \wedge \nu) \wedge [\mu \vee (\lambda \wedge \nu)] \wedge [\lambda \wedge (\mu \vee \nu)]$ and $\beta = \nu \wedge (\lambda \vee \nu) \wedge (\mu \vee \nu) \wedge [\nu \vee (\lambda \wedge \mu)] \wedge (\lambda \vee \mu \vee \nu)$ are fuzzy dense and fuzzy $G_\delta-$sets in (X, T). Also, $cl(\alpha \wedge \beta) = 1$. Hence the fuzzy topological space (X, T) is a fuzzy Volterra space.

Also, the fuzzy sets $\lambda \wedge \nu = \lambda \wedge (\lambda \wedge \nu) \wedge [\mu \vee (\lambda \wedge \nu)] \wedge (\lambda \vee \mu)$, $\nu \wedge (\lambda \vee \mu) = \nu \wedge (\lambda \vee \nu) \wedge (\mu \vee \nu) \wedge [\nu \vee (\lambda \wedge \mu)] \wedge [\lambda \vee (\mu \wedge \nu)]$ and $\lambda \wedge \mu \wedge \nu = \mu \wedge (\lambda \wedge \mu) \wedge (\mu \wedge \nu)$ are fuzzy pre-open and fuzzy G_δ-sets in (X, T). But $cl\,[(\lambda \wedge \nu) \wedge [\nu \wedge (\lambda \vee \mu)] \wedge (\lambda \wedge \mu \wedge \nu)] = 1 - (\lambda \wedge \mu) \neq 1$. Hence (X, T) is not a fuzzy ε_p-Volterra space.

Theorem 4.9 ([3])**.** *Let λ be a fuzzy set of a fuzzy topological space (X, T). Then $int(\lambda) \leq pint(\lambda) \leq \lambda \leq pcl(\lambda) \leq cl(\lambda)$.*

Proposition 4.10. *If $pint(\vee_{i=1}^{N}(\mu_i)) = 0$, where (μ_i)'s are fuzzy pre-closed sets in a fuzzy topological space (X, T), then (X, T) is a fuzzy ε_p-Volterra space.*

Proof. Let (λ_i)'s ($i = 1$ to N) be fuzzy pre-open and fuzzy G_δ-sets in (X, T). Since (λ_i)'s are fuzzy pre-open sets, $(1 - \lambda_i)$'s are fuzzy pre-closed sets in (X, T). By hypothesis, $pint(\vee_{i=1}^{N}(1 - \lambda_i)) = 0$. This implies that $1 - pcl(\wedge_{i=1}^{N}(\lambda_i)) = 0$ and hence $pcl(\wedge_{i=1}^{N}(\lambda_i)) = 1$. By theorem 4.9, $pcl(\wedge_{i=1}^{N}(\lambda_i)) \leq cl(\wedge_{i=1}^{N}(\lambda_i))$ implies that $1 \leq cl(\wedge_{i=1}^{N}(\lambda_i))$. That is, $cl(\wedge_{i=1}^{N}(\lambda_i) = 1$. Therefore $cl(\wedge_{i=1}^{N}(\lambda_i) = 1$, where (λ_i)'s are fuzzy pre-open and fuzzy G_δ-sets in (X, T), implies that (X, T) is a fuzzy ε_p-Volterra space. □

Proposition 4.11. *If $pcl(\wedge_{i=1}^{N}(\lambda_i)) = 1$, where (λ_i)'s are fuzzy pre-open sets in a fuzzy topological space (X, T), then (X, T) is a fuzzy ε_p-Volterra space.*

Proof. Suppose that $pcl(\wedge_{i=1}^{N}(\lambda_i)) = 1$, where (λ_i)'s are fuzzy pre-open sets in a fuzzy topological space (X, T). Then $1 - pcl(\wedge_{i=1}^{N}(\lambda_i)) = 0$ implies that $pint(1 - \wedge_{i=1}^{N}(\lambda_i)) = 0$. Hence $pint(\vee_{i=1}^{N}(1 - \lambda_i)) = 0$. Since (λ_i)'s are fuzzy pre-open sets, $(1 - \lambda_i)$'s are fuzzy pre-closed sets in (X, T). Since $pint(\vee_{i=1}^{N}(1 - \lambda_i)) = 0$, where $(1 - \lambda_i)$'s are fuzzy pre-closed sets in (X, T), by proposition 4.10, (X, T) is a fuzzy ε_p-Volterra space. □

Theorem 4.12 ([3]). *Let λ be a fuzzy set of a fuzzy topological space (X, T). Then*

(1) $pcl(\lambda) \geq \lambda \vee cl\ int(\lambda)$.
(2) $pint(\lambda) \leq \lambda \wedge int\ cl(\lambda)$.

Proposition 4.13. *If $\lambda = \vee_{i=1}^{N}(\mu_i)$, where (μ_i)'s are fuzzy pre-closed sets, is a fuzzy nowhere dense set in a fuzzy topological space (X, T), then (X, T) is a fuzzy ε_p-Volterra space.*

Proof. Suppose that $\lambda = \vee_{i=1}^{N}(\mu_i)$, where (μ_i)'s are fuzzy pre-closed sets and λ is a fuzzy nowhere dense set in (X, T). Then $intcl(\lambda) = 0$. By theorem 4.12, $pint(\lambda) \leq \lambda \wedge intcl(\lambda)$. This implies that $pint(\lambda) \leq \lambda \wedge 0 = 0$. Hence $pint(\lambda) = 0$. Therefore $pint(\vee_{i=1}^{N}(\mu_i)) = 0$, where (μ_i)'s are fuzzy pre-closed sets in a fuzzy topological space (X, T). Hence, by proposition 4.10, (X, T) is a fuzzy ε_p-Volterra space. □

Proposition 4.14. *If $\lambda = \vee_{i=1}^{N}(\mu_i)$, where (μ_i)'s are fuzzy pre-closed sets, is a fuzzy pre-nowhere dense set in a fuzzy topological space (X, T), then (X, T) is a fuzzy ε_p-Volterra space.*

Proof. Suppose that $\lambda = \vee_{i=1}^{N}(\mu_i)$, where (μ_i)'s are fuzzy pre-closed sets and λ is a fuzzy pre-nowhere dense set in (X, T). Then $pintpcl(\lambda) = 0$. Now $\lambda \leq pcl(\lambda)$ implies that $pint(\lambda) \leq pint\ pcl(\lambda)$ and $pintpcl(\lambda) = 0$ implies that $pint(\lambda) = 0$. Then $pint(\vee_{i=1}^{N}(\mu_i)) = 0$, where (μ_i)'s are fuzzy pre-closed sets in a fuzzy topological space (X, T). Hence, by proposition 4.10, (X, T) is a fuzzy ε_p-Volterra space. □

The following proposition gives a condition, for a fuzzy ε_r-Volterra space, to be a fuzzy ε_p-Volterra space.

Proposition 4.15. *If each fuzzy pre-closed set is a fuzzy nowhere dense set in a fuzzy ε_r-Volterra space (X, T), then (X, T) is a fuzzy ε_p-Volterra space.*

Proof. Let (X, T) be a fuzzy ε_r-Volterra space and (λ_i)'s $(i = 1$ to $N)$ be fuzzy pre-open and fuzzy G_δ-sets in (X, T). Since (λ_i)'s are fuzzy pre-open sets, $(1-\lambda_i)$'s are fuzzy pre-closed sets in (X, T). By hypothesis, the fuzzy pre-closed sets $(1-\lambda_i)$'s are fuzzy nowhere dense sets in (X, T). Since $(1-\lambda_i)$'s are fuzzy nowhere dense sets, by theorem 3.12, (λ_i)'s are fuzzy dense sets in (X, T). Hence (λ_i)'s are fuzzy dense and fuzzy G_δ-sets in (X, T). Then, by theorem 3.4, (λ_i)'s are fuzzy residual sets in (X, T). Therefore (λ_i)'s are fuzzy dense and fuzzy residual sets in (X, T). Since (X, T) is a ε_r-fuzzy Volterra space, $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 1$. Hence $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 1$, where (λ_i)'s are fuzzy pre-open and fuzzy G_δ-sets in (X, T), implies that (X, T) is a fuzzy ε_p-Volterra space. □

The following propositions yield the conditions for fuzzy ε_p-Volterra spaces to be fuzzy ε_r-Volterra spaces.

Proposition 4.16. *If a fuzzy ε_p-Volterra space (X, T) is a fuzzy submaximal space, then (X, T) is a fuzzy ε_r-Volterra space.*

Proof. Let (λ_i)'s $(i = 1$ to $N)$ be fuzzy dense and fuzzy residual sets in (X, T). Since (λ_i)'s are fuzzy dense sets in (X, T), $cl(\lambda_i) = 1$. Now $intcl(\lambda_i) = int(1) = 1$. Then $\lambda_i \leq intcl(\lambda_i)$. Hence (λ_i)'s are fuzzy pre-open sets in (X, T). Since (λ_i)'s are fuzzy residual sets implies that $(1-\lambda_i)$'s are fuzzy first category sets in (X,T). Therefore $1-\lambda_i = \vee_{j=1}^{\infty}(\mu_{ij})$, where (μ_{ij})'s are fuzzy nowhere dense sets in (X,T). Since (μ_{ij})'s are fuzzy nowhere dense sets, by theorem 3.12, $(1-\mu_{ij})$'s are fuzzy dense sets in (X,T). Since (X,T) is a fuzzy submaximal space, the fuzzy dense sets $(1-\mu_{ij})$'s are fuzzy open sets in (X,T). Now $\lambda_i = 1-(1-\lambda_i) = 1-\big(\vee_{j=1}^{\infty}(\mu_{ij})\big) = \wedge_{j=1}^{\infty}(1-\mu_{ij})$. Since $(1-\mu_{ij})$'s are fuzzy open sets, (λ_i)'s are fuzzy G_δ-sets in (X,T). Hence (λ_i)'s are fuzzy pre-open and fuzzy G_δ-sets in (X,T). Since (X,T) is a fuzzy ε_p-Volterra space, $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 1$. Hence, $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 1$, where (λ_i)'s are fuzzy dense and fuzzy residual sets in (X,T), implies that (X,T) is a fuzzy ε_r-Volterra space. □

Proposition 4.17. *If each fuzzy nowhere dense set is a fuzzy closed set in a fuzzy ε_p-Volterra space (X,T), then (X,T) is a fuzzy ε_r-Volterra space.*

Proof. Let (λ_i)'s $(i = 1$ to $N)$ be fuzzy dense and fuzzy residual sets in (X,T). Since (λ_i)'s are fuzzy dense sets in (X,T), $cl(\lambda_i) = 1$. Now $intcl(\lambda_i) = int(1) = 1$. Then $\lambda_i \leq intcl(\lambda_i)$. Hence (λ_i)'s are fuzzy pre-open sets in (X,T). Since (λ_i)'s are fuzzy residual sets implies that $(1-\lambda_i)$'s are fuzzy first category sets in (X,T). Therefore $1-\lambda_i = \vee_{j=1}^{\infty}(\mu_{ij})$, where (μ_{ij})'s are fuzzy nowhere dense sets in (X,T). By hypothesis, the fuzzy nowhere dense sets (μ_{ij})'s are fuzzy closed sets in (X,T). Then $(1-\mu_{ij})$'s are fuzzy open sets in (X,T). Now $\lambda_i = 1-(1-\lambda_i) = 1-\big(\vee_{j=1}^{\infty}(\mu_{ij})\big) = \wedge_{j=1}^{\infty}(1-\mu_{ij})$. Since $(1-\mu_{ij})$'s are fuzzy open sets, (λ_i)'s are fuzzy G_δ-sets in (X,T). Hence (λ_i)'s are fuzzy pre-open and fuzzy G_δ-sets in (X,T). Since (X,T) is a fuzzy ε_p-Volterra space, $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 1$. Hence $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 1$, where (λ_i)'s are fuzzy dense and fuzzy residual sets in (X,T), implies that (X,T) is a fuzzy ε_r-Volterra space. □

Proposition 4.18. *If a fuzzy ε_p-Volterra space (X,T) is a fuzzy nodec space, then (X,T) is a fuzzy ε_r-Volterra space.*

Proof. Let (X,T) be a fuzzy ε_p-Volterra space and a fuzzy nodec space. Since (X,T) is a fuzzy nodec space, each fuzzy nowhere dense set is a fuzzy closed set in (X,T). Hence each fuzzy nowhere dense set is a fuzzy closed set in the fuzzy ε_p-Volterra space (X,T). Therefore, by proposition 4.17, (X,T) is a fuzzy ε_r-Volterra space. □

5. Conclusions

The concepts of fuzzy ε_r-Volterra and fuzzy ε_p-Volterra spaces are introduced and studied in this paper. The inter relations between fuzzy ε_r-Volterra spaces and fuzzy ε_p-Volterra spaces are also investigated.

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