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On Zweier statistically convergent sequences of fuzzy numbers and de la Vallée-Poussin mean of order α

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Abstract. The main purpose of this paper is to introduced the notion of Zweier statistical convergence using de la Vallée-Poussin mean of order α of fuzzy numbers. Also we introduced the space $\mathcal{F}_\lambda^\alpha[\mathcal{Z}, M, p]$ defined by Orlicz function and Zweier operator, and derived some results between Zweier statistical convergence using de la Vallée-Poussin mean of order α and the space $\mathcal{F}_\lambda^\alpha[\mathcal{Z}, M, p]$.

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1. Introduction

The idea of statistical convergence first appeared, under the name of almost convergence, in the first edition Zygmund [35]. In 1951, Fast [12] introduced the concept of statistical convergence for real sequences, the statistical convergence has been further stuided by Steinhaus [32], Fridy [13], Šalát [28] and other authors. Mursaleen [24], introduced the notion λ-statistical convergence for real sequences. For more details on λ-statistical convergence we refer to [4] and many others. The notion of order statistical convergence was introduced by Gadjiev and Orhan [14]. The concept of statistical convergence of order α was studied by Çolak [5]. The concept of λ-statistical convergence of order α was introduced by Çolak and Bektaş [6], λ-statistical convergence of order α of sequence of functions studied by Et et al., [10, 11].

Şengönül [30] defined the sequence $y = (y_k)$ which is frequently used as the Z-transformation of the sequence $x = (x_k)$ i.e.

$$y_k = px_k + (1-p)x_{k-1}$$

where $x_{-1} = 0, 1 \leq k < \infty$ and Z denotes the matrix $Z = (z_{nk})$ defined by

$$z_{nk} = \begin{cases} p, & \text{if } n = k; \\ 1-p, & \text{if } n-1 = k; \\ 0, & \text{otherwise.} \end{cases}$$

Şengönül [30] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows

$$\mathcal{Z} = \{x = (x_k) \in w : Zx \in c\}$$

and

$$\mathcal{Z}_0 = \{x = (x_k) \in w : Zx \in c_0\}.$$

For details on Zweier sequence spaces we refer to [8, 9, 16, 17, 18, 19, 20].

Let $\lambda = (\lambda_r)$ be an increasing sequence of positive real numbers tending to ∞ such that $\lambda_r \leq \lambda_r + 1, \lambda_1 = 1$. We denote $\Lambda = \{\lambda = (\lambda_r) : \lambda_r \to \infty$ such that $\lambda_r \leq \lambda_r + 1, \lambda_1 = 1\}$. The generalized de la Vallée-Poussin mean is defined by $t_r(x) = \frac{1}{\lambda_r}\sum_{k\in I_r} x_k$ where $I_r = [r - \lambda_r + 1, r]$ for $r = 1, 2, 3, \ldots$. A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L if $t_r(x) \to L$ as $r \to \infty$ (see [21]). If $\lambda_r = r$, then (V, λ)-summability is reduced to Cesàro summability.

Fuzzy set theory proposed by Zadeh [34] which is a generalization of classical or crisp sets. As a suitable mathematical model to handle vagueness and uncertainty, fuzzy set theory is emerging as a powerful theory and has attracted the attention of many researchers for Cybernetics, Artificial Intelligence, Expert System and Fuzzy Control, Pattern recognition, Operation Research, Decision making, Image Analysis, Projectiles, Probabilty theory, Weather forecasting etc and practitioners who contributed to its develepoment and applications. Matloka [23] introduced the sets of bounded and convergent sequences of fuzzy numbers and studied their some properties. Later on sequences of fuzzy numbers have been discussed by Diamond and Kloeden [7], Mursaleen and Başarir [25], Altın et al. [1], Nanda [26], Çanak [3], Tripathy et al. [33], Hazarika and Savaş [15] and many others. The statistical convergence for a sequence of fuzzy numbers has been studied by several authors. In 2001, Savaş [29] discussed the statistical convergence for a sequence of fuzzy numbers and presented a characterization theorem. In 2012, Altınok [2] introduced the concept of the λ-statistical convergence of order β of sequences of fuzzy numbers. Recently, Srivastava and Ojha [31] discussed the λ-statistical convergence of fuzzy numbers and fuzzy functions of order θ.

A fuzzy number is a function $X : \mathbb{R} \to [0, 1]$ which is normal, fuzzy convex, upper semi-continuous and supp $[X]^0 = \overline{\{t \in \mathbb{R} : X(t) > 0\}}$ is compact. Here $\bar{S}$ denotes the closure of S.

We denote $L(\mathbb{R})$ the set of all fuzzy numbers, if X is a fuzzy number, then the level set $[X]^a = \{t \in \mathbb{R} : X(t) \geq a\} = [X_a^-, X_a^+]$ is a bounded, closed interval for any $a \in [0,1]$. The space $L(\mathbb{R})$ has a linear structure induced by the addition $X+Y$ and the scalar multiplication λX in terms of a-level sets, defined by

$$[X+Y]^a = [X]^a + [Y]^a \text{ and } [\lambda X]^a = \lambda[X]^a \text{ for each } 0 \leq a \leq 1.$$

Clearly $\mathbb{R}$ is embedded in $L(\mathbb{R})$ i.e. in this case for $t \in \mathbb{R}$, we define $\overline{r} \in L(\mathbb{R})$ by

$$\overline{r}(t) = \begin{cases} 1, & \text{if } t = r; \\ 0, & \text{if } t \neq r. \end{cases}$$

The additive identity and multiplicative identity of $L(\mathbb{R})$ are denoted by $\overline{0}$ and $\overline{1}$, respectively.

For r in $\mathbb{R}$ and X in $L(\mathbb{R})$, the product rX is defined as follows:

$$rX(t) = \begin{cases} X(r^{-1}t), & \text{if } r \neq 0; \\ 0, & \text{if } r = 0. \end{cases}$$

We use the Hausdorff distance between fuzzy numbers given by $d : L(\mathbb{R}) \times L(\mathbb{R}) \to [0, +\infty]$ as follows.

$$d(X,Y) = \sup_{0 \leq a \leq 1} \delta_\infty(X^a, Y^a) = \sup_{a \in [0,1]} \max\{|X_a^- - Y_a^-|, |X_a^+ - Y_a^+|\},$$

where δ_∞ is the Hausdorff metric. For $X, Y \in L(\mathbb{R})$ define $X \leq Y$ if and only if $X^a \leq Y^a$ for any $a \in [0,1]$. Then $L(\mathbb{R})$ is complete metric space with respect to the metric d (see [23]).

A sequence $u = (u_k)$ of fuzzy numbers is said to be

(i) bounded if the set $\{u_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded.

(ii) convergent to a fuzzy number u_0 if for every $\varepsilon > 0$ there is a positive integer k_0 such that $d(u_k, u_0) < \varepsilon$ for all $k > k_0$.

We denote w^F, ℓ_∞^F and c^F, the set of all, bounded and convergent sequences of fuzzy numbers, respectively. It is straightforward that $c^F \subset \ell_\infty^F \subset w^F$.

The characteristic function of a subset A of $\mathbb{N}$ is defined as follows:

$$\chi_A(k) = \begin{cases} 1, & \text{if } k \in A; \\ 0, & \text{if } k \notin A. \end{cases}$$

Let $T = (t_{nk})$ be a regular non-negative matrix. For $A \subset \mathbb{N}$, define $d_T^{(n)}(A) = \sum_{k=1}^{\infty} t_{nk}\chi_A(k)$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} d_T^{(n)}(A) = d_T(A)$ exists, then $d_T(A)$ is called as T-density of A. Clearly $I_{d_T} = \{A \subset \mathbb{N} : d_T(A) = 0\}$ is an ideal.

Note 1: Some particular cases of T-density:

(i) Asymptotic density, for

$$t_{nk} = \begin{cases} \frac{1}{n}, & \text{if } n \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Logarithmic density, for

$$t_{nk} = \begin{cases} \frac{k^{-1}}{s_n}, & \text{if } n \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.1 ([24]). A sequence $x = (x_k)$ of real numbers is said to be λ-statistically convergent to L if for every $\varepsilon > 0$

$$\lim_r \frac{1}{\lambda_r} \left|\{k \in I_r : \ |x_k - L| \geq \varepsilon\}\right| = 0.$$

In this case we write $S_\lambda - \lim x = L$ or $x_k \to L(S_\lambda)$.

Definition 1.2 ([6]). A sequence $x = (x_k)$ of real numbers is said to be λ-statistically convergent of order α to L or S_λ^α-convergent to L if for every $\varepsilon > 0$

$$\lim_r \frac{1}{\lambda_r^\alpha} \left|\{k \in I_r : \ |x_k - L| \geq \varepsilon\}\right| = 0.$$

In this case we write $S_\lambda^\alpha - \lim x = L$ or $x_k \to L(S_\lambda^\alpha)$.

2. Zweier statistical convergence

In this section, we define the concept of $\mathcal{Z}S_\lambda$-statistical convergence and established the relationship of $\mathcal{Z}S_\lambda$ with $\mathcal{F}_\lambda^\alpha[\mathcal{Z}]$. Also we introduced the notion of $\mathcal{Z}S_\lambda$-statistical convergence of order α of fuzzy number sequences and obtained some inclusion relations between the set of $\mathcal{Z}S_\lambda$-statistical convergence of order α and $\mathcal{F}_\lambda^\alpha[\mathcal{Z}, M, p]$.

Definition 2.1. Let $\lambda = (\lambda_r)$ be a sequence in Λ. A sequence $x = (x_k)$ of fuzzy numbers is said to be $\mathcal{Z}S_\lambda$-convergent to $x_0 \in L(\mathbb{R})$ if for every $\varepsilon > 0$

$$\lim_r \frac{1}{\lambda_r} \left|\{k \in I_r : \ d((Zx)_k, x_0) \geq \varepsilon\}\right| = 0.$$

In this case we write $\mathcal{Z}S_\lambda - \lim x = x_0$ or $x_k \to x_0(\mathcal{Z}S_\lambda)$.

Definition 2.2. Let $\lambda = (\lambda_r)$ be a sequence in Λ. A sequence $x = (x_k)$ of fuzzy numbers is said to be Zweier strong λ-summable to $x_0 \in L(\mathbb{R})$ if

$$\lim_{r\to\infty} \frac{1}{\lambda_r} \sum_{k \in I_r} d((Zx)_k, x_0) = 0.$$

We denote the set of all Zweier strong λ-summable sequences of fuzzy numbers by $\mathcal{F}_\lambda[\mathcal{Z}]$.

Theorem 2.3. *Let $\lambda = (\lambda_r)$ be a sequence in Λ.*

(a) *If $x_k \to x_0(\mathcal{F}_\lambda[\mathcal{Z}])$ then $x_k \to x_0(\mathcal{Z}S_\lambda)$,*
(b) *If $x \in l_\infty^F[\mathcal{Z}]$ and $x_k \to x_0(\mathcal{Z}S_\lambda)$ then $x_k \to x_0(\mathcal{F}_\lambda[\mathcal{Z}])$,*
(c) *$\mathcal{F}_\lambda[\mathcal{Z}] \cap l_\infty^F[\mathcal{Z}] = \mathcal{Z}S_\lambda \cap l_\infty^F[\mathcal{Z}]$, where*

$$l_\infty^F[\mathcal{Z}] = \left\{ x \in w^F : \sup_k d((Zx)_k, \bar{0}) < \infty \right\}.$$

Proof. **(a)** Suppose that $\varepsilon > 0$ and $x_k \to x_0(\mathcal{F}_\lambda[\mathcal{Z}])$. Then we have

$$\sum_{k\in I_r} d((Zx)_k, x_0) \geq \sum_{\substack{k\in I_r\\ d((Zx)_k,x_0)\geq\varepsilon}} d((Zx)_k, x_0) \geq \varepsilon\,|\{k\in I_r:\ d((Zx)_k, x_0)\geq\varepsilon\}|.$$

Thus $x_k \to x_0(\mathcal{Z}S_\lambda)$.

(b) Suppose that $x \in l_\infty^F[\mathcal{Z}]$ and $x_k \to x_0(\mathcal{Z}S_\lambda)$, i.e., for some $K > 0$, $d((Zx)_k, x_0) \leq K$ for all k. Given $\varepsilon > 0$, we get

$$\frac{1}{\lambda_r}\sum_{k\in I_r} d((Zx)_k, x_0) = \frac{1}{\lambda_r}\sum_{\substack{k\in I_r\\ d((Zx)_k,x_0)\geq\varepsilon}} d((Zx)_k, x_0) + \frac{1}{\lambda_r}\sum_{\substack{k\in I_r\\ d((Zx)_k,x_0)<\varepsilon}} d((Zx)_k, x_0) \leq \frac{K}{\lambda_r}|\{k\in I_r:\ d((Zx)_k, x_0)\geq\varepsilon\}| + \varepsilon$$

as $r\to\infty$, the right side goes to zero, which implies that $x_k \to x_0(\mathcal{F}_\lambda[\mathcal{Z}])$.

(c) Follows from (a) and (b). □

Definition 2.4. Let $0 < \alpha \leq 1$ be given. A sequence $x = (x_k)$ of fuzzy numbers is said to be Zweier statistically convergent of order α to $x_0 \in L(\mathbb{R})$ or $\mathcal{Z}S^\alpha$-convergent of order α to $x_0 \in L(\mathbb{R})$ if for every $\varepsilon > 0$

$$\lim_{n\to\infty}\frac{1}{n^\alpha}|\{k\leq n:\ d((Zx)_k, x_0)\geq\varepsilon\}| = 0.$$

In this case we write $\mathcal{Z}S^\alpha - \lim x = x_0$ or $x_k \to x_0(\mathcal{Z}S^\alpha)$.

Definition 2.5. Let $\lambda = (\lambda_r)$ be a sequence in Λ and $0 < \alpha \leq 1$ be given. A sequence $x = (x_k)$ of fuzzy numbers is said to be $\mathcal{Z}S_\lambda^\alpha-$convergent of order α to $x_0 \in L(\mathbb{R})$ if for every $\varepsilon > 0$

$$\lim_{r\to\infty}\frac{1}{\lambda_r^\alpha}|\{k\in I_r:\ d((Zx)_k, x_0)\geq\varepsilon\}| = 0. \tag{2.1}$$

In this case we write $\mathcal{Z}S_\lambda^\alpha - \lim x = x_0$ or $x_k \to x_0(\mathcal{Z}S_\lambda^\alpha)$.

Theorem 2.6. *For $0 < \alpha \leq 1$, if $\mathcal{Z}S^\alpha - \lim_k x_k = x_0$ then x_0 is unique.*

Proof. The proof of the result is easy, so omitted. □

Theorem 2.7. *Let $0 < \alpha \leq 1$ and $x = (x_k)$ and $y = (y_k)$ be sequences of fuzzy numbers.*

(a) *If $\mathcal{Z}S^\alpha - \lim_k x_k = x_0$ and $c \in \mathbb{C}$ then $\mathcal{Z}S^\alpha - \lim_k(cx_k) = cx_0$;*
(b) *If $\mathcal{Z}S^\alpha - \lim_k x_k = x_0$ and $\mathcal{Z}S^\alpha - \lim_k y_k = y_0$ then $\mathcal{Z}S^\alpha - \lim_k(x_k + y_k) = x_0 + y_0$.*

Proof. (a) For $c = 0$ the result is trivial. Suppose that $c \neq 0$, then for every $\varepsilon > 0$ the result follows form the following inequality

$$\frac{1}{n^\alpha}|\{k\leq n: d((Zcx)_k, cx_0)\geq\varepsilon\}| = \frac{1}{n^\alpha}\left|\left\{k\leq n: d((Zx)_k, x_0)\geq\frac{\varepsilon}{|c|}\right\}\right|.$$

(b) For every $\varepsilon > 0$. The result follows from the following inequality.

$$\begin{aligned}
&\frac{1}{n^\alpha}\left|\{k \le n : d((Z(x+y))_k, (x_0+y_0)) \ge \varepsilon\}\right| \\
&\le \frac{1}{n^\alpha}\left|\left\{k \le n : d((Zx)_k, x_0) \ge \frac{\varepsilon}{2}\right\}\right| + \frac{1}{n^\alpha}\left|\left\{k \le n : d((Zy)_k, y_0) \ge \frac{\varepsilon}{2}\right\}\right|.
\end{aligned}$$

□

Theorem 2.8. *Let $0 < \alpha \le 1$ and $x = (x_k)$ and $y = (y_k)$ be sequences of fuzzy numbers.*

(a) *If $\mathcal{Z}S_\lambda^\alpha - \lim_k x_k = x_0$ and $c \in \mathbb{C}$, then $\mathcal{Z}S_\lambda^\alpha - \lim_k (cx_k) = cx_0$;*
(b) *If $\mathcal{Z}S_\lambda^\alpha - \lim_k x_k = x_0$ and $\mathcal{Z}S_\lambda^\alpha - \lim_k y_k = y_0$, then $\mathcal{Z}S_\lambda^\alpha \lim_k (x_k + y_k) = x_0 + y_0$.*

Proof. (a) For $c = 0$, the result is trivial. Suppose that $c \neq 0$, then for every $\varepsilon > 0$ the result follows form the following inequality

$$\frac{1}{\lambda_r^\alpha}\left|\{k \in I_r : d((Zcx)_k, cx_0) \ge \varepsilon\}\right| = \frac{1}{\lambda_r^\alpha}\left|\left\{k \in I_r : d((Zx)_k, x_0) \ge \frac{\varepsilon}{|c|}\right\}\right|.$$

(b) For every $\varepsilon > 0$. The result follows from the following enequality.

$$\begin{aligned}
&\frac{1}{\lambda_r^\alpha}\left|\{k \in I_r : d((Z(x+y))_k, (x_0+y_0)) \ge \varepsilon\}\right| \\
&\le \frac{1}{\lambda_r^\alpha}\left|\left\{k \in I_r : d((Zx)_k, x_0) \ge \frac{\varepsilon}{2}\right\}\right| + \frac{1}{\lambda_r^\alpha}\left|\left\{k \in I_r : d((Zy)_k, y_0) \ge \frac{\varepsilon}{2}\right\}\right|.
\end{aligned}$$

□

Theorem 2.9. *If $0 < \alpha < \beta \le 1$ then $\mathcal{Z}S_\lambda^\alpha \subset \mathcal{Z}S_\lambda^\beta$ and the inclusion is strict.*

Proof. The proof of the result follows form the following inequality.

$$\frac{1}{\lambda_r^\beta}\left|\{k \in I_r : d((Zx)_k, x_0) \ge \varepsilon\}\right| = \frac{1}{\lambda_r^\alpha}\left|\{k \in I_r : d((Zx)_k, x_0) \ge \varepsilon\}\right|.$$

To prove the inclusion is strict, let (λ_r) be given and we consider the sequence $x = (x_k)$ be defined by

$$(Zx)_k = \begin{cases} \bar{1}, & \text{if } r - [\sqrt{\lambda_r}] + 1 \le k \le r; \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}
&\frac{1}{\lambda_r^\beta}\left|\{k \in I_r : d((Zx)_k, \bar{0}) \ge \varepsilon\}\right| \\
&= \frac{1}{\lambda_r^\beta}\left|\{k \in I_r : r - [\sqrt{\lambda_r}] + 1 \le k \le r\}\right| \le \frac{\sqrt{\lambda_r}}{\lambda_r^\beta}.
\end{aligned}$$

Thus $x \in \mathcal{Z}S_\lambda^\beta$ for $\frac{1}{2} < \beta \le 1$, but $x \notin \mathcal{Z}S_\lambda^\alpha$ for $0 < \alpha \le \frac{1}{2}$. □

Corollary 2.10. *If a sequence is $\mathcal{Z}S_\lambda^\alpha$-convergent to x_0 then it is $\mathcal{Z}S_\lambda$-convergent to x_0 for $0 < \alpha \le 1$.*

Theorem 2.11. *Let $0 < \alpha \leq 1$ and $\lambda = (\lambda_r) \in \Lambda$. Then $\mathcal{Z}S^{\alpha} \subset \mathcal{Z}S^{\alpha}_{\lambda}$ if*

$$\liminf_{r\to\infty} \frac{\lambda_r^{\alpha}}{r^{\alpha}} > 0.$$

Proof. If $x_k \to x_0\,(\mathcal{Z}S^{\alpha})$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\begin{aligned} &\frac{1}{r^{\alpha}} \left|\{k \leq r : d((Zx)_k, x_0) \geq \varepsilon\}\right| \\ \geq\ &\frac{1}{r^{\alpha}} \left|\{k \in I_r : d((Zx)_k, x_0) \geq \varepsilon\}\right| \\ \geq\ &\frac{\lambda_r^{\alpha}}{r^{\alpha}} \frac{1}{\lambda_r^{\alpha}} \left|\{k \in I_r : d((Zx)_k, x_0) \geq \varepsilon\}\right|. \end{aligned}$$

Taking the limit as $r \to \infty$ and using the given condition, we get $x_k \to x_0\,(\mathcal{Z}S^{\alpha}_{\lambda})$. This completes the proof of the theorem. □

Corollary 2.12. *Let $0 < \alpha \leq 1$ and $\lambda = (\lambda_r) \in \Lambda$. Then $\mathcal{Z}S^{\alpha}_{\lambda} \subset \mathcal{Z}S$.*

Theorem 2.13. *Let $0 < \alpha \leq 1$ and $\lambda = (\lambda_r) \in \Lambda$. Then $\mathcal{Z}S \subset \mathcal{Z}S^{\alpha}_{\lambda}$ if and only if*

$$\liminf_{r\to\infty} \frac{\lambda_r^{\alpha}}{r} > 0. \tag{2.2}$$

Proof. Let the condition (2.2) holds and $x = (x_k) \in \mathcal{Z}S$. For a given $\varepsilon > 0$ we have

$$\{k \leq r : d((Zx)_k, x_0) \geq \varepsilon\} \supset \{k \in I_r : d((Zx)_k, x_0) \geq \varepsilon\}.$$

Then we have

$$\begin{aligned} &\frac{1}{r^{\alpha}} \left|\{k \leq r : d((Zx)_k, x_0) \geq \varepsilon\}\right| \\ \geq\ &\frac{1}{r} \left|\{k \in I_r : d((Zx)_k, x_0) \geq \varepsilon\}\right| \\ =\ &\frac{\lambda_r^{\alpha}}{r} \frac{1}{\lambda_r^{\alpha}} \left|\{k \in I_r : d((Zx)_k, x_0) \geq \varepsilon\}\right|. \end{aligned}$$

By taking limit as $r \to \infty$ and from relation (2.2) we have

$$x_k \to L\,(\mathcal{Z}S) \Rightarrow x_k \to L\,(\mathcal{Z}S^{\alpha}_{\lambda}).$$

Next we suppose that

$$\liminf_{r\to\infty} \frac{\lambda_r^{\alpha}}{r} = 0.$$

Then we can choose a subsequence (r_i) such that $\frac{\lambda_{r_i}^{\alpha}}{r_i} < \frac{1}{i}$. Define a sequence $x = (x_k)$ as follows:

$$(x)_k = \begin{cases} \bar{1}, & \text{if } k \in I_{r_i}; \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Then clearly $x = (x_k) \in \mathcal{Z}S$ but $x = (x_k) \notin \mathcal{Z}S_{\lambda}$. Since $\mathcal{Z}S^{\alpha}_{\lambda} \subset \mathcal{Z}S_{\lambda}$, we have $x = (x_k) \notin \mathcal{Z}S^{\alpha}_{\lambda}$, which is a contradiction. Hence the relation (2.2) holds. □

Theorem 2.14. *Let $\lambda = (\lambda_r)$ and $\mu = (\mu_r)$ be two sequences in Λ such that $\lambda_r \leq \mu_r$ for all $r \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$. If*

$$\liminf_{r\to\infty} \frac{\lambda_r^{\alpha}}{\mu_r^{\beta}}, \tag{2.3}$$

then $\mathcal{Z}S_{\mu}^{\beta} \subseteq \mathcal{Z}S_{\lambda}^{\alpha}$.

Proof. Suppose that $\lambda_r \leq \mu_r$ for all $r \in \mathbb{N}$ and the condition (2.3) satisfied. Then $I_r \subset J_r$ and so that for $\varepsilon > 0$ we can write

$$\{k \in J_r : d((Zx)_k, x_0) \geq \varepsilon\} \supset \{k \in I_r : d((Zx)_k, x_0) \geq \varepsilon\}.$$

Then we have

$$\frac{1}{\mu_r^{\beta}}|\{k \in J_r : d((Zx)_k, x_0) \geq \varepsilon\}| \geq \frac{\lambda_r^{\alpha}}{\mu_r^{\beta}} \frac{1}{\lambda_r^{\alpha}}|\{k \in I_r : d((Zx)_k, x_0) \geq \varepsilon\}|,$$

for all $r \in \mathbb{N}$, where $J_r = [r - \mu_r + 1, r]$. Taking limit $r \to \infty$ in the last inequality and using (2.3), we have $\mathcal{Z}S_{\mu}^{\beta} \subseteq \mathcal{Z}S_{\lambda}^{\alpha}$. □

Corollary 2.15. *Let $\lambda = (\lambda_r)$ and $\mu = (\mu_r)$ be two sequences in Λ such that $\lambda_r \leq \mu_r$ for all $r \in \mathbb{N}$. If (2.3) holds, then*
- (a) $\mathcal{Z}S_{\mu}^{\alpha} \subseteq \mathcal{Z}S_{\lambda}^{\alpha}$ *for* $0 < \alpha \leq 1$,
- (b) $\mathcal{Z}S_{\mu} \subseteq \mathcal{Z}S_{\lambda}^{\alpha}$ *for* $0 < \alpha \leq 1$,
- (c) $\mathcal{Z}S_{\mu} \subseteq \mathcal{Z}S_{\lambda}$.

Theorem 2.16. *Let $\lambda = (\lambda_r)$ and $\mu = (\mu_r)$ be two sequences in Λ such that $\lambda_r \leq \mu_r$ for all $r \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$. If*

$$\lim_{r\to\infty} \frac{\mu_r}{\lambda_r^{\beta}} = 1, \tag{2.4}$$

then $\mathcal{Z}S_{\lambda}^{\alpha} \subseteq \mathcal{Z}S_{\mu}^{\beta}$.

Proof. Let $\mathcal{Z}S_{\lambda}^{\alpha} - \lim x = x_0$ and (2.4) be satisfied. Since $I_r \subset J_r$, for $\varepsilon > 0$ we can write

$$\begin{aligned}
&\frac{1}{\mu_r^{\beta}}|\{k \in J_r : d((Zx)_k, x_0) \geq \varepsilon\}| \\
&= \frac{1}{\mu_r^{\beta}}|\{r - \mu_r + 1 \leq k \leq r - \lambda_r : d((Zx)_k, x_0) \geq \varepsilon\}| \\
&\qquad\qquad + \frac{1}{\mu_r^{\beta}}|\{k \in I_r : d((Zx)_k, x_0) \geq \varepsilon\}| \\
&\leq \frac{\mu_r - \lambda_r}{\mu_r^{\beta}} + \frac{1}{\mu_r^{\beta}}|\{k \in I_r : d((Zx)_k, x_0) \geq \varepsilon\}| \\
&\leq \frac{\mu_r - \lambda_r^{\beta}}{\lambda_r^{\beta}} + \frac{1}{\mu_r^{\beta}}|\{k \in I_r : d((Zx)_k, x_0) \geq \varepsilon\}| \\
&\leq \left(\frac{\mu_r}{\lambda_r^{\beta}} - 1\right) + \frac{\lambda_r^{\alpha}}{\mu_r^{\beta}} \frac{1}{\lambda_r^{\alpha}}|\{k \in I_r : d((Zx)_k, x_0) \geq \varepsilon\}|.
\end{aligned}$$

Using the relation (2.4) and $\mathcal{Z}S_{\lambda}^{\alpha} - \lim x = x_0$ the right-hand side of the above inequality tends to zero as $r \to \infty$. This implies that $\mathcal{Z}S_{\lambda}^{\alpha} \subseteq \mathcal{Z}S_{\mu}^{\beta}$. □

Corollary 2.17. *Let $\lambda = (\lambda_r)$ and $\mu = (\mu_r)$ be two sequences in Λ such that $\lambda_r \leq \mu_r$ for all $r \in \mathbb{N}$. If (2.4) holds, then*
- (a) $\mathcal{Z}S_{\lambda}^{\alpha} \subseteq \mathcal{Z}S_{\mu}^{\alpha}$ *for* $0 < \alpha \leq 1$,
- (b) $\mathcal{Z}S_{\lambda} \subseteq \mathcal{Z}S_{\mu}^{\alpha}$ *for* $0 < \alpha \leq 1$,

(c) $\mathcal{Z}S_\lambda \subseteq \mathcal{Z}S_\mu$.

3. Zweier de la Vallée-Poussin mean of order α

A function $M : [0, \infty) \to [0, \infty)$ is said to be an Orlicz function if is continuous, convex, nondecreasing with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called the modulus function and it was characterized by Ruckle [27]. An Orlicz function M is said to satisfy Δ_2-condition for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$.

Lemma 3.1. *An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.*

Lindenstrauss and Tzafriri [22] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) < \infty, \text{ for some } r > 0 \right\}$$

which is a Banach space normed by

$$\|(x_k)\| = \inf\left\{ r > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) \leq 1 \right\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

Now we define a new sequence space as follows. Let M be an Orlicz function, $p = (p_k)$ be a sequence of positive real numbers, $\alpha \in (0, 1]$, $\lambda = (\lambda_r)$ be a sequence of positive reals, and for $\rho > 0$, we define

$$\mathcal{F}_\lambda^\alpha[\mathcal{Z}, M, p] = \left\{ x \in w^F : \lim_{r\to\infty} \frac{1}{\lambda_r^\alpha} \sum_{k \in I_r} \left[M\left(\frac{d((Zx)_k, l)}{\rho} \right) \right]^{p_k} = 0, \text{ for some } l \in L(\mathbb{R}) \right\}.$$

If $M(x) = x$ and $p_k = p$ for all $k \in \mathbb{N}$ then we shall write $\mathcal{F}_\lambda^\alpha[\mathcal{Z}, M, p] = \mathcal{F}_\lambda^\alpha[\mathcal{Z},](p)$ and if $M(x) = x$ then we shall write $\mathcal{F}_\lambda^\alpha[\mathcal{Z}, M, p] = \mathcal{F}_\lambda^\alpha[\mathcal{Z}, p]$.

Theorem 3.2. *Let (p_k) be a bounded and $0 < \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. If $0 < \alpha \leq \beta \leq 1$, M is an Orlicz function, and $\lambda = (\lambda_r)$ is a sequence of positive reals, then $\mathcal{F}_\lambda^\alpha[\mathcal{Z}, M, p] \subset \mathcal{Z}S_\lambda^\beta$.*

Proof. Let $x=(x_k)\in\mathcal{F}_\lambda^\alpha[\mathcal{Z},M,p]$. Let $\varepsilon>0$ be given. As $\lambda_r^\alpha\le\lambda_r^\beta$ for each r, we can write

$$\begin{aligned}
&\frac{1}{\lambda_r^\alpha}\sum_{k\in I_r}\left[M\left(\frac{d((Zx)_k,l}{\rho}\right)\right]^{p_k}\\
&=\frac{1}{\lambda_r^\alpha}\left[\sum_{\substack{k\in I_r\\ d((Zx)_k,l)\ge\varepsilon}}\left[M\left(\frac{d((Zx)_k,l)}{\rho}\right)\right]^{p_k}+\sum_{\substack{k\in I_r\\ d((Zx)_k,l)<\varepsilon}}\left[M\left(\frac{d((Zx)_k,l)}{\rho}\right)\right]^{p_k}\right]\\
&\ge\frac{1}{\lambda_r^\beta}\left[\sum_{\substack{k\in I_r\\ d((Zx)_k,l)\ge\varepsilon}}\left[M\left(\frac{d((Zx)_k,l)}{\rho}\right)\right]^{p_k}+\sum_{\substack{k\in I_r\\ d((Zx)_k,l)<\varepsilon}}\left[M\left(\frac{d((Zx)_k,l)}{\rho}\right)\right]^{p_k}\right]\\
&\ge\frac{1}{\lambda_r^\beta}\sum_{\substack{k\in I_r\\ d((Zx)_k,l)\ge\varepsilon}}\left[M\left(\frac{\varepsilon}{\rho}\right)\right]^{p_k}\ge\frac{1}{\lambda_r^\beta}\sum_{\substack{k\in I_r\\ d((Zx)_k,l)\ge\varepsilon}}\min\left([M(\varepsilon_1)]^h,[M(\varepsilon_1)]^H\right),\varepsilon_1=\frac{\varepsilon}{\rho}\\
&\ge\frac{1}{\lambda_r^\beta}|\{k\in I_r:d((Zx)_k,l)\ge\varepsilon\}|\min\left([M(\varepsilon_1)]^h,[M(\varepsilon_1)]^H\right).
\end{aligned}$$

From the above inequality, we have $(x_k)\in\mathcal{Z}S_\lambda^\beta$. □

Corollary 3.3. *If $0<\alpha\le1$, M is an Orlicz function, and $\lambda=(\lambda_r)$ is an element of Λ, then $\mathcal{F}_\lambda^\alpha[\mathcal{Z},M,p]\subset\mathcal{Z}S_\lambda^\alpha$.*

Theorem 3.4. *Let M be an Orlicz function, $x=(x_k)$ be a sequence in $l_\infty^F[\mathcal{Z}]$, and $\lambda=(\lambda_r)$ be an element of Λ. If $\lim_{r\to\infty}\frac{\lambda_r}{\lambda_r^\alpha}=1$, then $\mathcal{Z}S_\lambda^\alpha\subset\mathcal{F}_\lambda^\alpha[\mathcal{Z},M,p]$.*

Proof. Suppose that $x=(x_k)$ is in $l_\infty^F[\mathcal{Z}]$ and $\mathcal{Z}S^\alpha-\lim_k x_k=l$. Then there exists $K>0$ such that $d((Zx)_k,\bar{0})\le K$ for all k. For given $\varepsilon>0$, we have

$$\begin{aligned}
&\frac{1}{\lambda_r^\alpha}\sum_{k\in I_r}\left[M\left(\frac{d((Zx)_k,l)}{\rho}\right)\right]^{p_k}\\
&=\frac{1}{\lambda_r^\alpha}\sum_{\substack{k\in I_r\\ d((Zx)_k,l)\ge\varepsilon}}\left[M\left(\frac{d((Zx)_k,l)}{\rho}\right)\right]^{p_k}+\frac{1}{\lambda_r^\alpha}\sum_{\substack{k\in I_r\\ d((Zx)_k,l)<\varepsilon}}\left[M\left(\frac{d((Zx)_k,l)}{\rho}\right)\right]^{p_k}\\
&\le\frac{1}{\lambda_r^\alpha}\sum_{\substack{k\in I_r\\ d((Zx)_k,l)\ge\varepsilon}}\max\left\{\left[M\left(\frac{K}{\rho}\right)\right]^h,\left[M\left(\frac{K}{\rho}\right)\right]^H\right\}+\frac{1}{\lambda_r^\alpha}\sum_{\substack{k\in I_r\\ d((Zx)_k,l)<\varepsilon}}\left[M\left(\frac{\varepsilon}{\rho}\right)\right]^{p_k}\\
&\le\max\left\{\left[M\left(\frac{K}{\rho}\right)\right]^h,\left[M\left(\frac{K}{\rho}\right)\right]^H\right\}\frac{1}{\lambda_r^\alpha}|d((Zx)_k,l)\ge\varepsilon|\\
&\qquad+\frac{\lambda_r}{\lambda_r^\alpha}\max\left\{\left[M\left(\frac{\varepsilon}{\rho}\right)\right]^h,\left[M\left(\frac{\varepsilon}{\rho}\right)\right]^H\right\}.
\end{aligned}$$

Thus we have $(x_k)\in\mathcal{F}_\lambda^\alpha[\mathcal{Z},M,p]$. □

Theorem 3.5. *If $\lambda = (\lambda_r) \in \Lambda, 0 < \alpha \leq \beta \leq 1, p$ is a positive real number, then $\mathcal{F}_{\lambda}^{\alpha}[\mathcal{Z},](p) \subseteq \mathcal{F}_{\lambda}^{\beta}[\mathcal{Z}](p)$.*

Proof. The proof is easy, so omitted. □

Corollary 3.6. *If $\lambda = (\lambda_r) \in \Lambda$ and p is a positive real number, then $\mathcal{F}_{\lambda}^{\alpha}[\mathcal{Z}](p) \subseteq \mathcal{F}_{\lambda}[\mathcal{Z}](p)$.*

Theorem 3.7. *If $\lambda = (\lambda_r) \in \Lambda, 0 < \alpha \leq \beta \leq 1$ and p is a positive real number, then $\mathcal{F}_{\lambda}^{\alpha}[\mathcal{Z}](p) \subseteq \mathcal{Z}S_{\lambda}^{\beta}$.*

Proof. Let $x = (x_k) \in \mathcal{F}_{\lambda}^{\alpha}[\mathcal{Z}](p)$. Then we have for $\varepsilon > 0$ that

$$\begin{aligned}\sum_{k \in I_r} (d((Zx)_k, l))^p &= \sum_{\substack{k \in I_r \\ d((Zx)_k, l) \geq \varepsilon}} (d((Zx)_k, l))^p + \sum_{\substack{k \in I_r \\ d((Zx)_k, l) < \varepsilon}} (d((Zx)_k, l))^p \\ &\geq \sum_{\substack{k \in I_r \\ d((Zx)_k, l) \geq \varepsilon}} (d((Zx)_k, l))^p \\ &\geq |\{k \in I_r : d((Zx)_k, l) \geq \varepsilon\}| \, \varepsilon^p.\end{aligned}$$

Thus we have

$$\frac{1}{\lambda_r^{\alpha}} \sum_{k \in I_r} d(\left(Z^i x\right)_k, l)|^p \geq \frac{1}{\lambda_r^{\beta}} |\{k \in I_r : d(\left(Z^i x\right)_k, l) \geq \varepsilon\}| \, \varepsilon^p.$$

The last inequality implies that $x = (x_k) \in \mathcal{Z}S_{\lambda}^{\beta}$. This completes the proof of the theorem. □

Theorem 3.8. *Let $\lambda = (\lambda_r)$ and $\mu = (\mu_r)$ be two sequences in Λ such that $\lambda_r \leq \mu_r$ for all $r \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$. If (2.3) holds, then $\mathcal{F}_{\mu}^{\beta}[\mathcal{Z}](p) \subseteq \mathcal{F}_{\lambda}^{\alpha}[\mathcal{Z}](p)$.*

Proof. The proof is easy, so omitted. □

Corollary 3.9. *Let $\lambda = (\lambda_r)$ and $\mu = (\mu_r)$ be two sequences in Λ such that $\lambda_r \leq \mu_r$ for all $r \in \mathbb{N}$. If (2.3) holds, then*

(a) $\mathcal{F}_{\mu}^{\alpha}[\mathcal{Z}](p) \subseteq \mathcal{F}_{\lambda}^{\alpha}[\mathcal{Z}](p)$ *for* $0 < \alpha \leq 1$,
(b) $\mathcal{F}_{\mu}[\mathcal{Z}](p) \subseteq \mathcal{F}_{\lambda}^{\alpha}[\mathcal{Z}](p)$ *for* $0 < \alpha \leq 1$,
(c) $\mathcal{F}_{\mu}[\mathcal{Z}](p) \subseteq \mathcal{F}_{\lambda}[\mathcal{Z}](p)$.

Theorem 3.10. *Let $\lambda = (\lambda_r)$ and $\mu = (\mu_r)$ be two sequences in Λ such that $\lambda_r \leq \mu_r$ for all $r \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$. If (2.3) holds, then $\mathcal{F}_{\mu}^{\beta}[\mathcal{Z}](p) \subseteq \mathcal{Z}S_{\lambda}^{\alpha}$.*

Proof. Let $x = (x_k) \in \mathcal{F}_{\mu}^{\beta}[\mathcal{Z}](p)$. Then we have for $\varepsilon > 0$,

$$\begin{aligned}\sum_{k \in I_r} (d((Zx)_k, l))^p &= \sum_{\substack{k \in I_r \\ d((Zx)_k, l) \geq \varepsilon}} (d((Zx)_k, l))^p + \sum_{\substack{k \in I_r \\ d((Zx)_k, l) < \varepsilon}} (d((Zx)_k, l))^p \\ &\geq \sum_{\substack{k \in I_r \\ d((Zx)_k, l) \geq \varepsilon}} (d((Zx)_k, l))^p \\ &\geq |\{k \in I_r : d((Zx)_k, l) \geq \varepsilon\}| \, \varepsilon^p.\end{aligned}$$

Thus we have

$$\frac{1}{\mu_r^{\beta}}\sum_{k\in I_r}(d((Zx)_k\,,l))^p \geq \frac{\lambda_r^{\alpha}}{\mu_r^{\beta}}\frac{1}{\lambda_r^{\alpha}}|\{k\in I_r : d((Zx)_k\,,l)\geq\varepsilon\}|\;\varepsilon^p.$$

Since (2.3) holds and $x=(x_k)\in\mathcal{F}_{\mu}^{\beta}[\mathcal{Z}](p)$, the last inequality implies that $x=(x_k)\in\mathcal{Z}S_{\lambda}^{\alpha}$. This completes the proof of the theorem. □

Corollary 3.11. *Let $\lambda=(\lambda_r)$ and $\mu=(\mu_r)$ be two sequences in Λ such that $\lambda_r\leq\mu_r$ for all $r\in\mathbb{N}$ and $0<\alpha\leq 1$. If (2.3) holds, then*

(a) $\mathcal{F}_{\mu}^{\alpha}[\mathcal{Z}](p)\subseteq\mathcal{Z}S_{\lambda}^{\alpha}$,
(b) $\mathcal{F}_{\mu}[\mathcal{Z}](p)\subseteq\mathcal{Z}S_{\lambda}^{\alpha}$,
(c) $\mathcal{F}_{\mu}[\mathcal{Z}](p)\subseteq\mathcal{Z}S_{\lambda}$.

Theorem 3.12. *Let $\lambda=(\lambda_r)$ and $\mu=(\mu_r)$ be two sequences in Λ such that $\lambda_r\leq\mu_r$ for all $r\in\mathbb{N}$ and $0<\alpha\leq\beta\leq 1$. If (2.4) holds, then $\ell_{\infty}^F[\mathcal{Z}]\cap\mathcal{F}_{\lambda}^{\alpha}[\mathcal{Z}](p)\subseteq\mathcal{F}_{\mu}^{\beta}[\mathcal{Z}](p)$.*

Proof. Let $x=(x_k)\in\ell_{\infty}^F[\mathcal{Z}]\cap\mathcal{F}_{\lambda}^{\alpha}[\mathcal{Z}](p)$ and suppose that (2.4) holds. Since $(x_k)\in\ell_{\infty}^F[\mathcal{Z}]$, there esists $K>0$ such that $d((Zx)_k\,,\bar{0})\leq K$ for all k. Since $\lambda_r\leq\mu_r$ and $I_r\subset J_r$ for all $r\in\mathbb{N}$ we can write

$$\begin{aligned}\frac{1}{\mu_r^{\beta}}\sum_{k\in J_r}(d((Zx)_k\,,l))^p &= \frac{1}{\mu_r^{\beta}}\sum_{k\in J_r-I_r}(d((Zx)_k\,,l))^p+\frac{1}{\mu_r^{\beta}}\sum_{k\in I_r}(d((Zx)_k\,,l))^p\\ &\leq\left(\frac{\mu_r-\lambda_r}{\mu_r^{\beta}}\right)K^p+\frac{1}{\mu_r^{\beta}}\sum_{k\in I_r}(d((Zx)_k\,,l))^p\\ &\leq\left(\frac{\mu_r-\lambda_r^{\beta}}{\mu_r^{\beta}}\right)K^p+\frac{1}{\mu_r^{\beta}}\sum_{k\in I_r}(d((Zx)_k\,,l))^p\\ &\leq\left(\frac{\mu_r-\lambda_r^{\beta}}{\lambda_r^{\beta}}\right)K^p+\frac{\lambda_r^{\alpha}}{\mu_r^{\beta}}\frac{1}{\lambda_r^{\alpha}}\sum_{k\in I_r}(d((Zx)_k\,,l))^p\\ &\leq\left(\frac{\mu_r}{\lambda_r^{\beta}}-1\right)K^p+\frac{\lambda_r^{\alpha}}{\mu_r^{\beta}}\frac{1}{\lambda_r^{\alpha}}\sum_{k\in I_r}(d((Zx)_k\,,l))^p.\end{aligned}$$

This imples that $x=(x_k)\in\mathcal{F}_{\mu}^{\beta}[\mathcal{Z}](p)$. Hence $\ell_{\infty}^F[\mathcal{Z}]\cap\mathcal{F}_{\lambda}^{\alpha}[\mathcal{Z}](p)\subseteq\mathcal{F}_{\mu}^{\beta}[\mathcal{Z}](p)$. □

Corollary 3.13. *Let $\lambda=(\lambda_r)$ and $\mu=(\mu_r)$ be two sequences in Λ such that $\lambda_r\leq\mu_r$ for all $r\in\mathbb{N}$. If (2.4) holds, then*

(a) $\ell_{\infty}^F[\mathcal{Z}]\cap\mathcal{F}_{\lambda}^{\alpha}[\mathcal{Z}](p)\subseteq\mathcal{F}_{\mu}^{\alpha}[\mathcal{Z}](p)$ *for* $0<\alpha\leq 1$,
(b) $\ell_{\infty}^F[\mathcal{Z}]\cap\mathcal{F}_{\lambda}^{\alpha}[\mathcal{Z}](p)\subseteq\mathcal{F}_{\mu}[\mathcal{Z}](p)$ *for* $0<\alpha\leq 1$,
(c) $\ell_{\infty}^F[\mathcal{Z}]\cap\mathcal{F}_{\lambda}[\mathcal{Z}](p)\subseteq\mathcal{F}_{\mu}[\mathcal{Z}](p)$.

Theorem 3.14. *Let M be an Orlicz function and if $\inf_k p_k>0$, then limit of any sequence $x=(x_k)$ in $\mathcal{F}_{\lambda}^{\alpha}[\mathcal{Z},M,p]$ is unique.*

Proof. Let $\lim_k p_k=s>0$. Suppose that $(x_k)\to l_1\,(\mathcal{F}_{\lambda}^{\alpha}[\mathcal{Z},M,p])$ and $(x_k)\to l_2\,(\mathcal{F}_{\lambda}^{\alpha}[\mathcal{Z},M,p])$, as $k\to\infty$. Then there exist $\rho_1>0$ and $\rho_2>0$ such that

$$\lim_{r\to\infty}\frac{1}{\lambda_r^{\alpha}}\sum_{k\in I_r}\left[M\left(\frac{d\left((Zx)_k\,,l_1\right)}{\rho}\right)\right]^{p_k}=0$$

and

$$\lim_{r\to\infty}\frac{1}{\lambda_r^\alpha}\sum_{k\in I_r}\left[M\left(\frac{d\left((Zx)_k\,,l_2\right)}{\rho}\right)\right]^{p_k}=0.$$

Let $\rho=\max\{2\rho_1,2\rho_2\}$. As M is nondecreasing and convex, we have

$$\begin{aligned}
&\frac{1}{\lambda_r^\alpha}\sum_{k\in I_r}\left[M\left(\frac{d(l_1,l_2)}{\rho}\right)\right]^{p_k}\\
&\le\frac{D}{\lambda_r^\alpha}\sum_{k\in I_r}\frac{1}{2^{p_k}}\left(\left[M\left(\frac{d\left((Zx)_k\,,l_1\right)}{\rho}\right)\right]^{p_k}+\left[M\left(\frac{d\left((Zx)_k\,,l_2\right)}{\rho}\right)\right]^{p_k}\right)\\
&\le\frac{D}{\lambda_r^\alpha}\sum_{k\in I_r}\left(\left[M\left(\frac{d\left((Zx)_k\,,l_1\right)}{\rho}\right)\right]^{p_k}+\frac{D}{\lambda_r^\alpha}\sum_{k\in I_r}\left[M\left(\frac{d\left((Zx)_k\,,l_2\right)}{\rho}\right)\right]^{p_k}\right)\\
&\to 0 \text{ as } r\to\infty,
\end{aligned}$$

where $\sup_k p_k=H$ and $D=\max(1,2^{H-1})$. Therefore we get

$$\lim_{r\to\infty}\frac{1}{\lambda_r^\alpha}\sum_{k\in I_r}\left[M\left(\frac{d\left(l_1,l_2\right)}{\rho}\right)\right]^{p_k}=0.$$

As $\lim_k p_k=s$, we have

$$\lim_{k\to\infty}\left[M\left(\frac{d\left(l_1,l_2\right)}{\rho}\right)\right]^{p_k}=\left[M\left(\frac{d\left(l_1,l_2\right)}{\rho}\right)\right]^{s}$$

and so $l_1=l_2$. Hence the limit is unique. □

4. Conclusion

The present work is a generalization of ideal convergence using Zweier transformation. Also, we investigate some futher results on generalized form of ideal convergence. So that one may expect it to be more useful tool in the field of metric space theory in modelling various problems occurring in many areas of science, computer science, information theory, dynamical systems, biological science, geographic information systems, population modelling, medical sciences and motion planning in robotics. It seems that an investigation of the present work taking "nets" instead of "sequences" could be done using the properties of "nets" instead of using the properties of "sequences" in different abstract spaces.

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