Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 11, No. 4, (April 2016), pp. 527–539
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI

Some fixed point theorems of fuzzy non-expansive mapping in fuzzy inner product spaces

S. Mukherjee, T. BAG

Received 31 July 2015; Revised 31 August 2015; Accepted 5 October 2015

Abstract. In this paper some new concepts viz. strictly fuzzy pseudo-contractive mapping, T-quasi non expansive mapping etc are introduced and by using these concepts some fixed point theorems are established in fuzzy inner product spaces.

2010 AMS Classification: : 54A40, 03E72

Keywords: Strictly fuzzy pseudocontractive mapping, T-quasi non expansive mapping, Sectional fuzzy continuous.

Corresponding Author: Tarapada Bag (tarapadavb@gmail.com)

1. Introduction

A lot of work have been done in fuzzy metric spaces and fuzzy normed linear spaces. For reference please see [1, 3, 4, 8, 9, 16, 22]. In 1997, C. Alsina et al.[2] first introduced the idea of real probabilistic inner product space. Following their concept, R. Biswas[7], A. M. El-Abyed & H. M. Hamouly[10] were the first who gave a meaningful definition of fuzzy inner product spaces. On the other hand Kohli & Kumar[14], Majumdar & Samanta[15], Hasankhani, Nazari & Saheli[12], Goudarzi & Vaezpour[11], S. Vijayabalaji[23], Mukherjee & Bag[17], Ramakrishnan[21], Beula & Gifta[6] studied different properties of fuzzy inner product spaces and fixed point theorems.

In [18], we redefine the definition of fuzzy inner product introduced by M.Goudarzi & S.M.Vaezpour[11] and choose t-norm $*$ as 'min' to establish a decomposition theorem which help us to develop more results in fuzzy inner product spaces.

In [19], we consider complex linear space and introduce a definition of fuzzy inner product (complex) space. We establish a decomposition theorem from a fuzzy inner product into a family of crisp inner products. Using this concept, in this paper, we study some results on fuzzy inner product spaces and establish some fixed point theorems in such spaces.

The organization of the paper is as follows:

Section 2, provides some preliminary results which are used in this paper.
In Section 3, concept of α-fuzzy convergent sequence is given .
Some fixed point theorems in fuzzy inner product spaces are established in Section 4.

2. Preliminaries

In this section some definitions and preliminary results which are used in this paper are given.

Definition 2.1 ([3])**.** (Bag & Samanta)
Let U be a linear space over a field F(field of real / complex numbers). A fuzzy subset N of $U \times R$ (R is the set of real numbers) is called a fuzzy norm on U if $\forall x, u \in U$ and $c \in F$, following conditions are satisfied:
(N1) $\forall t \in R$ with $t \leq 0,\ N(x\ ,\ t) = 0$,
(N2) ($\forall t \in R,\ t > 0,\ N(x\ ,\ t) = 1$) iff $x = \underline{0}$,
(N3) $\forall t \in R,\ t > 0,\ N(cx\ ,\ t) = N(x\ ,\ \frac{t}{|c|})$ if $c \neq 0$,
(N4) $\forall s, t \in R,\ x, u \in U,\ N(x + u\ ,\ s + t) \geq min\{N(x\ ,\ s)\ ,\ N(u\ ,\ t)\}$,
(N5) $N(x\ ,\ .)$ is a non-decreasing function of R and $\lim\limits_{t\to\infty} N(x\ ,\ t) = 1$.
The pair $(U\ ,\ N)$ will be referred to as a fuzzy normed linear space.

Theorem 2.2 ([3])**.** *Let $(U\ ,\ N)$ be a fuzzy normed linear space.*
Assume further that,
(N6) $\forall t > 0,\ N(x\ ,\ t) > 0$ *implies* $x = \underline{0}$.
Define $||x||_\alpha = \wedge\{t > 0 :\ N(x\ ,\ t) \geq \alpha\},\ \alpha \in (0\ ,\ 1)$.
Then $\{||\ ||_\alpha :\ \alpha \in (0\ ,\ 1)\}$ *is an ascending family of norms on U and they are called α-norms on U corresponding to the fuzzy norm N on U.*

Definition 2.3 ([3])**.** Let $(U\ ,\ N)$ be a fuzzy normed linear space. A subset A of U is said to be bounded iff $\exists\ t > 0$ and $0 < r < 1$ such that $N(x,\ t) > 1 - r\ \forall\ x \in A$.

Definition 2.4 ([4])**.** Let $(U\ ,\ N)$ be a fuzzy normed linear space. A subset F of U is said to be l-fuzzy closed if for each $\alpha \in (0\ ,\ 1)$ and for any sequence $\{x_n\}$ in F and $x \in U, (\underline{lim}_{n\to\infty} N(x_n - x\ ,\ t) \geq \alpha\ \forall t > 0) \Rightarrow x \in F$.

Proposition 2.5 ([4])**.** *Let $(U\ ,\ N)$ be a fuzzy normed linear space satisfying (N6) and $F \subset U$. Then F is l-fuzzy closed iff F is closed w.r.t. $||\ ||_\alpha$ $(\alpha - norm\ of\ N)$ for each $\alpha \in (0\ ,\ 1)$.*

Definition 2.6 ([5])**.** Let U and V be two linear spaces over the same field of scalars. Let N_1 and N_2 be two fuzzy norms on U and V respectively. A mapping $T : (U,\ N_1) \to (V,\ N_2)$ is said to be sectional fuzzy continuous at $x_0 \in U$, if $\exists\ \alpha \in (0,\ 1)$ such that for each $\epsilon > 0, \exists\ \delta > 0$ such that
$N_1(x - x_0,\ \delta) \geq \alpha \Rightarrow N_2(T(x) - T(x_0),\ \epsilon) \geq \alpha\ \ \forall\ x \in U$.
If T is sectional fuzzy continuous at each point of U, then T is said to be sectional fuzzy continuous on U.

Lemma 2.7 ([5])**.** *Let U and V be two linear spaces over the same field of scalars. Let N_1 and N_2 be two fuzzy norms on U and V respectively. A mapping T : $(U,\ N_1) \to (V,\ N_2)$ is sectional fuzzy continuous iff $T : (U,\ ||\ ||^1_\alpha) \to (V,\ ||\ ||^2_\alpha)$ is continuous for some $\alpha \in (0,\ 1)$.*

Definition 2.8 ([19]). Let V be a linear space over $F(R \text{ or } C)$. Define $\mu : V \times V \times F \to [0,\ 1]$ such that $\forall x,\ y,\ z \in V,\ t \in F$, the following conditions hold:

(CFI-1) $\mu(x,\ x,\ t) = 0\ \ \forall\, t$ having $Rl\ t < 0$ and $Im\ t = 0$.

(CFI-2) ($\mu(x,\ x,\ t) = 1\ \ \forall\, t$ having $Rl\ t > 0$ and $Im\ t = 0$) iff $x = \theta$.

(CFI-3) $\mu(x,\ y,\ t) = \mu(y,\ x,\ \bar{t})$.

(CFI-4) For any scalar k having $Im\ k = 0$ and $t \neq 0$,

$$\mu(kx,\ y,\ t) = \begin{cases} 1 - \mu(x,\ y,\ \frac{t}{k}) & \text{if } k \in R^-, \\ H(t) & \text{if } k = 0, \\ \mu(x,\ y,\ \frac{t}{k}) & \text{otherwise.} \end{cases}$$

Where $H : F \to [0,\ 1]$ defined by

$$H(t) = \begin{cases} 1 & \text{if } \ t \in R^+ \\ 0 & \text{otherwise.} \end{cases}$$

(CFI-5) (a) For $Rl\ t,\ Rl\ s > 0$ $\mu(x + y,\ z,\ Rl\ t + Rl\ s) \geq \mu(x,\ z,\ Rl\ t) \wedge \mu(y,\ z,\ Rl\ s)$.

(b) For $Im\, t,\ Im\ s > 0$ $\mu(x+y,\ z,\ Im\ t + Ims) \geq \mu(x,\ z,\ Im\ t) \wedge \mu(y,\ z,\ Ims)$.
where $Rl\ t$ is the real part of t and $Im\ t$ is the imaginary part of t.

(CFI-6) $\lim\limits_{Rl\ t \to \infty} \mu(x,\ y,\ t) = 1$.

Then μ is said to be a fuzzy inner product and $(V,\ \mu)$ is a fuzzy inner product space.

Theorem 2.9 ([19]). *Let $(V,\ \mu)$ be a fuzzy inner product space. Further assume that $\forall\ s,\ t \in R$ and $\forall\ x,\ y \in X$,*

(CFI-7) $\mu(x,\ y,\ st) \geq \mu(x,\ x,\ s^2) \wedge \mu(y,\ y,\ t^2)$.
Define a function $N : X \times R \to [0,\ 1]$ by

$$N(cx,\ t) = \begin{cases} \mu(|c|x,\ |c|x,\ t^2) & \text{if } \ t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then N is a fuzzy norm on X. We call this norm as induced norm of μ.

Theorem 2.10 ([19]). *Let $(V,\ \mu)$ be a fuzzy inner product space. Further assume that for $x,\ y \in V$,*

(CFI-8) $\mu(x,\ x,\ t) = 0\ \forall\ Im\ t \neq 0$ *and* $\{\forall\ t > 0 : \mu(x,\ x,\ t) > 0\} \Rightarrow x = \theta\}$.
Also assume that

(CFI-9) $\mu(ix,\ y,\ Rl\ t) = \mu(x,\ y,\ Im\frac{t}{i}),\ \mu(ix,\ y,\ Im\ t) = \mu(x,\ y,\ Rl\frac{t}{i})$.
Define for $\alpha \in (0,\ 1)$,
$\langle x,\ y\rangle_\alpha = \wedge\{t > 0 : \mu(x,\ y,\ t) \geq \alpha\} + \vee\{t < 0 : \mu(x,\ y,\ t) \leq 1 - \alpha\}$ *if* $t \in R$
$\langle x,\ y\rangle_\alpha = \{\wedge(Rl\ t > 0) + \vee(Rl\ t < 0) + i \wedge (Im\ t > 0) + i \vee (Im\ t < 0) : \mu(x,\ y,\ Rl\ t) \geq \alpha,\ \mu(x,\ y,\ Im\ t) \geq \alpha,\ \mu(x,\ y,\ t) \geq \alpha\}$ *otherwise.*
Then $\{\langle\ ,\ \rangle_\alpha : \alpha \in (0,\ 1)\}$ is a family of inner product in V. We call these inner products as α-inner products corresponding to the fuzzy inner product μ.

Definition 2.11 ([20]). Let $(X,\ N)$ be an l-fuzzy complete normed linear space and C be an l-fuzzy closed convex subset of X. Then $f : C \to X$ is said to be fuzzy demicompact if it has the property that whenever $\{x_n\}$ is a fuzzy bounded sequence in C and there is $\alpha \in (0,\ 1)$, for which for any $t > 0\ \exists\ M(t)$ such that

$N(f(x_n) - (x_n),\ t) \geq \alpha\ \forall\ n \geq M(t)$, then $\exists$ a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which is convergent.

Proposition 2.12 ([20])**.** *Let $(X,\ N)$ be an l-fuzzy complete normed linear space satisfying (N6). Let C be an l-fuzzy closed, convex subset of X and $f : C \to X$ be fuzzy demicompact. Then f is demicompact w.r.t. $||\ ||_\alpha\ \forall\ \alpha \in (0,\ 1)$ where $||\ ||_\alpha$ is an α-norm of N.*

Theorem 2.13 ([13])**.** *Let C be a fuzzy bounded, closed and convex set of a Hilbert space X and $f : C \to C$ be a strictly pseudocontractive mapping. Then for any $x_0 \in C$ and any $r \in (k,\ 1)$ the sequence $\{f^n_{s'}(x_0)\}$ where*
$s' = 1 - (1 - k)s$ where $s \in (0,\ 1)$ is fixed and
$f_{s'} = s'I + (1 - s')f_s, \quad f_s = sf + (1 - s)I$
converges weakly to a fixed point of f and if f is demicompact then the convergence is strong.

Theorem 2.14 ([13])**.** *Let C be a closed subset of a Hilbert space X. Suppose $f : C \to X$ be a continuous T-quasi nonexpansive mapping satisfying the following conditions:*

1. For every $x \in C - F(f)$, $\exists\ p_x \in F(f)$ and $k \in (0,\ 1)$ such that

$$||f(x) - p_x|| < ||x - p_x||.$$

2. $\exists\ x_0 \in C$ such that $\{f^n(x_0)\}$ are in C, $\forall\ n \geq 1$ and the sequence $\{f^n(x_0)\}$ contains a convergent subsequence converging to a point x' in C.
Then $x' \in F(f)$ and $\lim\limits_{n\to\infty} x_n \to x'$.

Theorem 2.15 ([13])**.** *Let X be a Hilbert space and C be a set in X. Suppose that $f : C \to C$ is a nonexpansive mapping and $F(f)$ contains an open set. Then Cauchy-Picard sequence $\{f^n(x)\}$, $x \in C$ converges to a point in C.*

Theorem 2.16 ([13])**.** *Let X be a Hilbert space and C be closed, convex and bounded set in X. Suppose that $f : C \to C$ is a nonexpansive mapping and $F(f) = \{x'\}$. Then for any x_0 in C, the sequence $\{f^n(x_0)\}$ is weakly convergent and converges weakly to x'.*

Theorem 2.17 ([13])**.** *Let X be a Hilbert space and C be closed, convex and bounded set in X. Suppose that $f : C \to C$ is a nonexpansive mapping. Then for any x_0 in C, the sequence $\{f^n(x_0)\}$ converges weakly to a fixed point of f.*

3. Some observations in fuzzy inner product spaces

In this section we get a fuzzy norm on the field (R or C) and using this norm we give the definition of α-fuzzy weakly convergent sequences.

Example 3.1. Let $(V,\ \langle\ ,\ \rangle)$ be an inner product space. Define $\mu : V \times V \times F \to [0,\ 1]$ by,
For $c = 0$, $\mu(cx,\ y,\ t) = H(t)$.
For $c \neq 0$,

$$\mu(cx,\ y,\ t) = \begin{cases} 1 & \text{if } Rl\ t > Rl\langle cx,\ y\rangle \text{ or } Rl\ t = Rl\langle cx,\ y\rangle \text{ when } Rl\ t < 0 \\ 0 & \text{if } Rl\ t < Rl\langle cx,\ y\rangle \text{ or } Rl\ t = Rl\langle cx,\ y\rangle \text{ when } Rl\ t \geq 0. \end{cases}$$

Then μ is a fuzzy inner product on V satisfying (CFI-7)

Proof. **(CFI-1)** Let $Rl\ t < 0$ and $Im\ t = 0$. Since $\langle x,\ x\rangle \geq 0$, $Rl\ t < Rl\langle x,\ x\rangle$. Thus $\mu(x,\ x,\ t) = 0$.

(CFI-2) $\mu(x,\ x,\ t) = 1\ \ \forall\, t$ having $Rl\, t > 0$ and $Im\, t = 0$. Then $Rl\, t > Rl\langle x,\ x\rangle$ or $Rl\ t = Rl\langle x,\ x\rangle$ when $Rl\ t < 0$. Since $Rl\ t > 0$, $Rl\ t > Rl\langle x,\ x\rangle\ \ \forall\ Rl\ t > 0$. Thus $Rl\langle x,\ x\rangle = 0$. So $x = \theta$.

Conversely let $x = \theta$. Then $\langle x,\ x\rangle = 0$. Thus $\forall\ t$ having $Rl\ t > 0$ and $Im\ t = 0$, $Rl\, t > Rl\langle x,\ x\rangle$. So $\mu(x,\ x,\ t) = 1$.

(CFI-3) Suppose $\mu(x,\ y,\ t) = 1$. Then

$$Rl\ t > Rl\langle x,\ y\rangle \text{ or } Rl\ t = Rl\langle x,\ y\rangle \text{ when } Rl\ t < 0$$

$$\Rightarrow Rl\ \bar{t} > Rl\overline{\langle x,\ y\rangle} \text{ or } Rl\ \bar{t} = Rl\overline{\langle x,\ y\rangle} \text{ when } Rl\ \bar{t} < 0$$

$$\Rightarrow Rl\ \bar{t} > Rl\langle y,\ x\rangle \text{ or } Rl\ \bar{t} = Rl\langle y,\ x\rangle \text{ when } Rl\ \bar{t} < 0.$$

Thus $\mu(y,\ x,\ \bar{t}) = 1$.

Similarly we have the result for $\mu(x,\ y,\ t) = \frac{1}{2}$ and $\mu(x,\ y,\ t) = 0$.
So $\mu(x,\ y,\ t) = \mu(y,\ x,\ \bar{t})$ in any case.

(CFI-4) For $c = 0$, by definition $\mu(cx,\ y,\ t) = H(t)$. Let $c \in R^-$ and $t \neq 0$. Then $\mu(cx,\ y,\ t) = 1$. Thus

$$Rl\ t > Rl\langle cx,\ y\rangle \text{ or } Rl\ t = Rl\langle cx,\ y\rangle \text{ when } Rl\ t < 0$$

$$\Rightarrow Rl\tfrac{t}{c} < Rl\langle x,\ y\rangle \text{ or } Rl\tfrac{t}{c} = Rl\langle x,\ y\rangle \text{ when } Rl\tfrac{t}{c} > 0.$$

So $\mu(x,\ y,\ \frac{t}{c}) = 0$. Hence $\mu(cx,\ y,\ t) = 1 = 1 - 0 = 1 - \mu(x,\ y,\ \frac{t}{c})$.

Now $\mu(cx,\ y,\ t) = 0$. Then

$$Rl\ t < Rl\langle cx,\ y\rangle \text{ or } Rl\ t = Rl\langle cx,\ y\rangle \text{ when } Rl\ t > 0$$

$$\Rightarrow Rl\tfrac{t}{c} > Rl\langle x,\ y\rangle \text{ or } Rl\tfrac{t}{c} = Rl\langle x,\ y\rangle \text{ when } Rl\tfrac{t}{c} < 0.$$

Thus $\mu(x,\ y,\ \frac{t}{c}) = 1$. So $\mu(cx,\ y,\ t) = 0 = 1 - 1 = 1 - \mu(x,\ y,\ \frac{t}{c})$. Hence $\mu(cx,\ y,\ t) = 1 - \mu(x,\ y,\ \frac{t}{c})$.

Now let $c > 0$. Then $\mu(cx,\ y,\ t) = 1$. Thus

$$Rl\ t > Rl\langle cx,\ y\rangle \text{ or } Rl\ t = Rl\langle cx,\ y\rangle \text{ when } Rl\ t < 0$$

$$\Rightarrow Rl\tfrac{t}{c} > Rl\langle x,\ y\rangle \text{ or } Rl\tfrac{t}{c} = Rl\langle x,\ y\rangle \text{ when } Rl\tfrac{t}{c} < 0.$$

So $\mu(x,\ y,\ \frac{t}{c}) = 1$. Hence $\mu(cx,\ y,\ t) = 1 = \mu(x,\ y,\ \frac{t}{c})$.

Similarly $\mu(cx,\ y,\ t) = 0$. Thus $\mu(x,\ y,\ \frac{t}{c}) = 0$. Therefore $\mu(cx,\ y,\ t) = \mu(x,\ y,\ \frac{t}{c})$.

(CFI-5) (a) If $\mu(x,\ z,\ Rl\ t) \wedge \mu(y,\ z,\ Rl\ s) = 0$, then there is nothing to prove. If $\mu(x,\ z,\ Rl\ t) \wedge \mu(y,\ z,\ Rl\ s) = 1$, then $\mu(x,\ z,\ Rl\ t) = \mu(y,\ z,\ Rl\ s) = 1$. Thus

$$Rl\ t > Rl\langle x,\ z\rangle \text{ or } Rl\ t = Rl\langle x,\ z\rangle \text{ when } Rl\ t < 0$$

and

$$Rl\ s > Rl\langle y,\ z\rangle \text{ or } Rl\ s = Rl\langle y,\ z\rangle \text{ when } Rl\ s < 0.$$

So

$$Rl\ t + Rl\ s > Rl\langle x,\ z\rangle + Rl\langle y,\ z\rangle = Rl\langle x+y,\ z\rangle$$

or

$$Rl\ t + Rl\ s = Rl\langle x+y,\ z\rangle \text{ when } Rl\ t + Rl\ s < 0.$$

Hence $\mu(x+y,\ z,\ Rl\ t + Rl\ s) = 1 \geq \mu(x,\ z,\ Rl\ t) \wedge \mu(y,\ z,\ Rl\ s)$.
The proof of the part (b) is similar to (a) as $Im\ t$ is a real number.

(CFI-6) Clearly $\lim_{Rl\ t\to\infty} \mu(x,\ y, t) = 1$.

(CFI-7) If $\mu(x,\ x,\ s^2) \wedge \mu(y,\ y,\ t^2) = 0$, then there is nothing to prove.
If $\mu(x,\ x,\ s^2) \wedge \mu(y,\ y,\ t^2) = 1$, then $\mu(x,\ x,\ s^2) = \mu(y,\ y,\ t^2) = 1$. Thus $s^2 > \langle x,\ x\rangle$ and $t^2 > \langle y,\ y\rangle$
So $s^2t^2 > \langle x,\ x\rangle\ \langle y,\ y\rangle = ||x||^2\ ||y||^2 \geq |\langle x,\ y\rangle|^2$ (by Cauchy-Swartz inequality).
Hence $st > |\langle x,\ y\rangle| \geq Rl\langle x,\ y\rangle$(since $s,\ t > 0$) and thus
$\mu(x,\ y,\ st) = 1 \geq \mu(x,\ x,\ s^2) \wedge \mu(y,\ y,\ t^2)$.
So μ is a fuzzy inner product on V satisfying (CFI-7).
Therefore, by Theorem 2.9, we get a norm function(Bag & Samanta type) as

$$N(cx,\ t) = \begin{cases} \mu(|c|x,\ |c|x,\ t^2) & \text{if } t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

□

Remark 3.2. Take $V = R$ or C.
Then (V, μ_2) is a fuzzy inner product space satisfying (CFI-7) where μ_2 is given in Example 3.1.
Thus induced norm N_2 of μ_2 is given by

$$N_2(cx,\ t) = \begin{cases} \mu(|c|x,\ |c|x,\ t^2) & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

which is equivalent to

$$N_2(x,\ t) = \begin{cases} 1 & \text{if } t > |x| \\ 0 & \text{if } t \leq |x|. \end{cases} \tag{3.2.1}$$

Clearly N_2 satisfies (N6).

Definition 3.3. Let $(X,\ N)$ be a fuzzy normed linear space satisfying (N6) and $\alpha \in (0,\ 1)$. A sequence $\{x_n\}$ in X is said to be α-fuzzy weakly convergent and converges to x if $\forall\ f \in X^*_\alpha$, where $f : X \to Y$,
$\lim_{n\to\infty} N_2(f(x_n) - f(x),\ t) > \alpha\ \ \forall\, t > 0$.
where $(Y,\ N_2)$ is a fuzzy normed linear space and $Y = R$ or C and N_2 is given by (3.2.1).

Proposition 3.4. *Let $(X,\ N_1)$ be a fuzzy normed linear space satisfying (N6) and $\alpha \in (0, 1)$. Then a sequence $\{x_n\}$ in X is α-fuzzy weakly convergent iff $\{x_n\}$ is weakly convergent w.r.t. $||\ ||_\alpha$ (α-norm of N_1).*

Proof. First we suppose that $\{x_n\}$ is α-fuzzy weakly convergent sequence in X.
Then $\forall\ f \in X^*_\alpha,\ \lim_{n\to\infty} N_2(f(x_n) - f(x),\ t) > \alpha\ \ \forall\, t > 0$.
$\Rightarrow$ for $\epsilon > 0,\ \ \exists\ N(\epsilon)$ such that $\lim_{n\to\infty} N_2(f(x_n) - f(x),\ \epsilon) > \alpha\ \ \forall\ n \geq N(\epsilon)$

$\Rightarrow |f(x_n) - f(x)| \leq \epsilon \quad \forall\, n \geq N(\epsilon)$
$\Rightarrow \lim\limits_{n\to\infty} |(f(x_n) - f(x)| = 0$
$\Rightarrow x_n \to x$ weakly w.r.t. $||\ ||_\alpha$.

Conversely suppose that $\{x_n\}$ is weakly convergent and converges to x w.r.t. $||\ ||_\alpha$. Then $\forall\, f \in X^*_\alpha,\ \ \lim\limits_{n\to\infty} |(f(x_n) - f(x)| = 0$
$\Rightarrow$ for $\epsilon > 0,\ \ \exists\, N(\epsilon)$ such that $|f(x_n) - f(x)| < \epsilon \quad \forall\, n \geq N(\epsilon)$
$\Rightarrow N_2(f(x_n) - f(x),\ \epsilon) = 1 \ \ \forall\, n \geq N(\epsilon)$
$\Rightarrow \lim\limits_{n\to\infty} N_2(f(x_n) - f(x),\ t) = 1 \ \ \forall\, t > 0$
$\Rightarrow \lim\limits_{n\to\infty} N_2(f(x_n) - f(x),\ t) > \alpha \ \ \forall\, t > 0.$
Thus $\{x_n\}$ is α-fuzzy weakly convergent. □

4. Some fixed point results in fuzzy inner product spaces

In this section we establish some fixed point results in fuzzy setting.

Theorem 4.1. *Let $(X,\ \mu)$ be an l-fuzzy complete inner product space satisfying (CFI-7), (CFI-8) and (CFI-9). Let $f : F \to F$ be a fuzzy non-expansive mapping where F is an l-fuzzy closed, convex and fuzzy bounded subset of X. Suppose that $F(f) = \{x'\}$. Then for any $x_0 \in F$, the sequence $\{f^n(x_0)\}$ is α-fuzzy weakly convergent and converges to x' for some $\alpha \in (0,\ 1)$.*

Proof. Since $(X,\ \mu)$ is l-fuzzy complete satisfying (CFI-7), (CFI-8) and (CFI-9), X is complete w. r. t. $||\ ||_\alpha$ for each $\alpha \in (0,\ 1)$ (induced α-norm of μ).
Again since F is l-fuzzy closed, so F is closed w.r.t. $||\ ||_\alpha$ for each $\alpha \in (0,\ 1)$ (by Proposition 2.5).
Now F is fuzzy bounded
$\Rightarrow\ \exists\, r$ and t such that $N(x,\ t) > 1 - r \ \ \forall\, x \in F$
$\Rightarrow ||x||_{1-r} \leq t \ \ \forall\, x \in F$
$\Rightarrow ||x||_{\alpha_0} \leq t$ (take $1 - r = \alpha_0$).
Therefore F is bounded w.r.t α_0-fuzzy norm and hence C is bounded in $(X \langle\ ,\ \rangle_{\alpha_0})$.
Also $f : F \to F$ is non-expansive w.r.t. $||\ ||_\alpha$ for each $\alpha \in (0,\ 1)$.
Thus the statement of the Theorem 2.16 is valid for $\langle\ ,\ \rangle_{\alpha_0}$.
So $\{f^n(x_0)\}$ is weakly convergent w.r.t. $||\ ||_{\alpha_0}$.
Hence by Proposition 3.4, it follows that the sequence $\{f^n(x_0)\}$ is α_0-fuzzy weakly convergent and converges to x'.
This completes the proof. □

Theorem 4.2. *Let $(X,\ \mu)$ be an l-fuzzy complete inner product space satisfying (CFI-7), (CFI-8) and (CFI-9). Let $f : F \to F$ be a fuzzy non-expansive mapping where F is an l-fuzzy closed, convex and fuzzy bounded subset of X. Then for any $x_0 \in F$, the sequence $\{f^n(x_0)\}$ is α-fuzzy weakly convergent and converges to a fixed point of f for some $\alpha \in (0,\ 1)$.*

Proof. Since $(X,\ \mu)$ is l-fuzzy complete satisfying (CFI-7), (CFI-8)and (CFI-9), X is complete w. r. t. $||\ ||_\alpha$ for each $\alpha \in (0,\ 1)$ (induced α-norm of μ_1).
Again since F is l-fuzzy closed, so F is closed w.r.t. $||\ ||_\alpha$ for each $\alpha \in (0,\ 1)$ (by Proposition 2.5).
Now F is fuzzy bounded

$\Rightarrow \ \exists\, r$ and t such that $N(x,\ t) > 1-r \ \ \forall\, x \in F$
$\Rightarrow ||x||_{1-r} \leq t \ \ \forall\, x \in F$
$\Rightarrow ||x||_{\alpha_0} \leq t$ (take $1-r = \alpha_0$).
Therefore F is bounded w.r.t α_0-fuzzy norm and hence C is bounded in $(X \left\langle\ ,\ \right\rangle_{\alpha_0})$.
Also $f : F \to F$ is non-expansive w.r.t. $||\ ||_\alpha$ for each $\alpha \in (0,\ 1)$.
Thus the statement of the Theorem 2.17 is valid for $\left\langle\ ,\ \right\rangle_{\alpha_0}$.
So $\{f^n(x_0)\}$ is weakly convergent w.r.t. $||\ ||_{\alpha_0}$.
Hence by Proposition 3.4, it follows that the sequence $\{f^n(x_0)\}$ is α_0-fuzzy weakly convergent and converges to a fixed point of f. □

Definition 4.3. Let $(X,\ N)$ be a fuzzy normed linear space and C be a convex, l-fuzzy closed and fuzzy bounded set in X and $\alpha \in (0,\ 1)$. The mapping $f : C \to C$ is called strictly fuzzy pseudocontractive mapping if there exists a constant $k \in (0,\ 1)$ such that $\forall\ x,\ y \in C$
$N(f(x) - f(y),\ t) \geq N(x-y,\ t) \bigwedge N((1-f)(x) - (1-f)(y),\ \frac{t}{\sqrt{k}})$.

Proposition 4.4. *Let X be a fuzzy normed linear space satisfying (N6) and C be a convex, l-fuzzy closed and fuzzy bounded set in X. If $f : C \to C$ is strictly fuzzy pseudocontractive then it is strictly pseudocontractive w.r.t. every α-norm, $\alpha \in (0,\ 1)$.*

Proof. Let $\alpha_0 \in (0,\ 1)$ and
$||x-y||^2_{\alpha_0} + k||(1-f)(x) - (1-f)(y)||^2_{\alpha_0} < t^2$
$\Rightarrow ||x-y||^2_{\alpha_0} < t^2$ and $k||(1-f)(x) - (1-f)(y)||^2_{\alpha_0} < t^2$
$\Rightarrow ||x-y||_{\alpha_0} < t$ and $||(1-f)(x) - (1-f)(y)||_{\alpha_0} < \frac{t}{\sqrt{k}}$
$\Rightarrow N(x-y,\ t) \geq \alpha_0$ and $N((1-f)(x) - (1-f)(y),\ \frac{t}{\sqrt{k}}) \geq \alpha_0$
$\Rightarrow N(x-y,\ t) \bigwedge N((1-f)(x) - (1-f)(y),\ \frac{t}{\sqrt{k}}) \geq \alpha_0$
$\Rightarrow N(f(x) - f(y),\ t) \geq \alpha_0$ (since f is strictly fuzzy pseudocontractive)
$\Rightarrow ||f(x) - f(y)||^2_{\alpha_0} \leq t^2$.
So $||f(x) - f(y)||^2_{\alpha_0} \leq ||x-y||^2_{\alpha_0} + k||(1-f)(x) - (1-f)(y)||^2_{\alpha_0}$.
Since $\alpha_0 \in (0,\ 1)$ is arbtrary thus
$||f(x) - f(y)||^2_{\alpha} \leq ||x-y||^2_{\alpha} + k||(1-f)(x) - (1-f)(y)||^2_{\alpha} \ \ \forall\ \alpha \in (0,\ 1)$. □

Theorem 4.5. *Let C be a fuzzy bounded, l-fuzzy closed and convex set of an l-fuzzy complete fuzzy inner product space $(H,\ \mu)$ satisfying (CFI-7), (CFI-8), (CFI-9) and $f : C \to C$ be a strictly fuzzy pseudocontractive mapping. Then for any $x_0 \in C$ and any $r \in (k,\ 1)$ the sequence $\{f^n_{s'}(x_0)\}$ where*
$s' = 1 - (1-k)s$ where $s \in (0,\ 1)$ is fixed and
$f_{s'} = s'I + (1-s')f_s, \quad f_s = sf + (1-s)I$
converges α-fuzzy weakly to a fixed point of f for some $\alpha \in (0,\ 1)$.
Moreover if f is fuzzy demicompact then the convergence is strong w.r.t. $||\ ||_\alpha$.

Proof. Since $(H,\ \mu)$ is l-fuzzy complete satisfying (CFI-7), (CFI-8)and (CFI-9), H is complete w. r. t. $||\ ||_\alpha$ for each $\alpha \in (0,\ 1)$ (induced α-norm of μ).
Since C is fuzzy bounded thus $\exists\ t > 0$ and $\alpha_0 \in (0,\ 1)$ such that $N(x,\ t) > \alpha_0 \ \ \forall\ x \in C$.
This implies that $||x||_{\alpha_0} \leq t \ \ \forall\ x \in C$.

So C is bounded w.r.t. some $|| \ ||_{\alpha_0}$.
Again since C is l-fuzzy closed, so C is closed w.r.t. every α-norm, $\alpha \in (0,\ 1)$.
Since $f : C \to C$ is strictly fuzzy pseudocontractive then it is strictly pseudocontractive w.r.t. every α-norm, $\alpha \in (0,\ 1)$ (by Proposition 4.4).
Thus the hypothesis of the Theorem 2.13 is valid w.r.t. $|| \ ||_{\alpha_0}$.
Thus for any $x_0 \in C$ and for any $r \in (k,\ 1)$ the sequence $\{f^n_{s'}(x_0)\}$ is weakly convergent w.r.t. $|| \ ||_{\alpha_0}$ to a fixed point of f.
Hence by Proposition 3.4, it follows that the sequence $\{f^n_{s'}(x_0)\}$ converges α_0-fuzzy weakly to a fixed point of f.
If f is fuzzy demicompact then by Proposition 2.12 it is demicompact w.r.t. every α-norm, $\alpha \in (0,\ 1)$.
So by Theorem 2.13 we have the convergence is strong w.r.t. $|| \ ||_{\alpha_0}$. □

Definition 4.6. Let $(X,\ N)$ be a complete fuzzy normed linear space and C be a convex, l-fuzzy closed, l-fuzzy bounded subset of X. A mapping $f : C \to C$ is said to be T-quasi non-expansive mapping if the set of fixed points $F(f)$ is nonempty and for any $x \in C$ and $p \in F(f)$,
$N(f(x) - p,\ t) \geq N(x - p,\ t) \ \ \forall\ t \in R.$

Proposition 4.7. *Let $(X,\ N)$ be a fuzzy normed linear space satisfying (N6) and C be a convex, l-fuzzy closed, l-fuzzy bounded subset of X. If a mapping $f : C \to C$ where $F(f)$ is nonempty is T-quasi non-expansive mapping then f is T-quasi non-expansive mapping w.r.t. each α-norm of N, where $\alpha \in (0,\ 1)$.*

Proof. Since C is l-fuzzy closed, l-fuzzy bounded subset of X so it is closed and bounded w.r.t. each α-norm of N, $\forall\ \alpha \in (0,\ 1)$.
Since $f : X \to X$ is fuzzy T-quasi nonexpansive, we have
$N(f(x) - p,\ t) \geq N(x - p,\ t) \ \ \forall\ t \in R, \ \ \forall\ x \in C$ and $p \in F(f)$.
Let $x \in C$, $p \in F(p)$, $s \in R$ and $\alpha \in (0,\ 1)$ such that
$s > ||x - p||_\alpha$
$\Rightarrow s > \wedge\{t > 0 : N(x - p,\ t) \geq \alpha\}$
$\Rightarrow N(x - p,\ s) \geq \alpha$
$\Rightarrow N(f(x) - p,\ s) \geq \alpha$
$\Rightarrow s \geq \wedge\{t > 0 : N(f(x) - p,\ t) \geq \alpha\}$
$\Rightarrow s \geq ||f(x) - p||_\alpha$.
So $||f(x) - p||_\alpha \leq ||x - p||_\alpha$
Since $\alpha \in (0,\ 1)$ is arbitrary, so $||f(x) - p||_\alpha \leq ||x - p||_\alpha \ \ \forall\ \alpha \in (0,\ 1)$.
Thus f is T-quasi non-expansive mapping w.r.t. each α-norm of N, $\forall\ \alpha \in (0,\ 1)$. □

Theorem 4.8. *Let C be an l-fuzzy closed subset of an l-fuzzy complete inner product space $(H,\ \mu)$ satisfying (CFI-7), (CFI-8)and (CFI-9). Suppose $f : C \to H$ be a sectional fuzzy continuous T-quasi nonexpansive mapping satisfying the following conditions:*

1. For every $x \in C - F(f)$, $\exists\ p_x \in F(f)$ and $k \in (0,\ 1)$ such that
$\mu(f(x) - p_x,\ f(x) - p_x,\ \frac{t^2}{k^2}) \geq \mu(x - p_x,\ x - p_x,\ t^2)$.

2. $\exists\ x_0 \in C$ such that $\{f^n(x_0)\}$ are in C, $\forall\ n \geq 1$ and the sequence $\{f^n(x_0)\}$ contains a convergent subsequence converging to a point x' in C.

Then $x' \in F(f)$ and $\exists\ \alpha_0 \in (0,\ 1)$ such that for any $t > 0$, $\exists\ M(t)$ such that $\mu(x_n - x',\ x_n - x',\ t) \geq \alpha_0\ \ \forall\ n \geq M(t)$.

Proof. Since C is l-fuzzy closed then it is closed w.r.t every α-norm, $\alpha \in (0,\ 1)$. $f : C \to H$ is sectional fuzzy continuous implies f is continuous for $||\ ||_{\alpha_0}$ for some $\alpha_0 \in (0,\ 1)$ (by Lemma 2.7).
Since $f : C \to H$ is fuzzy T-quasi non-expansive so f is T-quasi non-expansive w.r.t. each α-norm, $\alpha \in (0,\ 1)$ (by Proposition 4.7).
Now $\mu(f(x) - p_x,\ f(x) - p_x,\ \frac{t^2}{k^2}) \geq \mu(x - p_x,\ x - p_x,\ t^2)$
$\Rightarrow N(f(x) - p_x,\ \frac{t}{k}) \geq N(x - p_x,\ t)$.
For $\alpha \in (0,\ 1)$, let $||x - p_x||_\alpha < t$. Then
$N(x - p_x,\ t) \geq \alpha$
$\Rightarrow N(f(x) - p_x,\ \frac{t}{k}) \geq \alpha$
$\Rightarrow ||f(x) - p_x||_\alpha \leq \frac{t}{k}$
$\Rightarrow k||f(x) - p_x||_\alpha \leq t$.
Thus $k||f(x) - p_x||_\alpha \leq ||x - p_x||_\alpha\ \ \forall\ \alpha \in (0,\ 1)$.
Since $k \in (0,\ 1)$, $||f(x) - p_x||_\alpha < ||x - p_x||_\alpha\ \ \forall\ \alpha \in (0,\ 1)$.
Also the subsequence of $\{f^n(x_0)\}$ is convergent w.r.t. each α-norm, $\alpha \in (0,\ 1)$.
Thus the hypotheses of Theorem 2.14 holds for $||\ ||_{\alpha_0}$ for some $\alpha_0 \in (0,\ 1)$.
So $x' \in F(f)$ and $||x_n - x'||_{\alpha_0} \to 0$ as $n \to \infty$.
$\Rightarrow ||x_n - x'||^2_{\alpha_0} \to 0$ as $n \to \infty$
$\Rightarrow \langle x_n - x',\ x_n - x' \rangle_{\alpha_0} \to 0$ as $n \to \infty$.
Thus for each $t > 0$, $\exists\ M(t)$ such that
$\langle x_n - x',\ x_n - x' \rangle_{\alpha_0} < t\ \ \forall\ n \geq M(t)$
$\Rightarrow \mu(x_n - x',\ x_n - x',\ t) \geq \alpha_0\ \ \forall\ n \geq M(t)$.
This completes the proof. □

Definition 4.9. Let $(X,\ N)$ be an l-fuzzy complete fuzzy normed linear space. A sectional fuzzy continuous mapping f on a subset F of X is said to be fuzzy T-mapping if the set of fixed points $F(f)$ is nonempty and for any $x \in F$ and $p \in F(f)$, $N(f(x) - p,\ t) \geq N(x - p,\ t)\ \ \forall\ t \in R$.

Proposition 4.10. *Let $(X,\ N)$ be an l-fuzzy complete normed linear space satisfying (N6) and f be a sectional fuzzy continuous mapping on a subset F of X where $F(f)$ is nonempty. If f is fuzzy T-mapping, then f is T-mapping w.r.t. some α-norm of N, where $\alpha \in (0,\ 1)$.*

Proof. Since $(X,\ N)$ is a l-fuzzy complete fuzzy normed linear space satisfying (N6), so it is Banach space w.r.t $||\ ||_\alpha,\ \ \forall\ \alpha \in (0,\ 1)$. f is sectional fuzzy continuous implies f is continuous for $||\ ||_{\alpha_0}$ for some $\alpha_0 \in (0,\ 1)$.
Now let $x \in C$, $p \in F(p)$, $s \in R$ and $\alpha \in (0,\ 1)$ such that
$s > ||x - p||_{\alpha_0}$
$\Rightarrow s > \wedge\{t > 0 : N(x - p,\ t) \geq \alpha_0\}$
$\Rightarrow N(x - p,\ s) \geq \alpha_0$
$\Rightarrow N(f(x) - p,\ s) \geq \alpha_0$
$\Rightarrow s \geq \wedge\{t > 0 : N(f(x) - p,\ t) \geq \alpha_0\}$
$\Rightarrow s \geq ||f(x) - p||_{\alpha_0}$.
So $||f(x) - p||_{\alpha_0} \leq ||x - p||_{\alpha_0}$

Thus f is continuous mapping mapping on a set of a Banach space X w.r.t. α_0-norm where $F(f)$ is nonempty and $||f(x) - p||_{\alpha_0} \leq ||x - p||_{\alpha_0}$.
Therefore f is T-mapping w.r.t. α_0-norm of N, where $\alpha_0 \in (0,\ 1)$. $\square$

Theorem 4.11. *Let $(X,\ \mathfrak{F})$ be an l-fuzzy complete real inner product space satisfying (CFI-7), (CFI-8) and C be a subset in H. Suppose that $f : C \to C$ is a fuzzy nonexpansive mapping and $X - F(f)$ is l-fuzzy closed set. Then for $\alpha \in (0,\ 1)$, $\exists\ x_\alpha \in C$ such that Cauchy-Picard sequence $\{f^n(x)\},\ x \in C$ satisfy the condition that for any $t > 0,\ \exists\ M(t,\ \alpha)$ such that $\mathfrak{F}(f^n(x) - x_\alpha,\ f^n(x) - x_\alpha,\ t) > \alpha\ \forall\ n \geq M(t,\ \alpha)$.*

Proof. Since $(X,\ \mathfrak{F})$ is an l-fuzzy complete real inner product space satisfying (CFI-7) and (CFI-8), it is real Hilbert space w.r.t $\langle\ ,\ \rangle_\alpha,\ \ \forall\ \alpha \in (0,\ 1)$. Since f is fuzzy nonexpansive mapping so it is nonexpansive mapping w.r.t.$\langle\ ,\ \rangle_\alpha$-inner product, $\forall\ \alpha \in (0,\ 1)$.
Now $X - F(f)$ is l-fuzzy closed, so it is closed w.r.t. α-inner product $\forall\ \alpha \in (0,\ 1)$. Therefore $F(f)$ is an open set w.r.t. α- inner product $\forall\ \alpha \in (0,\ 1)$.
So f satisfy all the conditions of Theorem 2.15, and hence for $\alpha \in (0,\ 1)$, $\exists\ x_\alpha \in C$ such that Cauchy-Picard sequence $\{f^n(x)\},\ x \in C$ converges to x_α w.r.t. α-inner product. So for any $t > 0,\ \exists\ M(t,\ \alpha)$ such that $\mathfrak{F}(f^n(x) - x_\alpha,\ f^n(x) - x_\alpha,\ t) > \alpha\ \forall\ n \geq M(t,\ \alpha)$. $\square$

Theorem 4.12. *Let $(X,\ \mathfrak{F})$ be an l-fuzzy complete real inner product space satisfying (CFI-7), (CFI-8) and C be a subset in H. Suppose that $f : C \to C$ is a T-mapping and $X - F(f)$ is l-fuzzy closed set. Then there is $x' \in C$ and $\alpha_0 \in (0,\ 1)$ such that Cauchy-Picard sequence $\{f^n(x)\},\ x \in C$ satisfy the condition that for any $t > 0,\ \exists\ M(t)$ such that $\mu(f^n(x) - x',\ f^n(x) - x',\ t) > \alpha_0\ \ \forall\ n \geq M(t)$.*

Proof. Since $(X,\ N)$ is an l-fuzzy complete fuzzy real inner product space satisfying (CFI-7), (CFI-8), it is real Hilbert space w.r.t $\langle\ ,\ \rangle_\alpha,\ \ \forall\ \alpha \in (0,\ 1)$. Since f is T-mapping so it is T-mapping w.r.t. some real α_0-inner product, $\alpha_0 \in (0,\ 1)$.
Now $X - F(f)$ is l-fuzzy closed, so it is closed w.r.t. α_0 inner product. Therefore $F(f)$ is an open set w.r.t. α_0 inner product.
So f satisfy all the conditions of Theorem 2.15, and hence Cauchy-Picard sequence $\{f^n(x)\},\ x \in C$ converges to a point (say x') in C w.r.t. α_0 inner product. So for any $t > 0\ \exists\ M(t)$ such that $\mathfrak{F}(x_n - x',\ x_n - x',\ t) \geq \alpha_0\ \ \forall\ n \geq M(t)$. $\square$

5. Conclusion.

Very few work have been done in the fuzzy inner product spaces on complex field. In [19], we try to give an idea of fuzzy inner product spaces on complex field and establish a decomposition theorem from a fuzzy inner product into a family of crisp inner products. In this paper some observations are made in the fuzzy inner product space (complex) and some fixed point theorems are established. We hope that researchers will be benefitted through our work. Applications of fuzzy inner product spaces and its fixed points may be studied as a future scope of research.

Acknowledgements. The authors are grateful to the referees for their valuable suggestions in rewriting the paper in the present form. The authors are also thankful to the Editor-in-Chief of the journal (AFMI) for his valuable comments which helped me to revise the paper.

The present work is partially supported by Special Assistance Programme (SAP) of UGC, New Delhi, India [Grant No. F. 510/4/DRS/2009 (SAP-I)].

References

[1] Cihangir Alaca and Hakan Efe, On intuitionistic fuzzy Banach spaces, International Journal of Pure and Applied Mathematics 32 (3) (2006) 345–361.

[2] C. Alsina, B. Schweizer, C. Sempi and A. Sklar, On the definition of a probabilistic inner product space, Rendiconti di Matematica, Serie VII 17 Roma(1997) 115–127.

[3] T. Bag and S. K. Samanta, Finite dimensional fuzzy normed linear spaces, J. Fuzzy Math. 11 (3) (2003) 687–705.

[4] T. Bag and S. K. Samanta, Fixed point theorems on fuzzy normed linear spaces, Inform. Sci. 176 (2006) 2910–2931.

[5] T. Bag and S. K. Samanta, Some fixed point theorems in fuzzy normed linear spaces, Inform. Sci. 177 (2007) 3271–3289.

[6] Thangaraj Beula and R. Angeline Sarguna Gifta, Some aspects of 2-fuzzy inner product spaces, Ann. Fuzzy Math. Inform. 4 (2) (2012) 335–342.

[7] R. Biswas, Fuzzy innerproduct spaces and fuzzy norm function, Inform. Sci. 53 (1991) 185–190

[8] Binayak S. Choudhury and Pradyut Das, A new contraction mapping principle in partially ordered fuzzy metric spaces, Annals of Fuzzy Mathematics and Informatics, 8(6) (2014) 889-901.

[9] Krishnapada Das, Binayak S. Choudhury and Pritha Bhattacharya, A common fixed point theorem for cyclic contractive mappings in fuzzy metric spaces, Ann. Fuzzy Math. Inform. 9 (4) (2015) 581–592.

[10] A. M. El-Abyad and H. M. El-Hamouly, Fuzzy inner product spaces, Fuzzy Sets and Systems 44 (1991) 309–326.

[11] M. Goudarzi and S. M. Vaezpour, On the definition of fuzzy Hilbert spaces and its application, The Journal of Nonlinear Science and Applications 2 (1) (2009) 46–59.

[12] A. Hasankhani, A. Nazari and M. Saheli, Some properties of fuzzy Hilbert spaces and norm of operators, Iranian Journal of Fuzzy Systems 7 (3) (2010) 129–157.

[13] Vasile I. Istratescu, Fixed point theory, D. Reidel Publishing Company (Edn 1981).

[14] J. K. Kohli and R. Kumar, On fuzzy inner product spaces and fuzzy co-inner product spaces, Fuzzy Sets and Systems 53 (2) (1993) 227–232.

[15] Pinaki Majumdar and S. K. Samanta, On fuzzy inner product spaces, J. Fuzzy Math. 16 (2) (2008) 377–392.

[16] Saurav Manro, S. S. Bhatia and Sanjay Kumar, Common fixed point theorems in fuzzy metric spaces, Ann. Fuzzy Math. Inform. 3 (1) (2012) 151–158.

[17] S. Mukherjee and T. Bag, Some properties of fuzzy Hilbert spaces and fixed point theorems in such spaces, J. Fuzzy Math. 20 (3) (2012) 539–550.

[18] S. Mukherjee and T. Bag, Fuzzy real inner product space and its properties, Ann. Fuzzy Math. Inform. 6 (2) (2013) 377–389.

[19] S. Mukherjee and T. Bag, Fuzzy inner product space and its properties, International Journal of Fuzzy Mathematics and Systems 5 (1) (2015) 57–69.

[20] S. Mukherjee and T. Bag, Some Fixed Point Results in Fuzzy Inner Product Spaces, International Journal of Mathematics and Scientific Computing 5 (1) (2015) 44–48.

[21] T. V. Ramakrishnan, Fuzzy semi-inner product of fuzzy points, Fuzzy Sets and Systems 89 (1997) 249–256.

[22] G. Rano, T. Bag and S. K. Samata, Asymptotically regular mapping and fixed point theorems in generating spaces of semi-norm family, Ann. Fuzzy Math. Inform. 8 (6) (2014) 977–986.

[23] S. Vijayabalaji, Fuzzy strong n-inner product space, International Journal of Applied Mathematics 1 (2) (2010) 176–185.

SOUGAT MUKHERJEE (mukherjee.sougat@gmail.com)
Department of Mathematics, Visva-Bharati, Santiniketan-731235, West Bengal, India

TARAPADA BAG (tarapadavb@gmail.com)
Department of Mathematics, Visva-Bharati, Santiniketan-731235, West Bengal, India