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# Monoids of intuitionistic fuzzy matrices

S. Sriram , J. Boobalan

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Abstract. In this paper, we study the algebraic sum and algebraic product of intuitionistic fuzzy matrices and prove that the set of all intuitionistic fuzzy matrices forms a commutative monoid. We prove that the DeMorgan's laws of intuitionistic fuzzy matrices and we also prove that the distributive laws of intuitionistic fuzzy matrices are satisfied.

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Corresponding Author: J. Boobalan (jboobalan@hotmail.com)

## 1. Introduction

Atanassov[1] introduced the concept of intuitionistic fuzzy set(IFS) which was the generalization of fuzzy set introduced by Zadeh[21]. Since its appearance ,IFS has been investigated by many researchers and applied it to many fields, such as Decision making, Clustering analysis etc., Using the idea of fuzzy sets, Kim and Roush[9] studied fuzzy matrices as a generalization of matrices over the two element Boolean algebra. Meenakshi[11] studied the theoretical developments fuzzy matrices. Using the theory of IFS, Im et.al[8] defined the notion of intuitionistic fuzzy matrix(IFM) as a generalization of fuzzy matrix. IFM is very useful in the discussion of intuitionistic fuzzy relation(IFR)[2, 4, 5, 6, 7]. Xu and Yager[20] defined intuitionistic fuzzy values(IFV). In a matrix if all the elements are IFVs then it is called an IFM[22]. Lee and Jeong[10] obtained a canonical form of the transitive IFM. Sriram and Murugadas[16] proved the set of all IFMs form a semiring with respect to Max-min composition of IFMs. They also investigated the Moore-Penrose inverse of IFM[17]. Xu[18] defined the Intuitionistic fuzzy similarity matrix and utilized it in clustering analysis. Mondal and Pal[12] studied the similarity relations,invertibility and eigenvalues of IFM. Murugadas and Lalitha[14] obtained a decomposition of rectangular IFM. In this paper, the systematically studied algebraic operations in[6] related to IFSs are extended to IFMs and studied its algebraic properties.

## 2. Preliminaries

In this section, we refer to some basic definitions of intuitionistic fuzzy matrix that are necessary for this paper.

**Definition 2.1** ([12, 15]). An intuitionistic fuzzy matrix(IFM) is a matrix of pairs $A = (\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle)$of a non negative real numbers satisfying $\mu_{a_{ij}} + \nu_{a_{ij}} \leq 1$ for all $i, j$.

**Definition 2.2** ([16]). Let $A$ and $B$ are two intuitionistic fuzzy matrices, such that $A = (\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle)$, $B = (\langle \mu_{b_{ij}}, \nu_{b_{ij}} \rangle)$. Then
$A \vee B = (\langle \max(\mu_{a_{ij}}, \mu_{b_{ij}}), \min(\nu_{a_{ij}}, \nu_{b_{ij}}) \rangle)$,
$A \wedge B = (\langle \min(\mu_{a_{ij}}, \mu_{b_{ij}}), \max(\nu_{a_{ij}}, \nu_{b_{ij}}) \rangle)$.

**Definition 2.3** ([16]). Let $A$ and $B$ be two IFMs such that$A = (\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle)$, $B = (\langle \mu_{b_{ij}}, \nu_{b_{ij}} \rangle)$. Then we write $A \leq B$ if $\mu_{a_{ij}} \leq \mu_{b_{ij}}$ and $\nu_{a_{ij}} \geq \nu_{b_{ij}}$ for all $i, j$.

**Definition 2.4** ([16]). The $m \times n$ zero IFM $O$ is an IFM all of whose entries are $\langle 0, 1 \rangle$.
The $m \times n$ universal IFM J is an IFM all of whose entries are $\langle 1, 0 \rangle$.

**Definition 2.5** ([19]). Let $Z = (z_{ij})_{n \times n}$ be a matrix, if all of its elements $z_{ij}(i, j = 1, 2, ..., n)$ are intuitionistic fuzzy values, then $Z$ is called an IFM.

**Definition 2.6** ([13, 19]). Let $Z_1 = (\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle)$ and $Z_2 = (\langle \mu_{b_{ij}}, \nu_{b_{ij}} \rangle)$ be two intuitionistic fuzzy matrices of order $n$. If $Z = Z_1 \circ Z_2$, then $Z$ is called the composition matrix of $Z_1$ and $Z_2$, where

$$Z = (\langle \bigvee_{k=1}^{n} (\mu_{a_{ik}} \wedge \mu_{b_{kj}}), \bigwedge_{k=1}^{n} (\nu_{a_{ik}} \vee \nu_{b_{kj}}) \rangle).$$

**Definition 2.7** ([15]). Let $A = (\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle)$ and $B = (\langle \mu_{b_{ij}}, \nu_{b_{ij}} \rangle)$ be two intuitionistic fuzzy matrices. Then

$$A \oplus B = (\mu_{a_{ij}} + \mu_{b_{ij}} - \mu_{a_{ij}}.\mu_{b_{ij}}, \nu_{a_{ij}}.\nu_{b_{ij}})$$

is called the algebraic sum of $A$ and $B$ and

$$A \otimes B = (\mu_{a_{ij}}\mu_{b_{ij}}, \nu_{a_{ij}} + \nu_{b_{ij}} - \nu_{a_{ij}}\nu_{b_{ij}})$$

is called the algebraic product of $A$ and $B$.

**Definition 2.8** ([3]). The complement of an intuitionistic fuzzy matrix $A$ which is denoted by $A^c$ and is defined by $A^c = (\langle \nu_{a_{ij}}, \mu_{a_{ij}} \rangle)$.

## 3. Major section

In this section, we prove some algebraic properties of arithmetic sum and arithmetic product.
The Proof of the following Theorems are obvious.

**Theorem 3.1.** *Let $A$ and $B$ are IFMs of same order. Then*
*(i)* $A \oplus B = B \oplus A$,
*(ii)* $A \otimes B = B \otimes A$.

**Theorem 3.2.** *Let $A, B$ be IFMs of same order. Then*
*(i)* $(A \oplus B)^c = A^c \otimes B^c$,
*(ii)* $(A \otimes B)^c = A^c \oplus B^c$.

**Theorem 3.3.** *Let $A$ be an IFM and let $O = (\langle 0,1\rangle)$ be the identity IFM with respect to $\oplus$ and $J = (\langle 1,0\rangle)$ be the identity IFM with respect to $\otimes$. Then*
*(i)* $A \oplus O = O \oplus A = A$,
*(ii)* $A \otimes J = J \otimes A = A$.

*Proof.* (i) It is obvious.
(ii)
$$\begin{aligned} A \otimes J &= \langle(\mu_{a_{ij}}, \nu_{a_{ij}}) \otimes (1,0)\rangle \\ &= \langle \mu_{a_{ij}}.1, \nu_{a_{ij}} + 0 - \nu_{a_{ij}}.0\rangle \\ &= \langle \mu_{a_{ij}}, \nu_{a_{ij}}\rangle \\ &= A. \end{aligned}$$
Similarly, we can prove $J \otimes A = A$. □

**Theorem 3.4.** *Let $A, B$ and $C$ are IFMs of same order. Then*
*(i)* $(A \oplus B) \oplus C = A \oplus (B \oplus C)$,
*(ii)* $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.

*Proof.* (i)
$$\begin{aligned} \text{LHS} &= (A \oplus B) \oplus C \\ &= \langle(\mu_{a_{ij}} + \mu_{b_{ij}} - \mu_{a_{ij}}.\mu_{b_{ij}}, \nu_{a_{ij}}.\nu_{b_{ij}}) \oplus (\mu_{c_{ij}}, \nu_{c_{ij}})\rangle \\ &= \langle(\mu_{a_{ij}} + \mu_{b_{ij}} - \mu_{a_{ij}}.\mu_{b_{ij}}) + \mu_{c_{ij}} - (\mu_{a_{ij}} + \mu_{b_{ij}} - \mu_{a_{ij}}.\mu_{b_{ij}}).\mu_{c_{ij}} \\ &\qquad , \nu_{a_{ij}}.\nu_{b_{ij}}.\nu_{c_{ij}}\rangle \\ &= \langle \mu_{a_{ij}} + \mu_{b_{ij}} + \mu_{c_{ij}} - \mu_{a_{ij}}.\mu_{b_{ij}} - \mu_{a_{ij}}.\mu_{c_{ij}} - \mu_{b_{ij}}.\mu_{c_{ij}} \\ &\qquad + \mu_{a_{ij}}.\mu_{b_{ij}}.\mu_{c_{ij}}, \nu_{a_{ij}}.\nu_{b_{ij}}.\nu_{c_{ij}}\rangle, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \text{RHS} &= A \oplus (B \oplus C) \\ &= \langle(\mu_{a_{ij}}, \nu_{a_{ij}}) \oplus (\mu_{b_{ij}} + \mu_{c_{ij}} - \mu_{b_{ij}}.\mu_{c_{ij}}, \nu_{a_{ij}}.\nu_{b_{ij}})\rangle \\ &= \langle \mu_{a_{ij}} + (\mu_{b_{ij}} + \mu_{c_{ij}} - \mu_{b_{ij}}.\mu_{c_{ij}}) - \mu_{a_{ij}}.(\mu_{b_{ij}} + \mu_{c_{ij}} - \mu_{b_{ij}}.\mu_{c_{ij}}), \nu_{a_{ij}}.\nu_{b_{ij}}.\nu_{c_{ij}}\rangle \\ &= \langle \mu_{a_{ij}} + \mu_{b_{ij}} + \mu_{c_{ij}} - \mu_{a_{ij}}.\mu_{b_{ij}} - \mu_{a_{ij}}.\mu_{c_{ij}}, -\mu_{b_{ij}}.\mu_{c_{ij}} + \mu_{a_{ij}}.\mu_{b_{ij}}.\mu_{c_{ij}} \\ &\qquad , \nu_{a_{ij}}.\nu_{b_{ij}}.\nu_{c_{ij}}\rangle. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), (i) follows.
The proof of (ii) is similar to (i). □

**Theorem 3.5.** *Let $A, B$ and $C$ are IFMs of same order. Then*
*(i)* $(A \vee B) \oplus C = (A \oplus C) \vee (B \oplus C)$,
*(ii)* $(A \wedge B) \oplus C = (A \oplus C) \wedge (B \oplus C)$.

*Proof.* (i)
$$\begin{aligned} \text{LHS} &= (A \vee B) \oplus C \\ &= \langle(\max(\mu_{a_{ij}}, \mu_{b_{ij}}), \min(\nu_{a_{ij}}, \nu_{b_{ij}})) \oplus (\mu_{c_{ij}}, \nu_{c_{ij}})\rangle \\ &= \langle \max(\mu_{a_{ij}}, \mu_{b_{ij}}) + \mu_{c_{ij}} - \max(\mu_{a_{ij}}, \mu_{b_{ij}})\mu_{c_{ij}}, \min(\nu_{a_{ij}} \\ &\qquad , \nu_{b_{ij}})\nu_{c_{ij}}\rangle, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \text{RHS} &= (A \oplus C) \vee (B \oplus C) \\ &= \langle(\mu_{a_{ij}} + \mu_{c_{ij}} - \mu_{a_{ij}}.\mu_{c_{ij}}, \nu_{a_{ij}}.\nu_{c_{ij}}) \vee (\mu_{b_{ij}} + \mu_{c_{ij}} - \mu_{b_{ij}}.\mu_{c_{ij}}, \nu_{b_{ij}}.\nu_{c_{ij}})\rangle \\ &= \langle \max(\mu_{a_{ij}} + \mu_{c_{ij}} - \mu_{a_{ij}}.\mu_{c_{ij}}, \mu_{b_{ij}} + \mu_{c_{ij}} - \mu_{b_{ij}}.\mu_{c_{ij}}), \min(\nu_{a_{ij}}.\nu_{c_{ij}}, \nu_{b_{ij}}.\nu_{c_{ij}})\rangle \\ &= \langle \max(\mu_{a_{ij}}(1 - \mu_{c_{ij}}) + \mu_{c_{ij}}, \mu_{b_{ij}}(1 - \mu_{c_{ij}}) + \mu_{c_{ij}}), \min(\nu_{a_{ij}}, \nu_{b_{ij}})\nu_{c_{ij}}\rangle \\ &= \langle \max(\mu_{a_{ij}}(1 - \mu_{c_{ij}}), \mu_{b_{ij}}(1 - \mu_{c_{ij}})) + \mu_{c_{ij}}, \min(\nu_{a_{ij}}, \nu_{b_{ij}})\nu_{c_{ij}}\rangle \\ &= \langle \max(\mu_{a_{ij}}, \mu_{b_{ij}})(1 - \mu_{c_{ij}}) + \mu_{c_{ij}}, \min(\nu_{a_{ij}}, \nu_{b_{ij}})\nu_{c_{ij}}\rangle \\ &= \langle \max(\mu_{a_{ij}}, \mu_{b_{ij}}) - \max(\mu_{a_{ij}}, \mu_{b_{ij}})\mu_{c_{ij}} + \mu_{c_{ij}}, \min(\nu_{a_{ij}}, \nu_{b_{ij}})\nu_{c_{ij}}\rangle \\ &= \langle \max(\mu_{a_{ij}}, \mu_{b_{ij}}) + \mu_{c_{ij}} - \max(\mu_{a_{ij}}, \mu_{b_{ij}})\mu_{c_{ij}}, \min(\nu_{a_{ij}}, \nu_{b_{ij}})\nu_{c_{ij}}\rangle. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), (i) follows.

The proof of (ii) is similar to (i). □

The proof of the following Theorem on Distributivity is obvious.

**Theorem 3.6.** *Let $A, B$ and $C$ are IFMs of same order. Then*
*(i)* $(A \vee B) \otimes C = (A \otimes C) \vee (B \otimes C)$,
*(ii)* $(A \wedge B) \otimes C = (A \otimes C) \wedge (B \otimes C)$.

**Theorem 3.7.** *Let $A, B$ and $C$ are IFMs of same order. Then*
*(i)* $A \oplus (B \vee C) = (A \oplus B) \vee (A \oplus C)$,
*(ii)* $A \oplus (B \wedge C) = (A \oplus B) \wedge (A \oplus C)$.

*Proof.* (i) $A \oplus (B \vee C) = \langle \mu_{a_{ij}} \mu_{b_{ij}} \rangle \oplus \langle \max(\mu_{b_{ij}}, \mu_{c_{ij}}), \min(\nu_{b_{ij}}, \nu_{c_{ij}}) \rangle$
$= \langle \mu_{a_{ij}} + \max(\mu_{b_{ij}}, \mu_{c_{ij}}) - \mu_{a_{ij}} \max(\mu_{b_{ij}}, \mu_{c_{ij}}), \nu_{a_{ij}} \min(\nu_{b_{ij}}, \nu_{c_{ij}}) \rangle$.

If $\mu_{b_{ij}} > \mu_{c_{ij}}$, then
$A \oplus (B \vee C) = \langle \mu_{a_{ij}} + \mu_{b_{ij}} - \mu_{a_{ij}} \mu_{b_{ij}}, \nu_{a_{ij}} \min(\nu_{b_{ij}}, \nu_{c_{ij}}) \rangle$.
Also, $(A \oplus B) \vee (A \oplus C) = \langle \mu_{a_{ij}} + \mu_{b_{ij}} - \mu_{a_{ij}} \mu_{b_{ij}}, \nu_{a_{ij}} \min(\nu_{b_{ij}}, \nu_{c_{ij}}) \rangle$.
Since, $\mu_{b_{ij}} > \mu_{c_{ij}}$ then $\mu_{b_{ij}}(1 - \mu_{a_{ij}}) > \mu_{c_{ij}}(1 - \mu_{a_{ij}})$,
i.e, $\mu_{a_{ij}} + \mu_{b_{ij}} - \mu_{a_{ij}} \mu_{b_{ij}} > \mu_{a_{ij}} + \mu_{c_{ij}} - \mu_{a_{ij}} \mu_{c_{ij}}$.
Similarly, if $\mu_{b_{ij}} \leq \mu_{c_{ij}}$ then
$A \oplus (B \vee C) = (A \oplus B) \vee (A \oplus C)$.
The proof of (ii) is similar to (i). □

The proof of the following Theorem is obvious.

**Theorem 3.8.** *Let $A, B$ and $C$ are IFMs of same order. Then*
*(i)* $A \otimes (B \vee C) = (A \otimes B) \vee (A \otimes C)$.
*(ii)* $A \otimes (B \wedge C) = (A \otimes B) \wedge (A \otimes C)$.

**Theorem 3.9.** *Let $A$ and $B$ be two IFMs with $A \leq B$. Then*
*(i)* $(A \wedge B) \oplus (A \vee B) = A \oplus B$.
*(ii)* $(A \wedge B) \otimes (A \vee B) = A \otimes B$.

*Proof.* (i) LHS$= (A \wedge B) \oplus (A \vee B)$
$= \langle (\min(\mu_{a_{ij}}, \mu_{b_{ij}}), \max(\nu_{a_{ij}}, \nu_{b_{ij}})) + (\max(\mu_{a_{ij}}, \mu_{b_{ij}}), \min(\nu_{a_{ij}}, \nu_{b_{ij}})) \rangle$
$= \langle \min(\mu_{a_{ij}}, \mu_{b_{ij}}) + \max(\mu_{a_{ij}}, \mu_{b_{ij}}) - \min(\mu_{a_{ij}}, \mu_{b_{ij}}) \max(\mu_{a_{ij}}, \mu_{a_{ij}}), \max(\nu_{a_{ij}}, \nu_{b_{ij}}) \min(\nu_{a_{ij}}, \nu_{b_{ij}}) \rangle$
$= \langle \mu_{a_{ij}} + \mu_{b_{ij}} - \mu_{a_{ij}}.\mu_{b_{ij}}, \nu_{a_{ij}}.\nu_{b_{ij}} \rangle$
$= A \oplus B$.
$= RHS$.
Hence, $(A \wedge B) \oplus (A \vee B) = A \oplus B$.
The proof of (ii) is similar to (i). □

## 4. Conclusions

The set of all IFMs with respect to the algebraic sum and algebraic product form a commutative monoid. The arithmetic sum and arithmetic product of IFMs are satisfy the De Morgan's laws. Distributive laws (i) joint over arithmetic sum and arithmetic product (ii) meet over arithmetic sum and arithmetic product are proved.

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S. SRIRAM (ssm_3096@yahoo.co.in)
Mathematics Section, Faculty of Engineering & Technology, Annamalai University, Annamalainagar - 608 002, India

J. BOOBALAN (jboobalan@hotmail.com)
Mathematics Wing, Directorate of Distance Education, Annamalai University,

Annamalainagar - 608 002, India