

## Soft residuated lattice based on intuitionistic fuzzy sets

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**ABSTRACT.** In this paper, the concept of intuitionistic fuzzy soft sets is applied to residuated lattices. The notion of intuitionistic fuzzy soft filters of a residuated lattice is introduced and some related properties are investigated. Then the representations of generated intuitionistic fuzzy soft filters are established. We prove that the lattice of all intuitionistic fuzzy soft filters of a residuated lattice is a complete lattice. We determine the compact elements of this lattice and show that it is a distributive algebraic lattice.

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### 1. INTRODUCTION

**M.** Ward and R.P. Dilworth [22] introduced the concept of residuated lattices as generalization of ideal lattices of rings. These algebras are a common structure among algebras associated with logical systems. The study of residuated lattices is motivated by their occurrence both in universal algebra and algebraic logic. The residuated lattices have interesting algebraic and logical properties. The properties of these structures were presented in [5, 19, 20, 21].

The filter theory plays an important role in studying these logical algebras and many authors discussed the notion of filters of logical algebras. From a logical point of view, a filter corresponds to a set of provable formulas. Many fields deal daily with the uncertain data that may not be successfully modeled by the classical mathematics. There are some mathematical tools for dealing with uncertainties; three of them are fuzzy set theory, developed by Zadeh (1965), intuitionistic fuzzy sets introduced by Atanassov in (1986) and soft set theory, introduced by Molodtsov (1999), that are related to this work.

Zadeh introduced the concept of the fuzzy set in 1965 [23]. Since then, the fuzzy sets theory have been developed in a wide variety of fields such as fuzzy mathematics and many applications has been founded in the domain of mathematics and elsewhere. In [24] the fuzzy filters in residuated lattices are studied.

After the introduction of the concept of fuzzy sets by Zadeh, several researches were conducted on the generalizations of the notion of fuzzy sets. The concept of intuitionistic fuzzy sets was first introduced by Atanassov in 1986 [1] which is a generalization of the fuzzy sets (See [2] and [3]). Biswas [4] applied the concept of intuitionistic fuzzy sets to the theory of groups and studied intuitionistic fuzzy subgroups of a group. Also, Jun and Kim [8] introduced the concept of intuitionistic fuzzy ideals of BCK-algebras. Recently, Choc et al. [7] introduced the concept of intuitionistic ideals of pseudo MV-algebras.

In [11] and [12] the concept of intuitionistic fuzzy sets is applied to the theory of residuated lattice and the concept of intuitionistic fuzzy filters of residuated lattice is introduced.

Molodtsov introduced the concept of soft set theory in 1999 [14] as a new mathematical tool for dealing with uncertainties. In [15], Maji et al. (2001) defined operations of soft sets to make a detailed theoretical study on the soft sets. By using these definitions, the applications of soft set theory have been studied increasingly. Maji et al. [16] described the application of soft set theory to a decision making problem. Maji et al. [17, 18] also studied several operations on the theory of soft sets. Jun [9] introduced the concept of soft BCK/BCI-algebras and then in [10] soft set theory is applied to d-algebras. In [13], the concept of soft sets is applied to the theory of residuated lattice and some of its properties is obtained.

In this paper, we deal with the algebraic structure of residuated lattice by applying the notion of intuitionistic fuzzy soft sets.

In section 2, some basic definitions and results are explained. In section 3, we introduce the notion of intuitionistic fuzzy soft filters of a residuated lattice and investigate some related properties. In section 4, we define generated intuitionistic fuzzy soft filters of residuated lattices. In section 5, we study the lattice of intuitionistic fuzzy soft filters of residuated lattices and obtain some related results.

## 2. PRELIMINARIES

We recall some definitions and theorems which will be needed in this paper.

**Definition 2.1.** (1) Let  $(P, \leq)$  be a poset. The subset  $X$  of  $P$  is called a directed subset, if it is not the empty set, and every pair of elements has an upper bound.

(2) In a partially ordered set  $(P, \leq)$  an element  $c$  is called compact (or finite) if it satisfies the following condition :

for every directed subset  $D$  of  $P$ , if  $D$  has a supremum  $\sup D$  and  $c \leq \sup D$ , then  $c \leq d$  for some element  $d$  of  $D$ .

(3) A poset in which every element is the supremum of the compact elements below it is called an algebraic poset.

**Definition 2.2** ([5]). A residuated lattice is an algebraic structure  $(L, \wedge, \vee, \rightarrow, *, 0, 1)$  such that

- (1)  $(L, \wedge, \vee, 0, 1)$  is a bounded lattice with the least element 0 and the greatest element 1,
- (2)  $(L, *, 1)$  is a commutative monoid where 1 is a unit element,
- (3)  $x * y \leq z$  iff  $x \leq y \rightarrow z$ , for all  $x, y, z \in L$ .

In the rest of this paper, we denote the residuated lattice  $(L, \wedge, \vee, *, \rightarrow, 0, 1)$  by  $L$ .

**Proposition 2.3** ([5, 6]). Let  $L$  be a residuated lattice. Then we have the following properties : for all  $x, y, z \in L$ ,

- (1)  $x \leq y$  if and only if  $x \rightarrow y = 1$ ,
- (2)  $x * y \leq x \wedge y$ ,
- (3) If  $x \leq y$ , then  $x * z \leq y * z$ .

**Definition 2.4** ([20, 21]). Let  $F$  be a non-empty subset of a residuated lattice  $L$ .  $F$  is called a filter if

- (1)  $x \leq y$  and  $x \in F$  imply  $y \in F$ ,
- (2) if  $x, y \in F$ , then  $x * y \in F$ ,

for all  $x, y \in L$ .

If  $X \subseteq L$ , the smallest filter containing  $X$  is called the generated filter by  $X$  and denoted by  $\langle X \rangle$ .

For sets  $X, Y$  and  $Z$ ,  $f = (f_1, f_2) : X \rightarrow Y \times Z$  is called a complex mapping if  $f_1 : X \rightarrow Y$  and  $f_2 : X \rightarrow Z$  are mappings. Throughout this paper, we will denote the unit interval  $[0, 1]$  as  $I$ .

**Definition 2.5** ([3]). Let  $X$  be a non-empty set. A complex mapping  $A = (\mu_A, \nu_A) : X \rightarrow I \times I$  is called an intuitionistic fuzzy set (in short, IFS) in  $X$  if  $0 \leq \mu_A + \nu_A \leq 1$  for each  $x \in X$ , where the mapping  $\mu_A : X \rightarrow I$  and  $\nu_A : X \rightarrow I$  denote the degree of membership (namely  $\mu_A(x)$ ) and the degree of non-membership (namely  $\nu_A(x)$ ) of each  $x \in X$  to  $A$  respectively.

**Definition 2.6** ([3]). Let  $X$  be a non-empty set and  $A = (\mu_A, \nu_A)$ ,  $B = (\mu_B, \nu_B)$  be IFS in  $X$ . Then

- (1)  $A \subseteq B$  iff  $\mu_A \leq \mu_B$  and  $\nu_B \leq \nu_A$ .
- (2)  $A = B$  iff  $A \subseteq B$  and  $B \subseteq A$ .
- (3)  $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$ .
- (4)  $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$ .

**Definition 2.7** ([11]). Let  $A = (\mu_A, \nu_A)$  be an IFS in a residuated lattice  $L$ .  $A = (\mu_A, \nu_A)$  is called an intuitionistic fuzzy filter in  $L$  if it satisfies the following conditions :

- (IF1)  $x \leq y$  implies  $\mu_A(x) \leq \mu_A(y)$ ,  $\nu_A(x) \geq \nu_A(y)$ ,
- (IF2)  $\mu_A(x * y) \geq \mu_A(x) \wedge \mu_A(y)$ ,
- (IF3)  $\nu_A(x * y) \leq \nu_A(x) \vee \nu_A(y)$ ,

for all  $x, y \in L$ .

For any  $\alpha \in [0, 1]$  and a fuzzy set  $\mu$  in a non-empty set  $X$ , the set

$$U(\mu, \alpha) = \{x \in X : \mu(x) \geq \alpha\},$$

is called an upper  $\alpha$ - level of  $\mu$  and the set

$$L(\mu, \alpha) = \{x \in X : \mu(x) \leq \alpha\}$$

is called a lower  $\alpha$ - level of  $\mu$ .

**Theorem 2.8** ([11]). Let  $A = (\mu_A, \nu_A)$  be an IFS in a residuated lattice  $L$ . Then  $A = (\mu_A, \nu_A)$  is an intuitionistic fuzzy filter in  $L$  if and only if for all  $\alpha, \beta \in [0, 1]$ , the sets  $U(\mu_A, \alpha)$  and  $L(\nu_A, \alpha)$  are either empty or filters in  $L$ .

**Definition 2.9** ([11]). Let  $A = (\mu_A, \nu_A)$  be an IFS in a residuated lattice  $L$ . An intuitionistic fuzzy filter  $B = (\mu_B, \nu_B)$  is called an intuitionistic fuzzy filter generated by  $A$ , if it satisfies :

- (i)  $A \subseteq B$ ,
- (ii)  $A \subseteq C$  implies  $B \subseteq C$ , for all intuitionistic fuzzy filters  $C$  in  $L$ .

The intuitionistic fuzzy filter generated by  $A$  is denoted by  $\langle A \rangle$ .

**Theorem 2.10** ([11]). Let  $A = (\mu_A, \nu_A)$  be an IFS in a residuated lattice  $L$ . Then  $\langle A \rangle = (\mu_{\langle A \rangle}, \nu_{\langle A \rangle})$  where

$$\begin{aligned} \mu_{\langle A \rangle}(x) &= \bigvee \{ \mu_A(a_1) \wedge \mu_A(a_2) \wedge \dots \wedge \mu_A(a_n) : x \geq a_1 * \dots * a_n \text{ where } a_1, \dots, a_n \in A \}, \\ \nu_{\langle A \rangle}(x) &= \bigwedge \{ \nu_A(a_1) \vee \nu_A(a_2) \vee \dots \vee \nu_A(a_n) : x \geq a_1 * \dots * a_n \text{ where } a_1, \dots, a_n \in A \}, \end{aligned}$$

for all  $x \in L$ .

**Definition 2.11** ([14]). Let  $U$  be an initial universe set and  $E$  be a set of parameters. Let  $P(U)$  denotes the power set of  $U$  and  $A \subseteq E$ . A pair  $(F, A)$  is called a soft set over  $U$ , where  $F$  is a mapping given by  $F : A \rightarrow P(U)$ . In other words, a soft set over  $U$  is a parameterized family of subsets of the universe  $U$ .

**Definition 2.12.** Let  $U$  be an initial universe set and  $E$  be a set of parameters. Let  $IFS(U)$  denote the set of all intuitionistic fuzzy sets in  $U$ . Then  $(F, A)$  is called an intuitionistic fuzzy soft set over  $U$  where  $A \subseteq E$  and  $F$  is a mapping given by  $F : A \rightarrow IFS(U)$ .

In general, for every  $e \in A$ ,  $F[e]$  is an intuitionistic fuzzy set in  $U$  and it is called intuitionistic fuzzy value set of parameter  $e$ . If for every  $e \in A$ ,  $F[e]$  is an intuitionistic fuzzy subset of  $U$ , then  $(F, A)$  is degenerated to be the intuitionistic fuzzy soft set. Thus, from the above definition, it is clear that intuitionistic fuzzy soft sets are a generalization of fuzzy soft sets.

### 3. INTUITIONISTIC FUZZY SOFT FILTERS OF RESIDUATED LATTICES

In what follows, let  $E$  be a set of parameters unless otherwise specified and  $L$  be a residuated lattice.

**Definition 3.1.** Let  $(F, A)$  be an intuitionistic fuzzy soft set over  $L$  where  $A$  is a subset of  $E$ . If there exists  $e \in A$  such that  $F[e]$  is an intuitionistic fuzzy filter in  $L$ , we say that  $(F, A)$  is an intuitionistic fuzzy soft filter of  $L$  based on a parameter  $e$ . If  $(F, A)$  is an intuitionistic fuzzy soft filter of  $L$  based on all parameters, we say that  $(F, A)$  is an intuitionistic fuzzy soft filter over  $L$ .

**Example 3.2.** Suppose that there are four players in the universe, that is

$$U = \{a, b, c, d\}.$$

Let  $\cap, \cup, *, \rightarrow$  be four soft game machines for two players to play accordingly in such a way, we have the following results :

$$\begin{aligned} a \cap x &= a \text{ and } d \cap x = x \text{ for all } x \in U, \\ b \cap x &= \begin{cases} b & \text{if } x \in \{b, d\} \\ a & \text{if } x \in \{a, c\} \end{cases} \quad c \cap x = \begin{cases} c & \text{if } x \in \{c, d\} \\ a & \text{if } x \in \{a, b\} \end{cases} \\ a \cup x &= x \text{ and } d \cup x = d \text{ for all } x \in U, \\ b \cup x &= \begin{cases} d & \text{if } x \in \{c, d\} \\ b & \text{if } x \in \{a, b\} \end{cases} \quad c \cup x = \begin{cases} d & \text{if } x \in \{b, d\} \\ c & \text{if } x \in \{a, c\} \end{cases} \\ a * x &= a \text{ and } d * x = x \text{ for all } x \in U, \\ b * x &= \begin{cases} a & \text{if } x \in \{a, c\} \\ b & \text{if } x \in \{b, d\} \end{cases} \quad c * x = \begin{cases} a & \text{if } x \in \{a, b\} \\ c & \text{if } x \in \{c, d\} \end{cases} \\ a \rightarrow x &= d \text{ and } d \rightarrow x = x \text{ for all } x \in U, \\ b \rightarrow x &= \begin{cases} d & \text{if } x \in \{b, d\} \\ c & \text{if } x \in \{a, c\} \end{cases} \quad c \rightarrow x = \begin{cases} b & \text{if } x \in \{a, b\} \\ d & \text{if } x \in \{c, d\} \end{cases} \end{aligned}$$

Then  $(U, \cap, \cup, *, \rightarrow, a, d)$  is a residuated lattice. Consider a set of parameters :

$$E = \{Clever, Agile, Brave\}.$$

(1) Let  $(F, E)$  be an intuitionistic fuzzy soft set over  $U$ . Then  $F[Clever]$ ,  $F[Agile]$  and  $F[Brave]$  are intuitionistic fuzzy sets in  $U$ . Define them as follows :

| $F$           | $a$        | $b$        | $c$        | $d$        |
|---------------|------------|------------|------------|------------|
| <i>Clever</i> | (0.5, 0.4) | (0.5, 0.4) | (0.5, 0.3) | (0.6, 0.3) |
| <i>Agile</i>  | (0.3, 0.6) | (0.3, 0.4) | (0.4, 0.6) | (0.7, 0.2) |
| <i>Brave</i>  | (0.4, .6)  | (0.4, .5)  | (0.7, 0.1) | (0.1, 0.4) |

Then  $(F, E)$  is an intuitionistic fuzzy soft filter based on parameters "Clever" and "Agile" over  $U$  but it is not an intuitionistic fuzzy soft filter based on a parameter "Brave" over  $U$ . Hence  $(F, E)$  is not an intuitionistic fuzzy soft filter over  $U$ .

(2) Let  $(G, E)$  be an intuitionistic fuzzy soft set over  $U$ . Then  $G[Clever]$ ,  $G[Agile]$  and  $G[Brave]$  are intuitionistic fuzzy sets in  $U$ . Define them as follows :

| $G$           | $a$        | $b$        | $c$        | $d$        |
|---------------|------------|------------|------------|------------|
| <i>Clever</i> | (0.2, 0.7) | (0.2, 0.6) | (0.3, 0.7) | (0.8, 0.1) |
| <i>agile</i>  | (0.6, 0.3) | (0.6, 0.3) | (0.6, 0.3) | (0.9, 0.1) |
| <i>brave</i>  | (0.8, .2)  | (0.7, .1)  | (0.8, .2)  | (1, 0)     |

Then  $G[Clever]$ ,  $G[Agile]$  and  $G[Brave]$  are intuitionistic fuzzy soft filters based on parameters "Clever", "Agile" and "Brave" over  $U$ , respectively. Hence  $(G, E)$  is an intuitionistic fuzzy soft filter over  $U$ .

**Definition 3.3.** Let  $(F, A)$  and  $(G, B)$  be two intuitionistic fuzzy sets over a common universe  $U$ . The extended intersection of  $(F, A)$  and  $(G, B)$  is defined to be the intuitionistic fuzzy soft set  $(H, C)$  satisfying the following conditions :

- (i)  $C = A \cup B$ ,
- (ii) For all  $e \in C$ ,

$$H[e] = \begin{cases} F[e] & \text{if } e \in A \setminus B \\ G[e] & \text{if } e \in B \setminus A \\ F[e] \cap G[e] & \text{if } e \in A \cap B. \end{cases}$$

In this case, we write  $(F, A) \sqcap_e (G, B) = (H, C)$ .

**Theorem 3.4.** *Let  $(F, A)$  and  $(G, B)$  be two intuitionistic fuzzy soft filters over  $L$ . Then the extended intersection of  $(F, A)$  and  $(G, B)$  is an intuitionistic fuzzy soft filter over  $L$ .*

*Proof.* Let  $(F, A) \sqcap_e (G, B) = (H, C)$  be the extended intersection of  $(F, A)$  and  $(G, B)$ . We have  $C = A \cup B$ . Suppose that  $e \in C$  be arbitrary.

- (1) If  $e \in A \setminus B$ , then  $H[e] = F[e]$  is an intuitionistic fuzzy filter in  $L$ .
- (2) If  $e \in B \setminus A$ , then  $H[e] = G[e]$  is an intuitionistic fuzzy filter in  $L$ .
- (3) If  $A \cap B \neq \emptyset$ , then  $H[e] = F[e] \cap G[e]$  is an intuitionistic fuzzy filter for all  $e \in A \cap B$ , because the intersection of two intuitionistic fuzzy filters in  $L$  is an intuitionistic fuzzy filter. Therefore  $(H, C)$  is an intuitionistic fuzzy soft filter over  $L$ .  $\square$

**Definition 3.5.** Let  $(F, A)$  and  $(G, B)$  be two intuitionistic fuzzy soft sets over a common universe  $U$  such that  $A \cap B \neq \emptyset$ . The restricted intersection of  $(F, A)$  and  $(G, B)$  is defined to be the soft intuitionistic fuzzy soft set  $(H, C)$  satisfying the following conditions :

- (i)  $C = A \cap B$ .
- (ii)  $H[e] = F[e] \cap G[e]$  for all  $e \in C$ .

In this case, we write  $(F, A) \sqcap (G, B) = (H, C)$ .

**Corollary 3.6.** *The restricted intersection of two intuitionistic fuzzy soft filters over  $L$  is an intuitionistic fuzzy soft filter over  $L$ .*

**Definition 3.7.** Let  $(F, A)$  and  $(G, B)$  be two intuitionistic fuzzy soft sets over a common universe  $U$ . The union of  $(F, A)$  and  $(G, B)$  is defined to be the intuitionistic fuzzy soft set  $(H, C)$  satisfying the following conditions :

- (i)  $C = A \cup B$ .
- (ii) For all  $e \in C$ ,

$$H[e] = \begin{cases} F[e] & \text{if } e \in A \setminus B \\ G[e] & \text{if } e \in B \setminus A \\ F[e] \cup G[e] & \text{if } e \in A \cap B. \end{cases}$$

In this case, we write  $(F, A) \sqcup (G, B) = (H, C)$ .

**Theorem 3.8.** *Let  $(F, A)$  and  $(G, B)$  be two intuitionistic fuzzy soft filters over  $L$ . If  $A$  and  $B$  are disjoint, then the union  $(F, A) \sqcup (G, B)$  is an intuitionistic fuzzy soft filter over  $L$ .*

*Proof.* Suppose that  $(F, A) \sqcup (G, B) = (H, C)$ , where  $C = A \cup B$  and for all  $e \in C$ ,

$$H[e] = \begin{cases} F[e] & \text{if } e \in A \setminus B \\ G[e] & \text{if } e \in B \setminus A \\ F[e] \cup G[e] & \text{if } e \in A \cap B. \end{cases}$$

By assumption,  $A \cap B = \emptyset$ . Hence we have either  $e \in A \setminus B$  or  $e \in B \setminus A$  for all  $e \in C$ . Consider the following cases :

(1) If  $e \in A \setminus B$ , then  $H[e] = F[e]$  is an intuitionistic fuzzy filter in  $L$  because  $(F, A)$  is an intuitionistic fuzzy soft filter over  $L$ .

(2) If  $e \in B \setminus A$ , then  $H[e] = G[e]$  is an intuitionistic fuzzy filter in  $L$  because  $(G, B)$  is an intuitionistic fuzzy soft filter over  $L$ .

Therefore  $(H, C) = (F, A) \sqcup (G, B)$  is an intuitionistic fuzzy soft filter over  $L$ .  $\square$

The following example shows that Theorem 3.8 is not valid if  $A$  and  $B$  are not disjoint.

**Example 3.9.** Let  $(U, \mathbb{M}, \mathbb{U}, \mathbb{K}, \rightarrow, a, d)$  is the residuated lattice in Example 3.2. Consider two sets of parameters :

$$A = \{Attentive, Smart\}, B = \{Clever, Skilful, Smart\}$$

Then  $A$  and  $B$  are not disjoint.

Let  $(F, A)$  be an intuitionistic fuzzy soft set over  $U$ . Then  $F[Attentive]$ ,  $F[Smart]$  are intuitionistic fuzzy sets in  $U$ . Define them as follows :

| $F$         | $a$        | $b$        | $c$        | $d$        |
|-------------|------------|------------|------------|------------|
| $Attentive$ | (0.2, 0.4) | (0.2, 0.3) | (0.6, 0.4) | (0.8, 0.1) |
| $Smart$     | (0.6, 0.2) | (0.6, 0.1) | (0.8, 0.2) | (0.9, 0.0) |

Then  $(F, A)$  is an intuitionistic fuzzy soft filter over  $U$ .

Let  $(G, B)$  be an intuitionistic fuzzy soft set over  $U$ . Then  $G[Clever]$ ,  $G[Skilful]$  and  $G[Smart]$  are intuitionistic fuzzy sets in  $U$ . Define them as follows :

| $G$       | $a$        | $b$        | $c$        | $d$        |
|-----------|------------|------------|------------|------------|
| $Clever$  | (0.1, 0.4) | (0.5, 0.4) | (0.1, 0.4) | (0.6, 0.3) |
| $Skilful$ | (0.4, 0.4) | (0.6, 0.4) | (0.4, 0.2) | (0.7, 0.1) |
| $Smart$   | (0.3, 0.2) | (0.7, 0.2) | (0.3, 0.1) | (0.9, 0.1) |

Then  $(G, B)$  is an intuitionistic fuzzy soft filter over  $U$ .

But the union of  $(F, A)$  and  $(G, B)$  is not an intuitionistic fuzzy soft filter over  $U$ . Suppose that  $e = Smart$ . Then  $F[e] \cup G[e] = (\mu_{F[e]} \vee \mu_{G[e]}, \nu_{F[e]} \wedge \nu_{G[e]})$  is not an intuitionistic fuzzy filter in  $U$ , since

$$\begin{aligned} (\mu_{F[e]} \vee \mu_{G[e]})(b * c) &= (\mu_{F[e]} \vee \mu_{G[e]})(a) = 0.6 \\ [(\mu_{F[e]} \vee \mu_{G[e]})(b)] \wedge [(\mu_{F[e]} \vee \mu_{G[e]})(c)] &= 0.7 \end{aligned}$$

but  $0.6 \not\geq 0.7$ .

**Definition 3.10.** Let  $(F, A)$  and  $(G, B)$  be two intuitionistic fuzzy soft sets over a common universe  $U$ . Then  $(F, A) \text{ AND } (G, B)$  denoted by  $(F, A) \bar{\wedge} (G, B)$  is a soft intuitionistic fuzzy set defined by  $(F, A) \bar{\wedge} (G, B) = (H, A \times B)$ , where  $H[(u, v)] = F[u] \cap G[v]$  for all  $(u, v) \in A \times B$ .

**Theorem 3.11.** Let  $(F, A)$  and  $(G, B)$  be two intuitionistic fuzzy soft filters over  $L$ . Then  $(F, A) \bar{\wedge} (G, B)$  is an intuitionistic fuzzy soft filter over  $L$ .

*Proof.* Suppose that  $(H, A \times B) = (F, A) \bar{\wedge} (G, B)$ , where  $H[(u, v)] = F[u] \cap G[v]$ ,  $F[u] = (\mu_{F[u]}, \nu_{F[u]})$  and  $G[v] = (\mu_{G[v]}, \nu_{G[v]})$  for all  $(u, v) \in A \times B$ . We have

$$H[(u, v)] = F[u] \cap G[v] = (\mu_{F[u]} \wedge \mu_{G[v]}, \nu_{F[u]} \vee \nu_{G[v]}).$$

For any  $(u, v) \in A \times B$ , we will show that  $H[(u, v)]$  is an intuitionistic fuzzy filter in  $L$ .

(IF1) Let  $x \leq y$ . Since  $F[u]$  and  $G[v]$  are intuitionistic fuzzy filters in  $L$ , we have

$$\begin{aligned}\mu_{F[u]}(x) &\leq \mu_{F[u]}(y), \quad \nu_{F[u]}(x) \geq \nu_{F[u]}(y), \\ \mu_{G[v]}(x) &\leq \mu_{G[v]}(y), \quad \nu_{G[v]}(x) \geq \nu_{G[v]}(y).\end{aligned}$$

Therefore

$$\begin{aligned}\mu_{F[u]}(x) \wedge \mu_{G[v]}(x) &\leq \mu_{F[u]}(y) \wedge \mu_{G[v]}(y), \\ \nu_{F[u]}(x) \vee \nu_{G[v]}(x) &\geq \nu_{F[u]}(y) \vee \nu_{G[v]}(y).\end{aligned}$$

(IF2) For all  $x, y \in L$ , we have

$$\begin{aligned}(\mu_{F[u]} \wedge \mu_{G[v]})(x * y) &\geq [(\mu_{F[u]}(x) \wedge \mu_{F[u]}(y)) \wedge (\mu_{G[v]}(x) \wedge \mu_{G[v]}(y))] \\ &= [(\mu_{F[u]}(x) \wedge \mu_{G[v]}(x)) \wedge (\mu_{F[u]}(y) \wedge \mu_{G[v]}(y))] \\ &= (\mu_{F[u]} \wedge \mu_{G[v]})(x) \wedge (\mu_{F[u]} \wedge \mu_{G[v]})(y).\end{aligned}$$

(IF3) For all  $x, y \in L$ , we have

$$\begin{aligned}(\nu_{F[u]} \vee \nu_{G[v]})(x * y) &\leq [(\nu_{F[u]}(x) \vee \nu_{F[u]}(y)) \vee (\nu_{G[v]}(x) \vee \nu_{G[v]}(y))] \\ &= [(\nu_{F[u]}(x) \vee \nu_{G[v]}(x)) \vee (\nu_{F[u]}(y) \vee \nu_{G[v]}(y))] \\ &= (\nu_{F[u]} \vee \nu_{G[v]})(x) \vee (\nu_{F[u]} \vee \nu_{G[v]})(y).\end{aligned}$$

Hence  $(H, A \times B) = (F, A) \bar{\wedge} (G, B)$  is an intuitionistic fuzzy soft filter based on  $(u, v)$ . Since  $(u, v) \in A \times B$  is arbitrary, then  $(H, A \times B)$  is an intuitionistic fuzzy soft filter over  $L$ .  $\square$

**Definition 3.12.** Let  $(F, A)$  and  $(G, B)$  be two intuitionistic fuzzy soft sets over a common universe  $U$ . Then  $(F, A) \text{OR} (G, B)$  denoted by  $(F, A) \vee (G, B)$  is an intuitionistic fuzzy soft set defined by  $(F, A) \vee (G, B) = (H, A \times B)$ , where  $H[(u, v)] = F[u] \cup G[v]$  for all  $(u, v) \in A \times B$ .

**Remark 3.13.** Let  $(F, A)$  and  $(G, B)$  be two intuitionistic fuzzy soft filters over  $L$ . Then  $(F, A) \vee (G, B)$  may not be an intuitionistic fuzzy soft filter over  $L$ . Consider the following example.

**Example 3.14.** Let  $(F, A)$  and  $(G, B)$  be two intuitionistic fuzzy soft filters over  $U$  in Example 3.9. Then  $(F, A) \vee (G, B)$  is not an intuitionistic fuzzy soft filter over  $U$ . Suppose that  $u = \text{Attentive}$ ,  $v = \text{Skilful}$ . Then  $F[u] \cup G[v] = (\mu_{F[u]} \vee \mu_{G[v]}, \nu_{F[u]} \wedge \nu_{G[v]})$  is not an intuitionistic fuzzy filter in  $U$ , since

$$\begin{aligned}\mu_{F[u]} \vee \mu_{G[v]}(b * c) &= \mu_{F[u]} \vee \mu_{G[v]}(a) = 0.4 < 0.6 \\ &= [\mu_{F[u]} \vee \mu_{G[v]}(b)] \wedge [\mu_{F[u]} \vee \mu_{G[v]}(c)].\end{aligned}$$

#### 4. GENERATED INTUITIONISTIC FUZZY SOFT FILTERS OF RESIDUATED LATTICES

In this section, we will define and study the generated intuitionistic fuzzy soft filters of residuated lattices.



**Definition 4.1.** Let  $(F, A)$  and  $(G, B)$  be two intuitionistic fuzzy soft sets over a common universe  $U$ . Then  $(F, A)$  is called a soft subset of  $(G, B)$  and denoted by  $(F, A) \sqsubseteq (G, B)$ , if it satisfies :

- (i)  $A \subseteq B$ ,
- (ii) for every  $e \in A$ ,  $F[e] \subseteq G[e]$ , that is  $\mu_{F[e]}(x) \leq \mu_{G[e]}(x)$  and  $\nu_{G[e]}(x) \leq \nu_{F[e]}(x)$ , for all  $x \in U$ .

**Definition 4.2.** Let  $(F, A)$  be an intuitionistic fuzzy soft set over  $L$ . An intuitionistic fuzzy soft filter  $(G, B)$  is called an intuitionistic fuzzy soft filter generated by  $(F, A)$ , if it satisfies :

- (i)  $(F, A) \sqsubseteq (G, B)$ ,
- (ii)  $(F, A) \sqsubseteq (H, C)$  implies  $(G, B) \sqsubseteq (H, C)$ , for all intuitionistic fuzzy soft filters  $(H, C)$  of  $L$ .

The intuitionistic fuzzy soft filter generated by  $(F, A)$  is denoted by  $\langle (F, A) \rangle$ .

**Example 4.3.** Let  $L$  be a residuated lattice with the universe  $\{0, a, b, c, 1\}$  such that  $0 < a, b < c < 1$  and  $a, b$  are incomparable. The operations  $*$  and  $\rightarrow$  are given by the tables below :

| $*$ | 0 | a | b | c | 1 |
|-----|---|---|---|---|---|
| 0   | 0 | 0 | 0 | 0 | 0 |
| a   | 0 | a | 0 | a | a |
| b   | 0 | 0 | b | b | b |
| c   | 0 | a | b | c | c |
| 1   | 0 | a | b | c | 1 |

| $\rightarrow$ | 0 | a | b | c | 1 |
|---------------|---|---|---|---|---|
| 0             | 1 | 1 | 1 | 1 | 1 |
| a             | b | 1 | b | 1 | 1 |
| b             | a | a | 1 | 1 | 1 |
| c             | 0 | a | b | 1 | 1 |
| 1             | 0 | a | b | c | 1 |

Consider a set of parameters  $E = \{1/2, 1/4\}$ .

Let  $(F, E)$  be an intuitionistic fuzzy soft set over  $L$ . Then  $F[1/2]$  and  $F[1/4]$  are intuitionistic fuzzy sets in  $L$ . Define them as follows :

| $F$ | 0          | a          | b          | c          | 1          |
|-----|------------|------------|------------|------------|------------|
| 1/2 | (0.2, 0.4) | (0.2, 0.4) | (0.5, 0.4) | (0.7, 0.2) | (0.9, 0.1) |
| 1/4 | (0.6, 0.3) | (0.5, 0.4) | (0.6, 0.3) | (0.9, 0.1) | (0.8, 0.1) |

Then  $F[1/2]$  is an intuitionistic fuzzy soft filter based on parameter “1/2” but it is not an intuitionistic fuzzy soft filter based on the parameter 1/4 over  $L$ . Hence  $(F, A)$  is not an intuitionistic fuzzy soft filter over  $L$ .

Let  $(G, B)$  be an intuitionistic fuzzy soft set over  $L$ . Then  $G[1/2]$  and  $G[1/4]$  are intuitionistic fuzzy sets in  $L$ . Define them as follows :

| $G$ | 0          | a          | b          | c          | 1          |
|-----|------------|------------|------------|------------|------------|
| 1/2 | (0.2, 0.4) | (0.2, 0.4) | (0.5, 0.4) | (0.7, 0.2) | (0.9, 0.1) |
| 1/4 | (0.6, 0.3) | (0.6, 0.3) | (0.6, 0.3) | (0.9, 0.1) | (0.9, 0.1) |

Then  $(G, E)$  is an intuitionistic fuzzy soft filter generated by  $(F, A)$ .

**Theorem 4.4.** Let  $(F, A)$  and  $(G, A)$  be intuitionistic fuzzy soft sets over  $L$ . If  $(G, A)$  is an intuitionistic fuzzy soft set over  $L$  such that for any  $e \in A$  and  $x \in L$

$$\mu_{G[e]}(x) = \bigvee \{ \mu_{F[e]}(a_1) \wedge \mu_{F[e]}(a_2) \wedge \dots \wedge \mu_{F[e]}(a_n) : x \geq a_1 * \dots * a_n \},$$

$$\nu_{G[e]}(x) = \bigwedge \{ \nu_{F[e]}(a_1) \vee \nu_{F[e]}(a_2) \vee \dots \vee \nu_{F[e]}(a_n) : x \geq a_1 * \dots * a_n \},$$

Then  $(G, A) = \langle (F, A) \rangle$  and hence  $G[e] = \langle F[e] \rangle$  for any  $e \in A$ .

*Proof.* Obviously  $G[e]$  is an intuitionistic fuzzy filter generated by  $F[e]$ , for any  $e \in A$ . Hence  $(G, A)$  is an intuitionistic fuzzy soft filter over  $L$ . Suppose that  $(H, C)$  is an intuitionistic fuzzy soft filter over  $L$  such that  $(F, A) \sqsubseteq (H, C)$ . Then we have  $A \subseteq C$ , and for every  $e \in A$ ,  $\mu_{F[e]}(x) \leq \mu_{H[e]}(x)$  and  $\nu_{H[e]}(x) \leq \nu_{F[e]}(x)$ , for all  $x \in L$ . We get that

$$\begin{aligned} \mu_{G[e]}(x) &= \bigvee \{ \mu_{F[e]}(a_1) \wedge \mu_{F[e]}(a_2) \wedge \dots \wedge \mu_{F[e]}(a_n) : x \geq a_1 * \dots * a_n \} \\ &\leq \bigvee \{ \mu_{H[e]}(a_1) \wedge \mu_{H[e]}(a_2) \wedge \dots \wedge \mu_{H[e]}(a_n) : x \geq a_1 * \dots * a_n \} \\ &\leq \mu_{H[e]}(x), \end{aligned}$$

and

$$\begin{aligned} \nu_{G[e]}(x) &= \bigwedge \{ \nu_{F[e]}(a_1) \vee \nu_{F[e]}(a_2) \vee \dots \vee \nu_{F[e]}(a_n) : x \geq a_1 * \dots * a_n \} \\ &\geq \bigwedge \{ \nu_{H[e]}(a_1) \vee \nu_{H[e]}(a_2) \vee \dots \vee \nu_{H[e]}(a_n) : x \geq a_1 * \dots * a_n \} \\ &\geq \nu_{H[e]}(x), \end{aligned}$$

for all  $x \in L$ , that is  $(G, A) \sqsubseteq (H, C)$ . Therefore  $(G, A) = < (F, A) >$  and hence  $G[e] = < F[e] >$  for any  $e \in A$ .  $\square$

**Theorem 4.5.** *Let  $(F, A)$  be an intuitionistic fuzzy soft set over a residuated lattice  $L$ . If  $(G, A)$  is an intuitionistic fuzzy soft set over  $L$  such that for any  $e \in A$  and  $x \in L$*

$$\begin{aligned} \mu_{G[e]}(x) &= \bigvee \{ \alpha \in [0, 1] : x \in < U(\mu_{F[e]}, \alpha) > \}, \\ \nu_{G[e]}(x) &= \bigwedge \{ \alpha \in [0, 1] : x \in < L(\nu_{F[e]}, \alpha) > \}. \end{aligned}$$

*Then  $(G, A) = < (F, A) >$ .*

*Proof.* Suppose that  $\alpha \in [0, 1]$ . Put  $\alpha_n = \alpha - \frac{1}{n}$  and  $\alpha'_n = \alpha + \frac{1}{n}$ .

(1) First, we will show that  $U(\mu_{G[e]}, \alpha)$  is either empty or a filter of  $L$ . Assume that  $U(\mu_{G[e]}, \alpha) \neq \emptyset$  and  $x \in U(\mu_{G[e]}, \alpha)$ . For every  $n \in \mathbb{N}$ , we have

$$\mu_{G[e]}(x) = \bigvee \{ \alpha \in [0, 1] : x \in < U(\mu_{F[e]}, \alpha) > \} \geq \alpha > \alpha_n.$$

Then for every  $n \in \mathbb{N}$ , there exists

$$\beta_n \in \{ \alpha \in [0, 1] : x \in < U(\mu_{F[e]}, \alpha) > \}$$

such that  $\beta_n > \alpha_n$ . We get that  $x \in < U(\mu_{F[e]}, \beta_n) >$ , for every  $n \in \mathbb{N}$ . Thus  $x \in \bigcap_{n \in \mathbb{N}} < U(\mu_{F[e]}, \beta_n) >$ , that is  $U(\mu_{G[e]}, \alpha) \subseteq \bigcap_{n \in \mathbb{N}} < U(\mu_{F[e]}, \beta_n) >$ .

Conversely, let  $x \in \bigcap_{n \in \mathbb{N}} < U(\mu_{F[e]}, \beta_n) >$ . So

$$\beta_n \in \{ \alpha \in [0, 1] : x \in < U(\mu_{F[e]}, \alpha) > \}$$

for every  $n \in \mathbb{N}$ . We get that

$$\alpha_n < \beta_n \leq \bigvee \{ \alpha \in [0, 1] : x \in < U(\mu_{F[e]}, \alpha) > \} = \mu_{G[e]}(x)$$

for every  $n \in \mathbb{N}$ . Hence  $\alpha \leq \mu_{G[e]}(x)$  and thus  $x \in U(\mu_{G[e]}, \alpha)$ . We obtain that  $\bigcap_{n \in \mathbb{N}} < U(\mu_{F[e]}, \beta_n) > \subseteq U(\mu_{G[e]}, \alpha)$ .

Therefore  $U(\mu_{G[e]}, \alpha) = \bigcap_{n \in \mathbb{N}} < U(\mu_{F[e]}, \beta_n) >$  is a filter of  $L$ .

(2) We will prove that  $L(\mu_{G[e]}, \alpha)$  is either empty or a filter of  $L$ . Assume that  $L(\mu_{G[e]}, \alpha) \neq \emptyset$  and  $x \in L(\mu_{G[e]}, \alpha)$ . For every  $n \in \mathbb{N}$ , we have

$$\nu_{G[e]}(x) = \bigwedge \{ \alpha \in [0, 1] : x \in < L(\nu_{F[e]}, \alpha) > \} \leq \alpha < \alpha'_n.$$

Thus for every  $n \in \mathbb{N}$ , there exists

$$\beta'_n \in \{\alpha \in [0, 1] : x \in L(\nu_{F[e]}, \alpha) >\}$$

such that  $\beta'_n < \alpha'_n$ . We get that  $x \in L(\nu_{F[e]}, \beta'_n) >$ , for every  $n \in \mathbb{N}$ . So  $x \in \bigcap_{n \in \mathbb{N}} L(\nu_{F[e]}, \beta'_n) >$ , that is  $L(\nu_{G[e]}, \alpha) \subseteq \bigcap_{n \in \mathbb{N}} L(\nu_{F[e]}, \beta'_n) >$ .

Conversely, let  $x \in \bigcap_{n \in \mathbb{N}} L(\nu_{F[e]}, \beta'_n) >$ . Then

$$\beta'_n \in \{\alpha \in [0, 1] : x \in L(\nu_{F[e]}, \alpha) >\}$$

for every  $n \in \mathbb{N}$ . We get that

$$\alpha'_n > \beta'_n \geq \bigwedge \{\alpha \in [0, 1] : x \in L(\nu_{F[e]}, \alpha) >\} = \nu_{G[e]}(x)$$

for every  $n \in \mathbb{N}$ . Thus  $\alpha \geq \nu_{G[e]}(x)$  and then  $x \in L(\nu_{G[e]}, \alpha)$ . We get that

$$\bigcap_{n \in \mathbb{N}} L(\nu_{F[e]}, \beta'_n) > \subseteq L(\nu_{G[e]}, \alpha).$$

So  $L(\nu_{G[e]}, \alpha) = \bigcap_{n \in \mathbb{N}} L(\nu_{F[e]}, \beta'_n) >$  is a filter of  $L$ .

Hence  $(G, A)$  is an intuitionistic fuzzy soft filter over  $L$  by (1) and (2) and Theorem 2.8.

(3) Now, we will show that  $(F, A) \sqsubseteq (G, A)$ . Let  $x \in L$  and

$$\gamma \in \{\alpha \in [0, 1] : x \in U(\mu_{F[e]}, \alpha) >\}.$$

We get that  $\gamma \in \{\alpha \in [0, 1] : x \in U(\mu_{F[e]}, \alpha) >\}$ . Thus

$$x \in U(\mu_{F[e]}, \gamma) \subseteq U(\mu_{F[e]}, \gamma) > .$$

So

$$\{\alpha \in [0, 1] : x \in U(\mu_{F[e]}, \alpha) >\} \subseteq \{\alpha \in [0, 1] : x \in U(\mu_{F[e]}, \alpha) >\}.$$

We obtain that

$$\begin{aligned} \mu_{F[e]}(x) &= \bigvee \{\alpha \in [0, 1] : x \in U(\mu_{F[e]}, \alpha) >\} \\ &\leq \bigvee \{\alpha \in [0, 1] : x \in U(\mu_{F[e]}, \alpha) >\} = \mu_{G[e]}(x). \end{aligned}$$

Similarly, we can show that  $\nu_{F[e]}(x) \geq \nu_{G[e]}(x)$ . Hence  $(F, A) \sqsubseteq (G, A)$ .

(4) Finally, suppose that  $(H, C)$  be an intuitionistic fuzzy soft filter over  $L$ . Such that  $(G, A) \sqsubseteq (H, C)$ . Assume that  $x \in L$  and  $\mu_{G[e]}(x) = \alpha$ . Then

$$x \in U(\mu_{G[e]}, \alpha) = \bigcap_{n \in \mathbb{N}} U(\mu_{F[e]}, \beta_n) > .$$

We have  $x \in U(\mu_{F[e]}, \beta_n) >$  for every  $n \in \mathbb{N}$ . Thus there exist  $a_1, a_2, \dots, a_n \in U(\mu_{F[e]}, \beta_n)$  such that  $x \geq a_1 * a_2 * \dots * a_n$ . We obtain that

$$\mu_{F[e]}(x) \geq \mu_{F[e]}(x) \wedge \mu_{F[e]}(a_1) \wedge \dots \wedge \mu_{F[e]}(a_n) \geq \beta_n.$$

So  $\mu_{H[e]}(x) \geq \mu_{F[e]}(x) \geq \beta_n \geq \alpha_n = \alpha - \frac{1}{n}$  for every  $n \in \mathbb{N}$ . We obtain that  $\mu_{H[e]}(x) \geq \alpha = \mu_{G[e]}(x)$ . Similarly, we can show that  $\nu_{H[e]}(x) \leq \nu_{G[e]}(x)$ . Hence we get that  $(G, B) \sqsubseteq (H, C)$ . Therefore  $(G, B)$  is an intuitionistic fuzzy soft filter generated by  $(F, A)$ .  $\square$

**Proposition 4.6.** Let  $(F, A)$  and  $(H, C)$  be intuitionistic fuzzy soft sets and  $(G, B)$  be an intuitionistic fuzzy soft filter over  $L$ . Then

- (1) if  $(F, A) \sqsubseteq (H, C)$ , then  $\langle (F, A) \rangle \sqsubseteq \langle (H, C) \rangle$ .
- (2)  $\langle (F, A) \rangle \wedge (G, B) \sqsubseteq \langle (F, A) \cap (G, B) \rangle$ .

*Proof.* (1) The proof is straightforward.

(2) Suppose that  $e \in A \cap B$  and  $x \in L$  be arbitrary. Then

$$\begin{aligned} & \mu_{G[e]}(x) \wedge \bigvee \{ \mu_{F[e]}(a_1) \wedge \dots \wedge \mu_{F[e]}(a_n) : x \geq a_1 * \dots * a_n \} \\ & \leq \bigvee \{ (\mu_{G[e]}(x) \wedge \mu_{F[e]}(a_1)) \wedge \dots \wedge (\mu_{G[e]}(x) \wedge \mu_{F[e]}(a_n)) : x \geq a_1 * \dots * a_n \} \\ & \leq \bigvee \{ (\mu_{G[e]} \wedge \mu_{F[e]})(a_1) \wedge \dots \wedge (\mu_{G[e]} \wedge \mu_{F[e]})(a_n) : x \geq a_1 * \dots * a_n \}. \end{aligned}$$

Also, we have

$$\begin{aligned} & \nu_{G[e]}(x) \vee \bigwedge \{ \nu_{F[e]}(a_1) \vee \nu_{F[e]}(a_2) \vee \dots \vee \nu_{F[e]}(a_n) : x \geq a_1 * \dots * a_n \} \\ & \geq \bigwedge \{ (\nu_{G[e]}(x) \vee \nu_{F[e]}(a_1)) \vee \dots \vee (\nu_{G[e]}(x) \vee \nu_{F[e]}(a_n)) : x \geq a_1 * \dots * a_n \} \\ & \geq \bigwedge \{ (\nu_{G[e]} \vee \nu_{F[e]})(a_1) \vee \dots \vee (\nu_{G[e]} \vee \nu_{F[e]})(a_n) : x \geq a_1 * \dots * a_n \}. \end{aligned}$$

Hence  $\langle (F, A) \rangle \wedge (G, B) \sqsubseteq \langle (F, A) \cap (G, B) \rangle$ .  $\square$

## 5. LATTICE OF INTUITIONISTIC FUZZY SOFT FILTERS OF RESIDUATED LATTICES

In this section, the set of all intuitionistic fuzzy soft filters of a residuated lattice  $L$ , will be denoted by  $IFSF(L)$ .

**Theorem 5.1.** Let  $L$  be a residuated lattice. Then  $(IFSF(L), \sqsubseteq)$  is a complete lattice.

*Proof.* Suppose that  $\{(F_i, A_i)\}_{i \in I} \subseteq IFSF(L)$ . Then the infimum of this family is  $\bigwedge_{i \in I} (F_i, A_i) = \langle \bigcap_{i \in I} (F_i, A_i) \rangle$  and the supremum is  $\bigvee_{i \in I} (F_i, A_i) = \langle \bigcup_{i \in I} (F_i, A_i) \rangle$ . It is easy to prove that  $(IFSF(L), \bigwedge, \bigvee)$  is a complete lattice.  $\square$

**Proposition 5.2.** Let  $\{(F_i, A_i)\}_{i \in I}$  be a directed subset of  $IFSF(L)$ . Then  $\bigsqcup_{i \in I} (F_i, A_i) \in IFSF(L)$ .

*Proof.* Suppose that  $(H, C) = \bigsqcup_{i \in I} (F_i, A_i)$ , where  $C = \bigcup_{i \in I} A_i$  and for all  $e \in C$ ,

$$H[e] = \begin{cases} F_i[e] & \text{if } e \in A_i \setminus \bigcup_{j \in I, j \neq i} A_j \\ \bigcup_{i \in I} F_i[e] & \text{if } e \in \bigcap_{i \in I', I' \subseteq I} A_i \end{cases}$$

where  $I'$  is the biggest subset of  $I$  which  $e \in A_i$ , for all  $i \in I'$ . Consider the following cases :

Case (1). If  $e \in A_i \setminus \bigcap_{j \in I, j \neq i} A_j$ , then  $H[e] = F_i[e]$  is an intuitionistic fuzzy filter in  $L$ .

Case (2). If  $e \in \bigcap_{i \in I} A_i$ , we will prove that  $H[e] = \bigcup_{i \in I} F_i[e]$  is an intuitionistic fuzzy filter in  $L$ .

(IF1) Suppose that  $x, y \in L$  such that  $x \leq y$ . For all  $j \in I$ , we have

$$\mu_{F_j[e]}(x) \leq \mu_{F_j[e]}(y) \leq \bigvee_{i \in I} \mu_{F_j[e]}(y)$$

and

$$\nu_{F_j[e]}(x) \geq \mu_{F_j[e]}(y) \geq \bigwedge_{i \in I} \mu_{F_j[e]}(y).$$

Since  $j \in I$  is arbitrary, we obtain that

$$\bigvee_{i \in I} \mu_{F_i[e]}(x) \leq \bigvee_{i \in I} \mu_{F_j[e]}(y)$$

and

$$\bigwedge_{i \in I} \mu_{F_i[e]}(x) \geq \bigwedge_{i \in I} \mu_{F_j[e]}(y).$$

Since  $\{(F_i, A_i)\}_{i \in I}$  is a directed subset of  $IFSF(L)$ , for all  $i, j \in I$ , there exists  $k \in I$  such that  $\mu_{F_i[e]}, \mu_{F_j[e]} \leq \mu_{F_k[e]}$  and  $\nu_{F_i[e]}, \nu_{F_j[e]} \geq \nu_{F_k[e]}$ .

(IF2) For all  $x, y \in L$ , we have

$$\mu_{F_i[e]}(x) \wedge \mu_{F_j[e]}(y) \leq \mu_{F_k[e]}(x) \wedge \mu_{F_k[e]}(y) \leq \mu_{F_k[e]}(x * y) \leq \bigvee_{i \in I} \mu_{F_k[e]}(x * y).$$

We obtain that  $\bigvee_{i \in I} \mu_{F_k[e]}(x) \wedge \bigvee_{i \in I} \mu_{F_k[e]}(y) \leq \bigvee_{i \in I} \mu_{F_k[e]}(x * y)$ .

(IF3) For all  $x, y \in L$ , we have

$$\nu_{F_i[e]}(x) \vee \nu_{F_j[e]}(y) \geq \nu_{F_k[e]}(x) \vee \nu_{F_k[e]}(y) \geq \nu_{F_k[e]}(x * y) \geq \bigwedge_{i \in I} \mu_{F_k[e]}(x * y).$$

We obtain that  $\bigwedge_{i \in I} \nu_{F_k[e]}(x) \vee \bigwedge_{i \in I} \nu_{F_k[e]}(y) \geq \bigwedge_{i \in I} \nu_{F_k[e]}(x * y)$ .

Hence  $(H, C)$  is an intuitionistic fuzzy soft filter over  $L$ .  $\square$

**Proposition 5.3.** Let  $X$  be a finite subset of  $L$ . Let  $t, s \in [0, 1]$  such that  $t + s = 1$ . The intuitionistic fuzzy soft set  $(F_{(X,t,s)}, A)$  defined to be such that for all  $e \in A$  and  $x \in L$ :

$$\mu_{F_{(X,t,s)}[e]}(x) = \begin{cases} t & \text{if } x \in \langle X \rangle \\ 0 & \text{if } x \in L \setminus \langle X \rangle \end{cases}$$

and

$$\nu_{F_{(X,t,s)}[e]}(x) = \begin{cases} s & \text{if } x \in \langle X \rangle \\ 1 & \text{if } x \in L \setminus \langle X \rangle. \end{cases}$$

Then  $(F_{(X,t,s)}, A)$  is an intuitionistic fuzzy soft filter over  $L$  which is called finite generated intuitionistic fuzzy soft filter induced by  $X$  over  $L$ .

*Proof.* For every  $\alpha, \beta \in [0, 1]$ , we have

$$U(\mu_{F_{(X,t,s)}[e]}, \alpha) = \begin{cases} \emptyset & \text{if } 1 \geq \alpha > t \\ \langle X \rangle & \text{if } t \geq \alpha > 0 \\ L & \text{if } \alpha = 0 \end{cases}$$

and

$$L(\nu_{F_{(X,t,s)}[e]}, \alpha) = \begin{cases} L & \text{if } \beta = 1 \\ < X > & \text{if } s < \beta < 1 \\ \emptyset & \text{if } 0 \leq \beta \leq s. \end{cases}$$

By Theorem 2.8, we have  $F_{(X,t,s)}[e]$  is an intuitionistic fuzzy filter in  $L$ , for all  $e \in A$ . Hence  $(F_{(X,t,s)}, A)$  is an intuitionistic fuzzy soft filter over  $L$ .  $\square$

**Theorem 5.4.** *Finite generated intuitionistic fuzzy soft filters over  $L$  are compact in  $IFSF(L)$ .*

*Proof.* Suppose that  $(F_{(X,t,s)}, A)$  is an arbitrary finite generated intuitionistic fuzzy soft filter induced by  $X$  over  $L$  where  $X$  is a finite subset of  $L$  and  $t, s \in [0, 1]$  such that  $t + s = 1$ .

Let  $\{(F_i, A_i)\}_{i \in I}$  be a directed subset of  $IFSF(L)$  such that it has the supremum  $\bigvee_{\alpha \in \Gamma} (F_\alpha, A_\alpha)$  and  $(F_{(X,t,s)}, A) \sqsubseteq \bigvee_{\alpha \in \Gamma} (F_\alpha, A_\alpha)$ . We will show that there exists  $\beta \in \Gamma$  such that  $(F_{(t,s)}, A) \sqsubseteq (F_\beta, A)$ .

By Proposition 5.2, we have  $\bigvee_{\alpha \in \Gamma} (F_\alpha, A_\alpha) = \bigsqcup_{\alpha \in \Gamma} (F_\alpha, A_\alpha) \in IFSF(L)$ .

If  $X = \emptyset$ , then it is clear that the assertion is true.

If  $X = \{x_1, \dots, x_n\} \neq \emptyset$ , then for  $x_i \in X$ , there exists  $\alpha_i \in \Gamma$  such that  $F_{(X,t,s)}[e](x_i) \subseteq F_{\alpha_i}[e](x_i)$ , that is

$$t = \mu_{F_{(X,t,s)}[e]}(x_i) \leq \mu_{F_{\alpha_i}[e]}(x_i) \text{ and } s = \nu_{F_{(X,t,s)}[e]}(x_i) \geq \nu_{F_{\alpha_i}[e]}(x_i).$$

Since  $\{(F_\alpha, A_\alpha)\}_{\alpha \in \Gamma} \in IFSF(L)$  is directed, there exists  $\beta \in \Gamma$  such that  $(F_{\alpha_i}, A) \sqsubseteq (F_\beta, A)$  for  $1 \leq i \leq n$ . Then for all  $x \in X$ , there exist  $x_{k_1}, \dots, x_{k_m} \in X$  such that  $x_{k_1} * \dots * x_{k_m} = x$ . We obtain that

$$t \leq \bigwedge_{i=1}^m \mu_{F_{\alpha_{k_i}}[e]}(x_{\alpha_{k_i}}) \leq \bigwedge_{i=1}^m \mu_{F_\beta[e]}(x_{\alpha_{k_i}}) \leq \mu_{F_\beta[e]}(x)$$

and

$$s \geq \bigvee_{i=1}^m \nu_{F_{\alpha_{k_i}}[e]}(x_{\alpha_{k_i}}) \geq \bigvee_{i=1}^m \nu_{F_\beta[e]}(x_{\alpha_{k_i}}) \geq \nu_{F_\beta[e]}(x).$$

Hence there exists  $\beta \in \Gamma$  such that  $(F_{(X,t,s)}, A) \sqsubseteq (F_\beta, A)$ , that is finite generated intuitionistic fuzzy soft filters over  $L$  are compact in  $IFSF(L)$ .  $\square$

**Proposition 5.5.** *Any intuitionistic fuzzy soft filters over  $L$  is a directed supremum of compact elements of  $IFSF(L)$ .*

*Proof.* Let  $(F, A) \in IFSF(L)$ . We show the set of all finite subset of  $L$  by  $P$ . We will prove that  $(F, A) = \bigvee_{X \in P} (F_{(X,t,s)}, A)$ . Suppose that  $x \in L$  be arbitrary. Put  $t = \mu_{F[e]}(x)$  and  $s = \nu_{F[e]}(x)$ . We get that

$$\mu_{F[e]}(x) = \mu_{F_{(X,t,s)}[e]}(x) \text{ and } \nu_{F_{(X,t,s)}[e]}(x) = \nu_{F[e]}(x).$$

Hence

$$\mu_{F[e]} \leq \bigvee_{X \in P} \mu_{F_{(X,t,s)}[e]} \text{ and } \nu_{F[e]} \geq \bigwedge_{X \in P} \nu_{F_{(X,t,s)}[e]}.$$

It is clear that

$$\mu_{F[e]} \geq \bigvee_{X \in P} \mu_{F_{X,t,s}[e]} \text{ and } \nu_{F[e]} \leq \bigwedge_{X \in P} \nu_{F_{X,t,s}[e]}.$$

We obtain that  $(F, A) = \bigvee_{X \in P} (F_{(X,t,s)}, A)$ . By Theorem 5.4,  $(F, A)$  is a directed supremum of compact elements of  $IFSF(L)$ .  $\square$

**Corollary 5.6.** *The complete lattice  $(IFSF(L), \sqsubseteq)$  is an algebraic lattice.*

*Proof.* It follows from Proposition 5.2, Proposition 5.3 and Theorem 5.4.  $\square$

**Theorem 5.7.** *Let  $(F, A)$  be an intuitionistic fuzzy soft set over  $L$ . Then for all  $e \in A$  and  $x \in L$ ,*

- (1)  $\mu_{F[e]}(x) = \bigvee \{\mu_{F[e]}(a_1 * \dots * a_n) : x \geq a_1 * \dots * a_n\},$
- (2)  $\nu_{F[e]}(x) = \bigwedge \{\nu_{F[e]}(a_1 * \dots * a_n) : x \geq a_1 * \dots * a_n\}.$

*Proof.* (1) Let  $x \geq a_1 * \dots * a_n$ . Then  $\mu_{F[e]}(x) \geq \mu_{F[e]}(a_1 * \dots * a_n)$ . Thus

$$\mu_{F[e]}(x) \geq \bigvee \{\mu_{F[e]}(a_1 * \dots * a_n) : x \geq a_1 * \dots * a_n\}.$$

On the other hand, we have  $x = 1 * x$ . We get that

$$\mu_{F[e]}(x) = \mu_{F[e]}(1 * x) \leq \bigvee \{\mu_{F[e]}(a_1 * \dots * a_n) : x \geq a_1 * \dots * a_n\}.$$

- (2) Let  $x \geq a_1 * \dots * a_n$ . We have  $\nu_{F[e]}(x) \leq \nu_{F[e]}(a_1 * \dots * a_n)$ . Thus

$$\nu_{F[e]}(x) \leq \bigwedge \{\nu_{F[e]}(a_1 * \dots * a_n) : x \geq a_1 * \dots * a_n\}.$$

On the other hand, we have  $x = 1 * x$ . We obtain that

$$\nu_{F[e]}(x) = \nu_{F[e]}(1 * x) \geq \bigwedge \{\nu_{F[e]}(a_1 * \dots * a_n) : x \geq a_1 * \dots * a_n\}.$$

$\square$

**Theorem 5.8.** *The complete lattice  $(IFSF(L), \sqsubseteq)$  is distributive.*

*Proof.* Let  $(F, A), (G, B), (H, C) \in IFSF(L)$  be arbitrary.

Suppose that  $(F, A) \sqcap (G, B) \sqcup (H, C) = (K_1, D_1)$ , where  $D_1 = A \cap (B \cup C)$ , and for all  $e \in D_1$ ,

$$K_1[e] = \begin{cases} F[e] & \text{if } e \in A \cap (B \setminus C) \\ G[e] & \text{if } e \in A \cap (C \setminus B) \\ F[e] \cap G[e] \cup H[e] & \text{if } e \in A \cap (B \cap C). \end{cases}$$

Assume that  $(F, A) \sqcap (G, B) \sqcup ((F, A) \sqcap (H, C)) = (K_2, D_2)$ , where  $D_2 = (A \cap B) \cup (A \cap C)$ , and for all  $e \in D_2$ ,

$$K_1[e] = \begin{cases} F[e] & \text{if } e \in (A \cap B) \setminus (A \cap C) \\ G[e] & \text{if } e \in (A \cap C) \setminus (A \cap B) \\ (F[e] \cap G[e]) \cup (F[e] \cap H[e]) & \text{if } e \in (A \cap B) \cap (A \cap C). \end{cases}$$

We know that  $F[e], G[e]$  and  $H[e]$  are intuitionistic fuzzy soft filters in  $L$  and  $D_1 = D_2$ . Consider the following cases :

Case 1 : If  $e \in A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$ , then  $K_1[e] = F[e] = K_2[e]$ .

Case 2 : If  $e \in A \cap (C \setminus B) = (A \cap C) \setminus (A \cap B)$ , then  $K_1[e] = G[e] = K_2[e]$ .

Case 3 : Let  $e \in A \cap (B \cap C) = (A \cap B) \cap (A \cap C)$ . We will show that  $F[e] \cap < G[e] \cup H[e] > = < (F[e] \cap G[e]) \cup (F[e] \cap H[e]) >$ . We have

$$\begin{aligned} \mu_{F[e] \cap < G[e] \cup H[e] >}(x) &= \mu_{F[e]}(x) \wedge (\mu_{< G[e] \cup H[e] >})(x) \\ &= \mu_{F[e]}(x) \wedge \bigvee \{ \mu_{G[e] \cup H[e]}(x) : x \geq a_1 * \dots * a_n \} \\ &= \bigvee \{ \mu_{F[e]}(x) \wedge \mu_{G[e] \cup H[e]}(x) : x \geq a_1 * \dots * a_n \} \\ &= \bigvee \{ \mu_{F[e]}(x) \wedge (\mu_{G[e]} \vee \mu_{H[e]})(x) : x \geq a_1 * \dots * a_n \} \\ &= \bigvee \{ ((\mu_{F[e]} \wedge \mu_{G[e]}) \vee (\mu_{F[e]} \wedge \mu_{H[e]}))(x) : x \geq a_1 * \dots * a_n \} \\ &= \mu_{< (F[e] \cap G[e]) \cup (F[e] \cap H[e]) >}(x). \end{aligned}$$

and

$$\begin{aligned} \nu_{F[e] \cap < G[e] \cup H[e] >}(x) &= \nu_{F[e]}(x) \vee (\nu_{< G[e] \cup H[e] >})(x) \\ &= \nu_{F[e]}(x) \vee \bigwedge \{ \nu_{G[e] \cup H[e]}(x) : x \geq a_1 * \dots * a_n \} \\ &= \bigwedge \{ \nu_{F[e]}(x) \vee \nu_{G[e] \cup H[e]}(x) : x \geq a_1 * \dots * a_n \} \\ &= \bigwedge \{ \nu_{F[e]}(x) \vee (\nu_{G[e]} \wedge \nu_{H[e]})(x) : x \geq a_1 * \dots * a_n \} \\ &= \bigwedge \{ ((\nu_{F[e]} \vee \nu_{G[e]}) \wedge (\nu_{F[e]} \vee \nu_{H[e]}))(x) : x \geq a_1 * \dots * a_n \} \\ &= \nu_{< (F[e] \cap G[e]) \cup (F[e] \cap H[e]) >}(x). \end{aligned}$$

Therefore  $(F, A) \wedge ((G, B) \vee (H, C)) = ((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C))$ .  $\square$

## 6. CONCLUSIONS

In this paper, we applied the notion of intuitionistic fuzzy soft set to residuated lattices and we have introduced the notion of (generated) intuitionistic fuzzy soft filters of residuated lattices. Also, We have presented some properties of them. Furthermore, we proved that the set of all intuitionistic fuzzy soft filters of a residuated lattice is a distributive algebraic lattice and we showed that finite generated intuitionistic fuzzy soft filters over a residuated lattice are compact element of this lattice.

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