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Some Reviews in fuzzy subgroups and anti fuzzy subgroups

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Abstract. This paper is basically a review of some fundamental works in fuzzy group theory. The aim is to provide some new or alternative methods to proving some existing theorems. It also seeks to provide proofs of some theorems which, to the very best of our knowledge, do not exist in the literature. Along the line, it shall include some new results. The major focus of this research is on the properties of fuzzy subgroup, fuzzy cosets, fuzzy conjugacy, fuzzy relations, fuzzy normal subgroups and some properties of anti fuzzy subgroup of a cyclic group.

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1. Introduction

Researches in fuzzy group theory are actually on the wings of the work of Zadeh [17]. In his paper of 1965, "Fuzzy Sets", he worked on the basic properties and operations on fuzzy sets in which every element in a set X is assigned a grade membership value within the interval [0, 1] via a membership function as opposed to the the crisp set theory in which every member is assigned either 1, if it belongs to, or 0, if it does not belong to X.

This idea was extended and given a group structure by Rosenfeld [11] in 1971. This led to the concept of fuzzy subgroupoid and fuzzy subgroup. Furthermore, Das [4] worked on fuzzy groups and developed the concept of level subgroups. Also, Wu Wangming (1981) introduced the concept of fuzzy normal subgroups.

In 1984, Mukherjee and Bhattacharya [8] introduced the concept of fuzzy left and right cosets. Many other authors like H. V. Kumbhojkar, M. Asaad and Malik et al have notable contributions to knowledge in the field of fuzzy group theory. The concept of fuzzy normal subgroup was improved on by the work of Vasantha and Meiyappan [16] in 1997. They also introduced new concepts such as fuzzy middle cosets, pseudo fuzzy cosets and pseudo fuzzy double coset. In 2010 and 2013, Nagarajan and Solairaju [9], Shobha [14] and K. H. Manikandan and R. Muthuraj [6] also studied pseudo fuzzy cosets and related concepts.

However, the whole idea of fuzzy subgroup was extended to anti fuzzy subgroup by Biswas [3] in 1990 and Saleem et al [2] in 2015 has found a new way to generalise fuzzy subgroup and level subsets.

2. Preliminaries

Definition 2.1 ([15])**.** Let X be a non-empty set. A fuzzy subset μ of the set X is a function $\mu : X \to [0, 1]$.

Definition 2.2 ([11, 15])**.** Let G be a group and μ a fuzzy subset of G. Then μ is called a fuzzy subgroup of G if

(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(x^{-1}) = \mu(x)$

(iii) μ is called a fuzzy normal subgroup of G if $\mu(xy) = \mu(yx)$ for all x and y in G.

Definition 2.3 ([15])**.** Let μ and λ be any two fuzzy subsets of a set X. Then

(i) λ and μ are equal if $\mu(x) = \lambda(x)$ for every x in X

(ii) λ and μ are disjoint if $\mu(x) \neq \lambda(x)$ for every x in X

(iii) $\lambda \subseteq \mu$ if $\mu(x) \geq \lambda(x)$

Definition 2.4 ([15, 16])**.** Let μ be a fuzzy subset (subgroup) of X. Then, for some $t \in [0, 1]$, the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a t-level subset (subgroup) of the fuzzy subset (subgroup) μ.

Remark 2.5. If μ is a fuzzy subgroup of G, then μ_t is a subgroup of G and the set μ_t can be represented as G^t_μ.

Definition 2.6 ([16])**.** Let μ be a fuzzy subgroup of a group G. The set $H = \{x \in G : \mu(x) = \mu(e)\}$ is such that $o(\mu) = o(H)$ and the order of μ is $\mid H \mid$.

Theorem 2.7 ([11])**.** *Let G be a group and μ a fuzzy subset of G. Then μ is called a fuzzy subgroup of G if and only if $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$.*

Definition 2.8 ([8, 15])**.** Let μ be a fuzzy subgroup of a group G. For a in G, the fuzzy coset $a\mu$ of G determined by a and μ is defined by $(a\mu)(x) = \mu(a^{-1}x)$ for all x in G.

Definition 2.9 ([15])**.** The Cartesian product $\lambda \times \mu : X \times Y \longrightarrow [0, 1]$ of two fuzzy subgroups of G is defined by $(\lambda \times \mu)(x, y) = min\{\lambda(x), \mu(y)\}$ for all $(x, y) \in X \times Y$, where X and Y are crisp subgroups of G. R_λ is a binary fuzzy relation defined by $R_\lambda(x, y) = min\{\lambda(x), \lambda(y)\}$ for all $(x, y) \in X \times X$ and if $min\{\lambda(x), \lambda(y)\} \leq R_\mu(x, y)$ $\forall (x, y) \in X \times X$, λ is said to be a pre class of R_μ. Furthermore, R_λ is said to be a similarity relation if

(i) $R_\lambda(x, x) = 1$

(ii) $R_\lambda(x,y) = R_\lambda(y,x)$
(iii) $\min\{R_\lambda(x,y),\ R_\lambda(y,z)\} \leq R_\lambda(x,z)$

Definition 2.10. Let G be a finite group of order n and μ a fuzzy subgroup of G. Then for $t_1, t_2 \in [0,1]$ such that $t_1 \leq t_2, \mu_{t_2} \subseteq \mu_{t_1}$.

Proposition 2.11 ([4])**.** *Let G be a group and μ a fuzzy subset of G. Then μ is a fuzzy subgroup of G if and only if G_μ^t is a subgroup of G for every t in $[0, \mu(e)]$, where e is the identity of G.*

Proposition 2.12 ([4])**.** *H as described in 2.6 can be realized as a level subgroup.*

Theorem 2.13 ([8])**.** *Let μ be a fuzzy normal subgroup of a group G. Let $t \in [0,1]$ such that $t \leq \mu(e)$, where e is the identity of G. Then G_μ^t is a normal subgroup of G.*

Theorem 2.14 ([11, 1])**.** *Let G be a cyclic group of prime order. Then there exists a fuzzy subgroup μ of G such that $\mu(e) = t_o$ and $\mu(x) = t_1$ for all $x \neq e$ in G and $t_1 < t_o$.*

Proposition 2.15 ([1])**.** *Let G be a group of prime power order. Then G is cyclic if and only if there exists a fuzzy subgroup μ of G such that for $x, y \in G$,*

(i) *if $\mu(x) = \mu(y)$ then $\langle x \rangle = \langle y \rangle$*
(ii) *$\mu(x) > \mu(y)$ then $\langle x \rangle \subset \langle y \rangle$*

Theorem 2.16 ([1])**.** *Let G be a group of square free order. Let μ be a normal fuzzy subgroup of G. Then for $x, y \in G$,*

(i) *if $o(x)|o(y)$ then $\mu(y) \leq \mu(x)$*
(ii) *$o(x) = o(y)$ then $\mu(y) = \mu(x)$*

Definition 2.17 ([7, 15])**.** The sets $\underline{\mu_t}(A) = \{x \in G : [x]_\mu \subseteq A\}$ and $\overline{\mu_t}(A) = \{x \in G : [x]_\mu \cap A \neq \varnothing\}$ are respectively called the lower and upper approximations of the set A with respect to μ, where μ_t is a normal subgroup of G and the congruence class of μ_t containing an element $x \in G$ is denoted by $[x]_\mu$.

Note that if $\underline{\mu_t}(A)$ and $\overline{\mu_t}(A)$ are subgroups of G they are called lower and upper rough subgroups of G respectively. We can denote rough upper subgroup of G by $G_\mu^{\hat{t}}(A)$.

Definition 2.18 ([3])**.** Let G be a group. A fuzzy subset μ is an anti fuzzy subgroup of G if $\forall x, y \in G, \mu(xy^{-1}) \leq max\{\mu(x), \mu(y)\}$.

Proposition 2.19 ([3])**.** *Let G be a group of finite order and μ an anti fuzzy subgroup of G. Then, for $t_1, t_2 \in [0,1]$ such that $t_1 \leq t_2$, $\mu_{\underline{t_1}} \subseteq \mu_{\underline{t_2}}$, where $\mu_{\underline{t}} = \{x \in G : \mu(x) \leq t\}$ is called a lower level subgroup of G.*

Definition 2.20 ([15])**.** Let λ and μ be any two fuzzy subgroups of a group G. They are said to be conjugate fuzzy subgroups of G if for some $g \in G$, $\lambda(x) = \mu(g^{-1}xg)$ for all $x \in G$.

3. Some properties of fuzzy subgroups and relations

Proposition 3.1. *Every subgroup H of G can be realised as a level subgroup of some fuzzy subgroup μ of G.*

Das (1981) worked on this but this is an alternative way to doing the same thing. As opposed to Das' definition of membership value of members of H, elements of H may not have the same membership value $t \in [0,1]$ since μ is a fuzzy subset.

Proof. Let $H \leq G$. Let μ be a fuzzy subset of G such that if $x_i \in H, \mu(x_i) = t_i \in (0,1]$ and if $x_j \notin H, \mu(x_j) = 0$. So we can always have $t \in (0,1]$ such that $t \leq t_i \ \forall i$. Such $t = min\{t_i = \mu(x_i)\} \ \forall x_i \in G$. This means that $H = \{x \in G : \mu(x) \geq t\}$. For any $x, y^{-1} \in H$, $xy^{-1} \in H$ and $\mu(xy^{-1}) \geq t = \min\{\mu(x), \mu(y)\}$. Then μ is a fuzzy subgroup of G and $H = H_\mu^t$ by proposition 2.11 is a level subgroup. □

Remark 3.2. If we choose $t = max\{t_i = \mu(x_i)\} \ \forall x_i \in G$, then $H = \{x \in G : \mu(x) \leq t\}$ and it can be shown that μ is an anti fuzzy subgroup. In that case, it is easy to show that any subgroup H of G can be realised as a lower level subgroup of an anti fuzzy subgroup μ of G. Since for any $x, y^{-1} \in H$, $xy^{-1} \in H$ and $\mu(xy^{-1}) \leq t = \max\{\mu(x), \mu(y)\}$, by 2.18, μ is an anti fuzzy subgroup of G. By 2.19 H is a lower level subgroup of G.

The following is the proof of a problem posed by [15]. The fact that the intersection of any two fuzzy subgroups is also a fuzzy subgroup by [12] was used in proving this theorem.

Proposition 3.3. *Let λ and μ be any two fuzzy subgroups of G and*

$$R_{\mu\cap\lambda} : G \times G \longrightarrow [0,1]$$

be given by $R_{\mu\cap\lambda}(x,y) = (\mu \cap \lambda)(xy^{-1})$ for every $x, y \in G$. If $\rho \subseteq \mu \cap \lambda$ is any fuzzy subset of G, then ρ is a pre class of $R_{\mu\cap\lambda}$.

Proof.

$$\begin{aligned}
&min\{\rho(x), \rho(y)\} \\
&\leq min\{(\mu\cap\lambda)(x), (\mu\cap\lambda)(y)\} \\
&= min\{min\{\mu(x), \lambda(x)\}, min\{\mu(y), \lambda(y)\}\} \\
&= min\{min\{\mu(x), \lambda(x)\}, min\{\mu(y^{-1}), \lambda(y^{-1})\}\} \\
&= min\{(\mu\cap\lambda)(x), (\mu\cap\lambda)(y^{-1})\} \\
&\leq (\mu\cap\lambda)(xy^{-1}) = R_{\mu\cap\lambda}(x,y).
\end{aligned}$$

Then $min\{\rho(x), \rho(y)\} \leq R_{\mu\cap\lambda}(x,y)$. □

4. Fuzzy cosets and conjugates

Proposition 4.1. *The fuzzy coset $a\mu$ is not a fuzzy subgroup of G for all $x \in G$ and so the order of $a\mu$, $o(a\mu)$ is not defined.*

[15] made this claim but did not verify.

Proof. If on the contrary it is assumed that $a\mu$ is a fuzzy subgroup in general. If it can be shown by a counter example that the order of $a\mu$, which is $o(a\mu)$, is not defined then it suffices. Let $G = S_3$ and $p_1, p_2, p_3 \in [0, 1]$ such that $1 \geq p_1 \geq p_2 \geq p_3 \geq 0$ and define $\mu : G \longrightarrow [0, 1]$ by

$$\mu(x) = \begin{cases} p_1, \text{ if } x = e \\ \\ p_2, \text{ if } x = (12) \\ \\ p_3, \ otherwise \end{cases}$$

Let $a = (13)$. Then

$$(a\mu)(x) = \begin{cases} p_1, \text{ if } x = (13) \\ \\ p_2, \text{ if } x = (132) \\ \\ p_3, \ otherwise. \end{cases}$$

Obviously, the order of set $H = \{x \in G : (a\mu)(x) = (a\mu)(e) = p_3\}$ is 4. Note that $|G| = 6$ and H is supposed to be a subgroup of G. By Lagrange Theorem, $|H|$ divides $|G|$. But no such $n \in \mathbf{Z}$ such that $|G| = n|H|$. □

The following theorem is an extension of THEOREM [139] of page 14 and THEOREM 1.2.13 of [15]. Note that a fuzzy subgroup is improper if it is constant on the group.

Proposition 4.2. *Let μ and λ be any two improper conjugate fuzzy subgroups of G. The $o(\lambda) = o(\mu) = o(G)$*

Proof. Since μ and λ are fuzzy conjugates and constant on G, there is a $g \in G$ such that $\mu(x) = \lambda(g^{-1}xg) = \lambda(x) = \lambda(e)$ is true for some g and all $x \in G$. Thus $o(\lambda) = o(\mu) = o(G)$, by 2.6. □

The following is an extension of THEOREM 1.2.1.4 as stated by [15]. It was stated for fuzzy subsets of an abelian group but herein generalized to any two fuzzy subgroups on any group.

Theorem 4.3. *Let μ and λ be any two fuzzy subgroup of any group G. Then, μ and λ are conjugate fuzzy subgroup of G if and only if $\mu = \lambda$.*

Proof. Assume that μ and λ are conjugate fuzzy subgroups of G. Then for $y, g \in G$, $\lambda(x) = \mu((gy)^{-1}x(gy)) = \mu(y^{-1}(g^{-1}xg)y) = \lambda(g^{-1}xg) = \mu(x)$. Thus $\mu = \lambda$.
Conversely, assume that $\mu = \lambda$. For some $g = e \in G$, $\mu(x) = \lambda(x) = \lambda(e^{-1}xe)$. Then μ and λ are conjugate fuzzy subgroups of G. □

The foregoing shows that such λ or μ is self conjugate. This leads to the result that a fuzzy subgroup μ of a group G is self conjugate if and only if G is an abelian group.

5. Approximations, fuzzy conjugates and fuzzy normal subgroups

In this section, A is referred to as a none empty subset of G. The congruence class of μ_t containing an element $x \in G$ is denoted by $[x]_\mu$. This is an aspect of interplay between rough set and fuzzy set.

Theorem 5.1. *Let μ be a fuzzy subgroup of G. The congruence class of μ_t containing an element $x \in G$ denoted by $[x]_\mu$ exists only when μ is a fuzzy normal subgroup of G.*

Kuroki and Wang (1991) have shown that indeed the t-level relation on $[x]_\mu$ is a congruence relation. [16] by an example showed that if μ is not fuzzy normal the congruence class $[x]_\mu$ does not exist. This is a general proof of the theorem.

Proof. Let μ be fuzzy normal on G. Note that every element of a subgroup N of G is also in G. Then μ is fuzzy normal on N. Thus N_μ^t is a normal subgroup of N, by 2.13. So there is an $x \in N_\mu^t$ and $g \in G$ such that $gxg^{-1} = x$ is in N_μ^t. Hence gN_μ^t is a coset of N_μ^t. These cosets are the congruence classes, that is $[x]_\mu = gN_\mu^t$ (for the level relation μ_t) containing x only if μ is fuzzy normal. □

Consequent upon this theorem 5.1, the following theorem can be proved. [15] illustrates it by an example but here is provided a general proof.

Theorem 5.2. *Let μ be a fuzzy subgroup of G and $t \in [0, 1]$. Then for every $x \in G$, $[x]_\mu = xG_\mu^t$ and G_μ^t is a normal subgroup of G.*

Proof. Since the classes $[x]_\mu$ exist, μ is a fuzzy normal subgroup of G and $[x]_\mu = xG_\mu^t$, by theorem 5.1. Also, this is only when μ is fuzzy normal on G_μ^t so that $G_\mu^t x = xG_\mu^t$. Hence G_μ^t is a normal subgroup of G. □

Theorem 5.3. *Let μ be a fuzzy normal subgroup of a group G and $t \in [0, 1]$ and A be a non-empty subset of G. Then*

(i) $\underline{\mu_t}(A) = G_\mu^t(A)$,

(ii) $\hat{\mu}_t(A) = G_\mu^{\hat{t}}(A)$,

where G_μ^t is a normal subgroup of G.

The following is an independent proof of Theorem 5.3.

Proof. (i) Define $\underline{\mu_t}(A) = \{x \in G : [x]_\mu \subseteq A\}$ and $G_\mu^t(A) = \{x \in G : \mu(x) \geq t \text{ and } x \in A\}$. Hence, $x \in G_\mu^t$ and $G_\mu^t = xG_\mu^t = [x]_\mu$ is a normal subgroup by theorem 5.2. Since $x \in [x]_\mu, G_\mu^t$ and also in xG_μ^t, then they are all the same t-level subgroup partition of G. Hence, $G_\mu^t(A) = \{x \in G : \mu(x) \geq t \text{ and } x \in [x]_\mu, x \in A, [x]_\mu \subseteq A\}$ in which case $\underline{\mu_t}(A) = G_\mu^t(A)$.

(ii) Note that $\hat{\mu}_t(A)$ is a rough subgroup of G. Define $\overline{\mu_t}(A) = \{x \in G : [x]_\mu \cap A \neq \varnothing\}$ and $\overline{G_\mu^t}(A) = \{x \in G : xG_\mu^t \cap A \neq \varnothing\}$. Since $[x]_\mu = G_\mu^t = xG_\mu^t$, then $\overline{G_\mu^t}(A) = \overline{\mu_t}(A)$. Thus, $\hat{\mu}_t(A) = G_\mu^{\hat{t}}(A)$. □

Theorem 5.4. *Let μ be a fuzzy normal subgroup of G, Then for any $g \in G$, $\mu(gxg^{-1}) = \mu(g^{-1}xg)$ for every $x \in G$.*

The following is an independent proof of Theorem 5.4.

Proof. G is a group and so associativity holds. Then, $\mu(gxg^{-1}) = \mu((gx)g^{-1})$ so that $\mu(gxg^{-1}) = \mu(g^{-1}(gx)) = \mu(x) = \mu(ex) = \mu((gg^{-1})x) = \mu(g(g^{-1}x)) = \mu((g^{-1}x)g) = \mu(g^{-1}xg)$ by 2.2(iii). □

6. Some properties of anti fuzzy subgroup of a group

The following two theorems are the anti fuzzy versions of some theorems proposed by [11] and [1]

Theorem 6.1. *Let G be a cyclic group of prime order. Then there exists an anti fuzzy subgroup μ of G such that $\mu(e) = t_o$ and $\mu(x) = t_1$ for all $x \neq e$ in G and $t_1 > t_o$.*

Proof. Let $x \neq e$ be in G and p a prime order of G.
If $p = 2$, $x = x^{-1}$ and $\mu(e) = \mu(x^2) \leq max\{\mu(x), \mu(x)\} = \mu(x)$.
Thus,

$$t_1 = \mu(x) \geq \mu(e) = t_o.$$

For $p > 2$, $e = x^p = x^{p+1}x^{-1}$. Hence, $x^{p+1} = x \neq x^{-1}$ and $x \neq e \neq x^{-1}$.
But

$$t_1 = \mu(x^{-1}) = \mu(x) \geq \mu(e) = t_o.$$

Thus, for $p \geq 2$, $t_1 = \mu(x) \geq \mu(e) = t_o$. Indeed, for $x \neq e$, $t_1 = \mu(x) > \mu(e) = t_o$ is the appropriate choice. For if $\mu(x) = \mu(e)$, we can find some $t \in (0, 1]$ such that $\mu(x) = \mu(e) = t_o \leq t$. Then x and e are in the same lower level subgroup of G, say $\mu_{\underline{t_o}} \subseteq \mu_{\underline{t}}$, which also must be a (cyclic) subgroup of G. Hence, there is an $n \in \mathbf{Z}$, a set of integers, such that $e = x^n$, where n is the order of $\mu_{\underline{t}}$ and $n \neq p$. By Lagrange theorem, $n \mid p$, which is not possible since p is prime. Alternatively, note that since $x \in \mu_{\underline{t}}$ so is x^{-1} and we can have $(x^{-1})^n = e$ also. Then $x^{-n} = e = x^n$ and if $y = x^n$, it implies that $y^{-1} = x^{-n} = x^n = y$. This contradicts our assumption that for $p > 2, x \neq x^{-1}$. □

Theorem 6.2. *Let G be a finite group of prime power order. If G is cyclic and there is an anti fuzzy subgroup of G such that for $x, y \in G$, we have*

(i) $\mu(x) = \mu(y)$, *then* $\langle x \rangle = \langle y \rangle$
(ii) $\mu(x) > \mu(y)$, *then* $\langle x \rangle \supset \langle y \rangle$.

Proof. (i) Let μ be an anti fuzzy subgroup of G and $\mu_{\underline{t_x}}$ and $\mu_{\underline{t_y}}$ be the subgroups generated by x and y respectively. Since G is cyclic, they are both cyclic and we can write $\mu_{\underline{t_x}} = \langle x \rangle$ and $\mu_{\underline{t_y}} = \langle y \rangle$. Hence, $\exists\, n \in \mathbf{Z}$ such that $x^{p^n} = e \in \mu_{\underline{t}}$, where p is a prime. Thus, $\mu(x^{p^n}) = \mu(e)$. Let $y \in G$ be such that $\mu(x) = \mu(y)$. Hence, $\mu(y^{p^n}) = \mu(x^{p^n}) = \mu(e) \leq t$. For any $x \in \langle x \rangle$, $\mu(x) = \mu(y) \leq t_y$. Thus, $x \in \mu_{\underline{t_y}} = \langle y \rangle$ which implies that $\langle x \rangle \subseteq \langle y \rangle$. Also, for any $y \in \langle y \rangle$, $\mu(y) = \mu(x) \leq t_x$. Thus, $y \in \mu_{\underline{t_x}} = \langle x \rangle$ which implies that $\langle y \rangle \subseteq \langle x \rangle$. Then, $\langle x \rangle = \langle y \rangle$ so that x and y generate the same (lower level) subgroup.

(ii) Assume $\mu(x) > \mu(y)$ for $x, y \in G$. Let $\mu_{\underline{t_x}}$ and $\mu_{\underline{t_y}}$ be the subgroups generated by x and y respectively. Then, $t_x \geq \mu(x) > \mu(y) \Longrightarrow t_x > t_y \Longrightarrow \mu_{\underline{t_x}} \supset \mu_{\underline{t_y}}$ and $\langle x \rangle \supset \langle y \rangle$ by proposition 2.19.

□

Remark 6.3. It is a known fact that the intersection of any two fuzzy (or anti fuzzy) subgroups of a group is also a fuzzy (or an anti fuzzy)subgroup by [12] in 2005 (by [13] in 2008). Also, it is a known fact that the union of any two non equivalent fuzzy subgroups of a group is a fuzzy subgroup as illustrated with an example by [5].

The result of [3] that μ is a fuzzy subgroup if and only if its complement is an anti fuzzy subgroup can be used to investigate whether the union of any two anti fuzzy subgroups of a group is an anti fuzzy subgroup or not.

Proposition 6.4. *Let μ and λ be any anti fuzzy subgroups of a group. Then, $\mu \cup \lambda$ is an anti fuzzy subgroup.*

Proof. By 6.3, $1 - \mu = \mu'$ and $1 - \lambda = \lambda'$ are fuzzy subgroups. Hence,

$$(\mu' \cap \lambda')(xy^{-1}) \geq min\{(\mu' \cap \lambda')(x), (\mu' \cap \lambda')(y)\}$$

and by De Morgan's law,

$$(\mu \cup \lambda)'(xy^{-1}) \geq min\{(\mu \cup \lambda)'(x), (\mu \cup \lambda)'(y)\}$$

which leads to

$$1 - (\mu \cup \lambda)(xy^{-1}) \geq min\{1 - (\mu \cup \lambda)(x), 1 - (\mu \cup \lambda)(y)\}$$

and

$$(\mu \cup \lambda)(xy^{-1}) \leq 1 - min\{1 - (\mu \cup \lambda)(x), 1 - (\mu \cup \lambda)(y)\}$$

so that

$$(\mu \cup \lambda)(xy^{-1}) \leq max\{(\mu \cup \lambda)(x), (\mu \cup \lambda)(y)\}.$$

It can be concluded that the union of any two anti fuzzy subgroups is also an anti fuzzy subgroup. □

Proposition 6.5. *Let μ be a fuzzy subgroup of a group G. Both μ and its complement (which is an anti fuzzy subgroup of G) have the same normalizer.*

Proof. By [8], $N(\mu) = \{a \in G : \mu(axa^{-1}) = \mu(x)\}$ is the normalizer of the fuzzy subgroup μ. From $\mu(axa^{-1}) = \mu(x)$ we have $1 - \mu(axa^{-1}) = 1 - \mu(x)$. This leads to $\mu'(axa^{-1}) = \mu\prime(x)$, where μ' is an anti fuzzy subgroup of G by 6.3.
Thus,

$$N(\mu) = \{a \in G : \mu(axa^{-1}) = \mu(x)\} = \{a \in G : \mu'(axa^{-1}) = \mu'(x)\} = N(\mu').$$

□

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