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Fuzzy pairwise e-continuous mappings on fuzzy bitopological spaces

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Abstract. In this paper, we define the concept of (τ_i, τ_j)-fuzzy e-open ((τ_i, τ_j)- fuzzy e-closed) set on fuzzy bitopological spaces and study some of their properties. And we introduce the (τ_i, τ_j)-fuzzy e-interiors ((τ_i, τ_j)-fuzzy e-closures) and discuss the characteristic properties of fuzzy pairwise e-continuous (fuzzy pairwise e-open, fuzzy pairwise e-closed) mapping on fuzzy bitopological spaces.

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1. Introduction

Kandil [5] introduced the notion of fuzzy bitopological space as a generalization of a fuzzy topological space. Thakur and Malviya [10] defined the concepts of (τ_i, τ_j)-fuzzy semiopen ((τ_i, τ_j)-fuzzy semiclosed) set and (τ_i, τ_j)-fuzzy preopen ((τ_i, τ_j)-fuzzy preclosed) set, and studied fuzzy pairwise semicontinuous (fuzzy pairwise semiopen, fuzzy pairwise semiclosed) mappings and fuzzy pairwise precontinuous (fuzzy pairwise preopen, fuzzy pairwise preclosed) mapping on fuzzy bitopological space. Also, Im et al. [4, 7] defined fuzzy pairwise β-continuous mapping on fuzzy bitopological spaces and studied some of their properties. It was shown that every fuzzy pairwise semicontinuous mapping is a fuzzy pairwise β-continuous mapping and every fuzzy pairwise precontinuous mapping is a fuzzy pairwise β-continuous mapping. But the converses are not true in general. Seenivasan and Kamala [9] defined the concept of fuzzy e-open set, and studied fuzzy e-continuous mappings on fuzzy topological spaces.

2. Preliminaries

Let λ be a fuzzy subset of a space X. The fuzzy closure of λ, fuzzy interior of λ, fuzzy δ-closure of λ and the fuzzy δ-interior of λ with respect to topology τ_i (i =1, 2) are denoted by τ_i-$Cl(\lambda)$, τ_i-$Int(\lambda)$, τ_i-$Cl_\delta(\lambda)$ and τ_i-$Int_\delta(\lambda)$ respectively. A fuzzy subset λ of space X is called fuzzy regular open [3] (resp. fuzzy regular closed) if $\lambda =\tau_i$- $Int(\tau_i$-$Cl(\lambda))$ (resp. $\lambda = \tau_i$-$Cl(\tau_i$-$Int(\lambda))$. Now τ_i-$Cl(\lambda)$ and τ_i-$Int(\lambda)$ are defined as follows τ_i-$Cl(\lambda) = \wedge\{\mu : \mu \geq \lambda,\ \mu$ is τ_i-fuzzy closed in $X\}$ and τ_i-$Int(\lambda) = \vee\{\mu : \mu \leq \lambda,\ \mu$ is τ_i-fuzzy open in $X\}$. The fuzzy δ-interior of a fuzzy subset λ of X is the union of all fuzzy regular open sets contained in λ. A fuzzy subset λ is called fuzzy δ-open [6] if $\lambda = \tau_i$-$Int_\delta(\lambda)$. The complement of fuzzy δ-open set is called fuzzy δ-closed (i.e, $\lambda = \tau_i$-$Cl_\delta(\lambda)$).

In this paper, we define (τ_i, τ_j)-fuzzy e-open ((τ_i, τ_j)-fuzzy e-closed) set weaker than the concepts of (τ_i, τ_j)-fuzzy δ-semiopen ((τ_i, τ_j)-fuzzy δ-semiclosed) set or (τ_i, τ_j)-fuzzy δ-preopen ((τ_i, τ_j)-fuzzy δ-preclosed) set and stronger than the concept of (τ_i, τ_j)-fuzzy β-open ((τ_i, τ_j)-fuzzy β-closed) set on fuzzy bitopological spaces and study some of their properties. We introduce the (τ_i, τ_j)-fuzzy e-interiors ((τ_i, τ_j)-fuzzy e-closures) and investigate the characteristic properties of fuzzy pairwise e-continuous (fuzzy pairwise e-open and fuzzy pairwise e-closed) mapping on fuzzy bitopological spaces.

3. (τ_i, τ_j)-fuzzy e-open sets

A system (X, τ_1, τ_2) consisting of a set X with two fuzzy topologies τ_1 and τ_2 on X is called a fuzzy bitopological space (fbts for short) [5]. Throughout this paper, the induces $\{i, j\}$ take values in $\{1, 2\}$ with $i \neq j$.

For a fuzzy set λ in a fbts (X, τ_1, τ_2), τ_i-fo set λ and τ_j-fc set λ mean τ_i-fuzzy open set λ and τ_j-fuzzy closed set λ respectively. Also, τ_i-$Int(\lambda)$ and τ_j-$Cl(\lambda)$ mean the interior and closure of λ for the fuzzy topologies τ_i and τ_j respectively.

A mapping $f : (X, \tau_i, \tau_j) \to (X, \tau_i^*, \tau_j^*)$ is said to be fuzzy pairwise continuous (fpc-for short) (respectively fuzzy pairwise open (fp-open)) if and only if the induced mappings $f : (X, \tau_k) \to (Y, \tau_k^*)$ $(k = 1, 2)$ are fuzzy continuous (respectively fuzzy open) [1].

Definition 3.1. Let λ be a fuzzy set of a fbts X. Then λ is called,

(1) a (τ_i, τ_j)-fuzzy semiopen [3] ((τ_i, τ_j)-fso) set of X, if
$$\lambda \leq \tau_j\text{-}Cl(\tau_i\text{-}Int(\lambda)),$$
(2) a (τ_i, τ_j)-fuzzy semiclosed [3] ((τ_i, τ_j)-fsc) set of X, if
$$\tau_j\text{-}Int(\tau_i\text{-}Cl(\lambda)) \leq \lambda,$$
(3) a (τ_i, τ_j)-fuzzy preopen [8] ((τ_i, τ_j)-fpo) set of X, if
$$\lambda \leq \tau_i\text{-}Int(\tau_j\text{-}Cl(\lambda)),$$
(4) a (τ_i, τ_j)-fuzzy preclosed [8] ((τ_i, τ_j)-fpc) set of X, if
$$\tau_i\text{-}Cl(\tau_j\text{-}Int(\lambda)) \leq \lambda,$$
(5) a (τ_i, τ_j)-fuzzy β-open [7] ((τ_i, τ_j)-$f\beta o$) set of X, if
$$\lambda \leq \tau_j\text{-}Cl(\tau_i\text{-}Int(\tau_j\text{-}Cl(\lambda))),$$
(6) a (τ_i, τ_j)-fuzzy β-closed [7] ((τ_i, τ_j)-$f\beta c$) set of X, if

$$\tau_j\text{-}Int(\tau_i\text{-}Cl(\tau_j\text{-}Int(\lambda))) \leq \lambda,$$

(7) a (τ_i, τ_j)-fuzzy δ-semiopen [6] ((τ_i, τ_j)-$f\delta so$) set of X, if
$$\lambda \leq \tau_j\text{-}Cl(\tau_i\text{-}Int_\delta(\lambda)),$$

(8) a (τ_i, τ_j)-fuzzy δ-semiclosed [6] ((τ_i, τ_j)-$f\delta sc$) set of X, if
$$\lambda \geq \tau_j\text{-}Int(\tau_i\text{-}Cl_\delta(\lambda)),$$

(9) a (τ_i, τ_j)-fuzzy δ-preopen [2] ((τ_i, τ_j)-$f\delta po$) set of X, if
$$\lambda \leq \tau_i\text{-}int(\tau_j\text{-}Cl_\delta(\lambda)),$$

(10) a (τ_i, τ_j)-fuzzy δ-preclosed [2] ((τ_i, τ_j)-$f\delta pc$) set of X, if
$$\lambda \geq \tau_i\text{-}Cl(\tau_j\text{-}Int_\delta(\lambda)),$$

Definition 3.2. Let λ be a fuzzy set of a fbts X.

(1) The (τ_i, τ_j)-fuzzy β-interior of λ [4] ((τ_i, τ_j)-$\beta Int(\lambda)$) is defined by
$$(\tau_i, \tau_j)\text{-}\beta Int(\lambda) = Sup\{\mu | \mu \leq \lambda,\ \mu \text{ is a } (\tau_i, \tau_j)\text{-}f\beta o \text{ set } \}$$

(2) The (τ_i, τ_j)-fuzzy β-closure of λ [4] ((τ_i, τ_j)-$\beta Cl(\lambda)$) is defined by
$$(\tau_i, \tau_j)\text{-}\beta Cl(\lambda) = Inf\{\mu | \mu \geq \lambda,\ \mu \text{ is a } (\tau_i, \tau_j)\text{-}f\beta c \text{ set } \}$$

(3) The (τ_i, τ_j)-fuzzy δ-semi-interior of λ [6] ((τ_i, τ_j)-$\delta sInt(\lambda)$) is defined by
$$(\tau_i, \tau_j)\text{-}\delta sInt(\lambda) = Sup\{\mu | \mu \leq \lambda,\ \mu \text{ is a } (\tau_i, \tau_j)\text{-}f\delta so \text{ set } \}$$

(4) The (τ_i, τ_j)-fuzzy δ-semi-closure of λ [6] ((τ_i, τ_j)-$\delta sCl(\lambda)$) is defined by
$$(\tau_i, \tau_j)\text{-}\delta sCl(\lambda) = Inf\{\mu | \mu \geq \lambda,\ \mu \text{ is a } (\tau_i, \tau_j)\text{-}f\delta sc \text{ set } \}$$

(5) The (τ_i, τ_j)-fuzzy δ-preinterior of λ [2] ((τ_i, τ_j)-$\delta pInt(\lambda)$) is defined by
$$(\tau_i, \tau_j)\text{-}\delta pInt(\lambda) = Sup\{\mu | \mu \leq \lambda,\ \mu \text{ is a } (\tau_i, \tau_j)\text{-}f\delta po \text{ set } \}$$

(6) The (τ_i, τ_j)-fuzzy δ-preclosure of λ [2] ((τ_i, τ_j)-$\delta pCl(\lambda)$) is defined by
$$(\tau_i, \tau_j)\text{-}\delta pCl(\lambda) = Inf\{\mu | \mu \geq \lambda,\ \mu \text{ is a } (\tau_i, \tau_j)\text{-}f\delta pc \text{ set } \}$$

Definition 3.3. Let λ be a fuzzy set of a fbts X. Then λ is called,

(1) a (τ_i, τ_j)-fuzzy e-open ((τ_i, τ_j)-feo) set of X, if
$$\lambda \leq \tau_j\text{-}Cl(\tau_i\text{-}Int_\delta(\lambda)) \vee (\tau_i\text{-}Int(\tau_j\text{-}Cl_\delta(\lambda))),$$

(2) a (τ_i, τ_j)-fuzzy e-closed ((τ_i, τ_j)-fec) set of X, if
$$\tau_i\text{-}Cl(\tau_j\text{-}Int_\delta(\lambda)) \wedge (\tau_j\text{-}Int(\tau_i\text{-}Cl_\delta(\lambda))) \leq \lambda,$$

From the above definition it is clear that a (τ_i, τ_j)-feo set is weaker than the concepts of (τ_i, τ_j)-$f\delta so$ set or (τ_i, τ_j)-$f\delta po$ set and stronger than the concept of (τ_i, τ_j)-$f\beta o$ set. That is, every (τ_i, τ_j)-$f\delta so$ set is a (τ_i, τ_j)-feo set and every (τ_i, τ_j)-$f\delta po$ set is a (τ_i, τ_j)-feo set . Also, every (τ_i, τ_j)-feo sets is a (τ_i, τ_j)-$f\beta o$ set. The converses need not be true in general.

Example 3.4. Let λ_1, λ_2, λ_3, λ_4, η_1, η_2 and η_3 be fuzzy sets of $X = \{a, b\}$ define as follows, $\lambda_1 = \frac{0.2}{a} + \frac{0.1}{b}$, $\lambda_2 = \frac{0.3}{a} + \frac{0.5}{b}$, $\lambda_3 = \frac{0.7}{a} + \frac{0.7}{b}$, $\lambda_4 = \frac{0.2}{a} + \frac{0.8}{b}$, $\eta_1 = \frac{0.3}{a} + \frac{0.2}{b}$, $\eta_2 = \frac{0.3}{a} + \frac{0.4}{b}$ and $\eta_3 = \frac{0.7}{a} + \frac{0.7}{b}$. Consider fuzzy topologies $\tau_1 = \{0, 1, \lambda_1, \lambda_2, \lambda_3\}$ and $\tau_2 = \{0, 1, \eta_1, \eta_2, \eta_3\}$. Then the fuzzy set λ_4 is (τ_i, τ_j)-$f\beta o$, but λ_4 is not a (τ_i, τ_j)-feo set.

Example 3.5. Let λ_1, λ_2, λ_3, λ_4, λ_5, η_1, η_2 and η_3 be fuzzy sets of $X = \{a, b\}$ define as follows, $\lambda_1 = \frac{0.1}{a} + \frac{0.1}{b}$, $\lambda_2 = \frac{0.2}{a} + \frac{0.2}{b}$, $\lambda_3 = \frac{0.3}{a} + \frac{0.3}{b}$, $\lambda_4 = \frac{0.7}{a} + \frac{0.7}{b}$, $\lambda_5 = \frac{0.2}{a} + \frac{0.5}{b}$, $\eta_1 = \frac{0.2}{a} + \frac{0.3}{b}$, $\eta_2 = \frac{0.3}{a} + \frac{0.4}{b}$ and $\eta_3 = \frac{0.6}{a} + \frac{0.6}{b}$. Consider fuzzy topologies $\tau_1 = \{0, 1, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and $\tau_2 = \{0, 1, \eta_1, \eta_2, \eta_3\}$. Then the fuzzy set λ_5 is (τ_i, τ_j)-feo, but λ_5 is not a (τ_i, τ_j)-fso set and also not a (τ_i, τ_j)-$f\delta so$ set.

Example 3.6. Let λ_1, λ_2, λ_3, λ_4, η_1, η_2 and η_3 be fuzzy sets of $X = \{a, b\}$ define as follows, $\lambda_1 = \frac{0.7}{a} + \frac{0}{b}$, $\lambda_2 = \frac{0}{a} + \frac{0.2}{b}$, $\lambda_3 = \frac{0.7}{a} + \frac{0.2}{b}$, $\lambda_4 = \frac{0}{a} + \frac{0.3}{b}$, $\eta_1 = \frac{0.2}{a} + \frac{0.3}{b}$ and $\eta_3 = \frac{0.7}{a} + \frac{0.7}{b}$. Consider fuzzy topologies $\tau_1 = \{0, 1, \lambda_1, \lambda_2, \lambda_3\}$ and $\tau_2 = \{0, 1, \eta_1, \eta_2, \eta_3\}$. Then the fuzzy set λ_4 is (τ_i, τ_j)-feo, but λ_4 is not a (τ_i, τ_j)-fpo set and also not a (τ_i, τ_j)-$f\delta po$ set.

Theorem 3.7. (1) *Any union of (τ_i, τ_j)-feo sets is a (τ_i, τ_j)-feo set.*
(2) *Any intersection of (τ_i, τ_j)-fec sets is a (τ_i, τ_j)-fec set.*

Proof. (1) let $\{\lambda_\mu\}$ be a family of (τ_i, τ_j)-feo sets in a fbts X. Then $\lambda_\mu \leq \tau_j\text{-}Cl(\tau_i\text{-}Int_\delta(\lambda_\mu)) \vee \tau_i\text{-}Int(\tau_j\text{-}Cl_\delta(\lambda_\mu))$ for each μ. We have

$$\begin{aligned} \bigvee \lambda_\mu &\leq \bigvee[\tau_j\text{-}Cl(\tau_i\text{-}Int_\delta(\lambda_\mu)) \vee \tau_i\text{-}Int(\tau_j\text{-}Cl_\delta(\lambda_\mu))] \\ &\leq \tau_j\text{-}Cl(\tau_i\text{-}Int_\delta(\bigvee(\lambda_\mu))) \vee \tau_i\text{-}Int(\tau_j\text{-}Cl_\delta(\bigvee(\lambda_\mu))) \end{aligned}$$

Hence $\bigvee \lambda_\mu$ is a (τ_i, τ_j)-feo set.

(2) Follows easily by taking complements.

The intersection (union) of any two (τ_i, τ_j)-feo $\big((\tau_i, \tau_j)$-$fec\big)$ sets need not be a (τ_i, τ_j)-feo $\big((\tau_i, \tau_j)$-$fec\big)$ set. □

Example 3.8. Let λ_1, λ_2, λ_3, λ_4, λ_5, λ_6, η_1, η_2, η_3 and η_4 be fuzzy sets of $X = \{a, b, c\}$ define as follows, $\lambda_1 = \frac{0.3}{a} + \frac{0.5}{b} + \frac{0.5}{c}$, $\lambda_2 = \frac{0.4}{a} + \frac{0.2}{b} + \frac{0.6}{c}$, $\lambda_3 = \frac{0.4}{a} + \frac{0.5}{b} + \frac{0.6}{c}$, $\lambda_4 = \frac{0.3}{a} + \frac{0.2}{b} + \frac{0.5}{c}$, $\lambda_5 = \frac{0.7}{a} + \frac{0.6}{b} + \frac{0.9}{c}$, $\lambda_6 = \frac{0.7}{a} + \frac{0.8}{b} + \frac{0.5}{c}$, $\eta_1 = \frac{0.3}{a} + \frac{0.4}{b} + \frac{0.5}{b}$, $\eta_2 = \frac{0.6}{a} + \frac{0.5}{b} + \frac{0.5}{b}$, $\eta_3 = \frac{0.6}{a} + \frac{0.5}{b} + \frac{0.4}{b}$ and $\eta_4 = \frac{0.3}{a} + \frac{0.4}{b} + \frac{0.4}{b}$. Consider fuzzy topologies $\tau_1 = \{0, 1, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and $\tau_2 = \{0, 1, \eta_1, \eta_2, \eta_3, \eta_4\}$. Then λ_5 and λ_6 are (τ_i, τ_j)-feo sets, but $\lambda_5 \wedge \lambda_6$ is not a (τ_i, τ_j)-feo set.

Theorem 3.9. *Let λ be a fuzzy set of a fbts X.*

(1) *If λ is a (τ_i, τ_j)-feo and τ_j-$f\delta c$ set, then λ is a (τ_i, τ_j)-$f\delta so$ set.*
(2) *If λ is a (τ_i, τ_j)-fec and τ_j-$f\delta o$ set, then λ is a (τ_i, τ_j)-$f\delta sc$ set.*

Proof. (1) Let λ be (τ_i, τ_j)-feo and τ_j-$f\delta c$ of a fbts X. Then
$\lambda \leq \tau_j\text{-}Cl(\tau_i\text{-}Int_\delta(\lambda)) \vee \tau_i\text{-}Int(\tau_j\text{-}Cl_\delta(\lambda))$
$= \tau_j\text{-}Cl(\tau_i\text{-}Int_\delta(\lambda)) \vee \tau_i\text{-}Int(\lambda)$
$= \tau_j\text{-}Cl(\tau_i\text{-}Int_\delta(\lambda))$.
Hence λ is a (τ_i, τ_j)-$f\delta so$ set.
(2) Similar to (1). □

Theorem 3.10. *Let λ be a fuzzy set of fbts X.*

(1) *If λ is a (τ_i, τ_j)-$f\beta o$, τ_j-fc set and τ_j-$f\delta c$ set, then λ is a (τ_i, τ_j)-feo set.*
(2) *If λ is a (τ_i, τ_j)-$f\beta c$, τ_j-fo and τ_j-$f\delta o$ set, then λ is a (τ_i, τ_j)-fec set.*

Proof. (1) Let λ be (τ_i, τ_j)-$f\beta o$ and τ_j-fc of a fbts X. Then
$\lambda \leq \tau_j\text{-}Cl(\tau_i\text{-}Int(\tau_j\text{-}Cl(\lambda))) = \tau_j\text{-}Cl(\tau_i\text{-}Int(\lambda))$

$= \tau_j\text{-}Cl(\tau_i\text{-}Int(\lambda) \vee \tau_i\text{-}Int(\lambda)) \leq \tau_j\text{-}Cl(\tau_i\text{-}Int_\delta(\lambda)) \vee \tau_i\text{-}Int(\lambda)$
$= \tau_j\text{-}Cl(\tau_i\text{-}Int_\delta(\lambda)) \vee \tau_i\text{-}Int(\tau_j\text{-}Cl_\delta(\lambda))$.
Hence λ is a (τ_i, τ_j)-feo set.
(2) Similar to (1). □

Remark 3.11. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two fbts's. Then the product $\lambda \times \mu$ of a (τ_i, τ_j)-feo set λ of X and (σ_i, σ_j)-feo set μ of Y need not be feo set in the product space $(X \times Y, \eta_1, \eta_2)$, where η_k is the fuzzy product topology generated by $\tau_k,\ \sigma_k$ $(K = 1,\ 2)$.

Example 3.12. Let $\lambda_1,\ \lambda_2,\ \lambda_3$ and λ be fuzzy sets on $X = \{a, b\}$, defined as $\lambda_1 = \frac{0.6}{a} + \frac{0.6}{b}$, $\lambda_2 = \frac{0.5}{a} + \frac{0.4}{b}$, $\lambda_3 = \frac{0.3}{a} + \frac{0.3}{b}$, $\lambda = \frac{0.3}{a} + \frac{0.5}{b}$. Let (X, τ_1, τ_2) be a fbts where, $\tau_1 = \{0, 1, \lambda_1, \lambda_2\}$, $\tau_2 = \{0, 1, \lambda_1, \lambda_3\}$. The fuzzy set λ is (τ_1, τ_2)-feo. Let λ_1, λ_4, λ_5 and μ be fuzzy sets on $Y = \{a, b\}$ defined as $\lambda_4 = \frac{0.5}{a} + \frac{0.5}{b}$, $\lambda_5 = \frac{0.3}{a} + \frac{0.2}{b}$, $\mu = \frac{0.5}{a} + \frac{0.4}{b}$. Let (Y, σ_1, σ_2) be fbts where, $\sigma_1 = \{0, 1, \lambda_1, \lambda_4\}$, $\sigma_2 = \{0, 1, \lambda_1, \lambda_5\}$. The fuzzy set μ is (σ_1, σ_2)-feo, but $\lambda \times \mu$ is not feo in $X \times Y$.

Definition 3.13. Let λ be a fuzzy set of a fbts X.

(1) The (τ_i, τ_j)-fuzzy e-interior of $\lambda((\tau_i, \tau_j)\text{-}eInt(\lambda))$ is defined by
$$(\tau_i, \tau_j)\text{-}eInt(\lambda) = Sup\{\mu | \mu \leq \lambda,\ \mu \text{ is a } (\tau_i, \tau_j)\text{-}feo \text{ set }\}$$
(2) The (τ_i, τ_j)-fuzzy e-closure of $\lambda((\tau_i, \tau_j)\text{-}eCl(\lambda))$ is defined by
$$(\tau_i, \tau_j)\text{-}eCl(\lambda) = Inf\{\mu | \mu \geq \lambda,\ \mu \text{ is a } (\tau_i, \tau_j)\text{-}fec \text{ set }\}$$

Obviously, (τ_i, τ_j)-$eCl(\lambda)$ is the smallest (τ_i, τ_j)-fec set which contains λ, and (τ_i, τ_j)-$eInt\lambda$ is the largest (τ_i, τ_j)-feo set which is contained in λ. Also (τ_i, τ_j)-$eCl(\lambda) = \lambda$ for any (τ_i, τ_j)-fec set λ and (τ_i, τ_j)-$eInt(\lambda) = \lambda$. for any (τ_i, τ_j)-feo set λ.

Hence we have
$$\tau_i\text{-}Int(\lambda) \leq (\tau_i, \tau_j)\text{-}\delta sInt(\lambda) \leq (\tau_i, \tau_j)\text{-}eInt(\lambda) \leq (\tau_i, \tau_j)\text{-}\beta Int(\lambda) \leq \lambda,$$
$$\lambda \leq (\tau_i, \tau_j)\text{-}\beta Cl(\lambda) \leq (\tau_i, \tau_j)\text{-}eCl(\lambda) \leq (\tau_i, \tau_j)\text{-}\delta sCl(\lambda) \leq \tau_i\text{-}Cl(\lambda).$$
and
$$\tau_i\text{-}Int(\lambda) \leq (\tau_i, \tau_j)\text{-}\delta pInt(\lambda) \leq (\tau_i, \tau_j)\text{-}eInt(\lambda) \leq (\tau_i, \tau_j)\text{-}\beta Int(\lambda) \leq \lambda,$$
$$\lambda \leq (\tau_i, \tau_j)\text{-}\beta Cl(\lambda) \leq (\tau_i, \tau_j)\text{-}eCl(\lambda) \leq (\tau_i, \tau_j)\text{-}\delta pCl(\lambda) \leq \tau_i\text{-}Cl(\lambda).$$

4. Fuzzy pairwise e-continuous mappings

Definition 4.1. Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then f is called,

(1) a fuzzy pairwise semicontinuous ($fpsc$) mapping [10] if $f^{-1}(\lambda)$ is a (τ_i, τ_j)-fso set of X for each τ_i^*-fo set λ of Y,
(2) a fuzzy pairwise precontinuous ($fppc$) mapping [8] if $f^{-1}(\lambda)$ is a (τ_i, τ_j)-fpo set of X for each τ_i^*-fo set λ of Y,
(3) a fuzzy pairwise δ-precontinuous ($f\delta pc$) mapping [3] if $f^{-1}(\lambda)$ is a (τ_i, τ_j)-$f\delta po$ set of X for each τ_i^*-fo set λ of Y,
(4) a fuzzy pairwise δ-semicontinuous ($fp\delta sc$) mapping [6] if $f^{-1}(\lambda)$ is a (τ_i, τ_j)-$f\delta so$ set of X for each τ_i^*-fo set λ of Y,
(5) a fuzzy pairwise β-continuous ($fp\beta c$) mapping [7] if $f^{-1}(\lambda)$ is a (τ_i, τ_j)-$f\beta o$ set of X for each τ_i^*-fo set λ of Y.

Definition 4.2. Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then f is called a fuzzy pairwise e-continuous ($fpec$) mapping if $f^{-1}(\lambda)$ is a (τ_i, τ_j)-feo set of X for each τ_i^*-fo set λ of Y.

From the above definitions it is clear that every $fp\delta sc$ is a $fpec$ mapping and every $fp\delta pc$ is a $fpec$ mapping. Also, every $fpec$ is a $fp\beta c$ mapping. But the converses are not true in general.

Example 4.3. Let λ_1, λ_2, η_1, η_2, η_3, η_4 and μ_1 be fuzzy sets of X. Consider fuzzy topology $\tau_1 = \{0,\ 1,\ \lambda_1 = \frac{0.2}{a} + \frac{0.3}{b},\ \lambda_2 = \frac{0.7}{a} + \frac{0.7}{b}\}$, $\tau_2 = \{0,\ 1,\ \eta_1 = \frac{0.7}{a} + \frac{0}{b},\ \eta_2 = \frac{0.1}{a} + \frac{0.2}{b},\ \eta_3 = \frac{0.7}{a} + \frac{0.2}{b},\ \eta_4 = \frac{0.1}{a} + \frac{0}{b}\}$, $\tau_1^* = \{0,\ 1,\ \mu_1 = \frac{0.3}{a} + \frac{0.2}{b}\}$ and $\tau_2^* = \{0, 1\}$ and the identity mapping $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$. Then f is $fp\beta c$, but f is not $fpec$, since for the fuzzy open set μ_1 in τ_1^*, $f^{-1}(\mu_1) = \mu_1$ is fuzzy β-open in (X, τ_1, τ_2) but not fuzzy e-open in (X, τ_1, τ_2).

Example 4.4. Let λ_1, λ_2, λ_3, λ_4, η_1, η_2, η_3 and μ_1 be fuzzy sets of X. Consider fuzzy topology $\tau_1 = \{0, 1, \lambda_1 = \frac{0.1}{a} + \frac{0.1}{b}, \lambda_2 = \frac{0.2}{a} + \frac{0.2}{b}, \lambda_3 = \frac{0.3}{a} + \frac{0.3}{b}, \lambda_4 = \frac{0.6}{a} + \frac{0.6}{b}\}$, $\tau_2 = \{0, 1, \eta_1 = \frac{0.2}{a} + \frac{0.3}{b}, \eta_2 = \frac{0.3}{a} + \frac{0.4}{b}, \eta_3 = \frac{0.6}{a} + \frac{0.6}{b}\}$, $\tau_1^* = \{0, 1, \mu_1 = \frac{0.2}{a} + \frac{0.5}{b}\}$ and $\tau_2^* = \{0, 1\}$ and the identity mapping $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*,\ \tau_2^*)$. Then f is $fpec$, but f is not $fpsc$ and also f is not $fp\delta sc$, since for the fuzzy open set μ_1 in τ_1^*, $f^{-1}(\mu_1) = \mu_1$ is fuzzy e-open in (X, τ_1, τ_2) but not fuzzy semiopen in (X, τ_1, τ_2) and also not fuzzy δ-semiopen in (X, τ_1, τ_2).

Example 4.5. Let λ_1, λ_2, λ_3, η_1, η_2 and μ_1 be fuzzy sets of X. Consider fuzzy topology $\tau_1 = \{0,\ 1,\ \lambda_1 = \frac{0.7}{a} + \frac{0}{b},\ \lambda_2 = \frac{0}{a} + \frac{0.2}{b},\ \lambda_3 = \frac{0.7}{a} + \frac{0.2}{b}\}$, $\tau_2 = \{0,\ 1,\ \eta_1 = \frac{0.2}{a} + \frac{0.3}{b},\ \eta_2 = \frac{0.7}{a} + \frac{0.7}{b}\}$ $\tau_1^* = \{0,\ 1,\ \mu_1 = \frac{0.2}{a} + \frac{0.5}{b}\}$ and $\tau_2^* = \{0,\ 1\}$ and the identity mapping $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$. Then f is $fpec$, but f is not $fppc$ and also f is not $fp\delta pc$, since for the fuzzy open set μ_1 in τ_1^*, $f^{-1}(\mu_1) = \mu_1$ is fuzzy e-open in (X, τ_1, τ_2) but not fuzzy preopen in (X, τ_1, τ_2) and also not fuzzy δ-preopen in (X, τ_1, τ_2).

Theorem 4.6. *Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then the followings are equivalent:*

(1) *f is $fpec$.*
(2) *The inverse image of each τ_i^*-fc set of Y is a (τ_i, τ_j)-fec set of X.*
(3) *$f((\tau_i, \tau_j)$-$eCl(\lambda)) \leq \tau_i^*$-$Cl(f(\lambda))$ for each fuzzy set λ of X.*
(4) *(τ_i, τ_j)-$eCl(f^{-1}(\mu)) \leq f^{-1}(\tau_i^*$-$Cl(\mu))$ for each fuzzy set μ of Y.*
(5) *$f^{-1}(\tau_i^*$-$Int(\mu)) \leq (\tau_i, \tau_j)$-$eInt(f^{-1}(\mu))$ for each fuzzy set μ of Y.*

Proof. (1) implies (2) Let μ be a τ_i^*-fc set of Y. Then μ^c is a τ_i^*-fo set of Y and so $f^{-1}(\mu^c) = (f^{-1}(\mu))^c$ is a (τ_i, τ_j)-feo set of X. Hence $f^{-1}(\mu)$ is a (τ_i, τ_j)-fec set of X.

(2) implies (3) Let λ be any fuzzy set of X. Then τ_i^*-$Cl(f(\lambda))$ is a τ_i^*-fc set of Y and so $f^{-1}[\tau_i^*$-$Cl(f(\lambda))]$ is a (τ_i, τ_j)-fec set of X. Hence

$$\begin{aligned}(\tau_i, \tau_j)\text{-}eCl(\lambda) \leq (\tau_i, \tau_j)\text{-}eCl(f^{-1}(f(\lambda))) &\leq (\tau_i, \tau_j)\text{-}eCl(f^{-1}[\tau_i^*\text{-}Cl(f(\lambda))]) \\ &= f^{-1}[\tau_i^*\text{-}Cl(f(\lambda))].\end{aligned}$$

We have

$$f((\tau_i, \tau_j)\text{-}eCl(\lambda)) \leq f(f^{-1}[\tau_i^*\text{-}Cl(f(\lambda))]) \leq \tau_i^*\text{-}Cl(f(\lambda)).$$

(3) implies (4) Let μ be any fuzzy set of Y. Then

$$f[(\tau_i,\tau_j)\text{-}eCl(f^{-1}(\mu))] \leq \tau_i^*\text{-}Cl(f(f^{-1}(\mu))) \leq \tau_i^*\text{-}Cl(\mu).$$

Hence

$$(\tau_i,\tau_j)\text{-}eCl(f^{-1}(\mu)) \leq f^{-1}(f[(\tau_i,\tau_j)\text{-}eCl(f^{-1}(\mu))]) \leq f^{-1}(\tau_i^*\text{-}Cl(\mu)).$$

(4) implies (5) Let μ be any fuzzy set of Y. Then

$$(\tau_i,\tau_j)\text{-}eCl(f^{-1}(\mu^c)) \leq f^{-1}(\tau_i^*\text{-}Cl(\mu^c)).$$

Hence

$$\begin{aligned} f^{-1}(\tau_i^*\text{-}Int(\mu)) = f^{-1}[(\tau_i^*\text{-}Cl(\mu^c))^c] &\leq ((\tau_i,\tau_j)\text{-}eCl(f^{-1}(\mu^c)))^c \\ &= (\tau_i,\tau_j)\text{-}eInt(f^{-1}(\mu)). \end{aligned}$$

(5) implies (1) Let μ be a τ_i^*-fo set of Y. Then

$$f^{-1}(\mu) = f^{-1}(\tau_i^*\text{-}Int(\mu)) \leq (\tau_i,\tau_j)\text{-}eInt(f^{-1}(\mu)).$$

Hence $f^{-1}(\mu)$ is a (τ_i,τ_j)-feo set of X. Therefore, f is $fpec$. □

Theorem 4.7. *Let $f:(X,\tau_1,\tau_2)\to(Y,\tau_1^*,\tau_2^*)$ be a $fpec$ mapping. Then for each fuzzy set λ of Y,*

$$f^{-1}(\tau_i^*\text{-}Int(\lambda)) \leq \tau_j\text{-}Cl(\tau_i\text{-}Int_\delta(f^{-1}(\lambda))) \vee \tau_i\text{-}Int(\tau_j\text{-}Cl_\delta(f^{-1}(\lambda))).$$

Proof. Let λ be any fuzzy set of Y. Then τ_i^*-$Int(\lambda)$ is a τ_i^*-fo set of Y and so $f^{-1}(\tau_i^*\text{-}Int(\lambda))$ is a (τ_i,τ_j)-feo set of X. Hence

$$\begin{aligned} f^{-1}(\tau_i^*\text{-}Int(\lambda)) &\leq \tau_j\text{-}Cl(\tau_i\text{-}Int_\delta(f^{-1}(\tau_i^*\text{-}Int(\lambda))) \vee \tau_i\text{-}Int(\tau_j\text{-}Cl_\delta(f^{-1}(\tau_i^*\text{-}Int(\lambda)) \\ &\leq \tau_j\text{-}Cl(\tau_i\text{-}Int_\delta(f^{-1}(\lambda)) \vee \tau_i\text{-}Int(\tau_j\text{-}Cl_\delta(f^{-1}(\lambda))). \end{aligned}$$ □

Corollary 4.8. *Let $f:(X,\tau_1,\tau_2)\to(Y,\tau_1^*,\tau_2^*)$ be a $fpec$ mapping. Then for each fuzzy set λ of Y,*

$$\tau_i\text{-}Cl(\tau_j\text{-}Int_\delta(f^{-1}(\lambda))) \wedge \tau_j\text{-}Int(\tau_i\text{-}Cl_\delta(f^{-1}(\lambda))) \leq f^{-1}(\tau_i^*\text{-}Cl(\lambda)).$$

Theorem 4.9. *Let $f:(X,\tau_1,\tau_2)\to(Y,\tau_1^*,\tau_2^*)$ be a bijection. Then f is $fpec$ if and only if τ_i^*-$Int(f(\lambda)) \leq f((\tau_i,\tau_j)\text{-}eInt(\lambda))$ for each fuzzy set λ of X.*

Proof. Let λ be any fuzzy set of X. Then by Theorem 4.6

$$f^{-1}[\tau_i^*\text{-}Int(f(\lambda))] \leq (\tau_i,\tau_j)\text{-}eInt(f^{-1}(f(\lambda))).$$

Since f is a bijection,

$$\tau_i^*\text{-}Int(f(\lambda)) = f(f^{-1}[\tau_i^*\text{-}Int(f(\lambda))]) \leq f((\tau_i,\tau_j)\text{-}eInt(\lambda)).$$

Conversely, let μ be any fuzzy set of Y. Then

$$\tau_i^*\text{-}Int(f(f^{-1}(\mu))) \leq f[(\tau_i,\tau_j)\text{-}eInt(f^{-1}(\mu))].$$

Since f is a bijection,

$$\tau_i^*\text{-}Int(\mu) = \tau_i^*\text{-}Int(f(f^{-1}(\mu)) \leq f[(\tau_i,\tau_j)\text{-}eInt(f^{-1}(\mu))]$$

and

$$f^{-1}(\tau_i^*\text{-}Int(\mu)) \leq f^{-1}[f((\tau_i,\tau_j)\text{-}eInt(f^{-1}(\mu)))] = (\tau_i,\tau_j)\text{-}eInt(f^{-1}(\mu)).$$

Therefore, by Theorem 4.6, f is $fpec$. □

Theorem 4.10. *Let (X_1, τ_1, τ_2), $(X_2, \tau_1^*, \tau_2^*)$, $(Y_1, \sigma_1, \sigma_2), (Y_2, \sigma_1^*, \sigma_2^*)$, be fbts's such that X_1 is product related to X_2. Then the product $f_1 \times f_2 : (X_1 \times X_2, \theta_1, \theta_2) \to (Y_1 \times Y_2, \eta_1, \eta_2)$, where θ_k (respectively η_k) is the fuzzy product topology generated by τ_k and τ_k^* (respectively σ_k and σ_k^*) $(k = 1, 2)$ of $fpec$ mappings $f_1 : (X, \tau_1, \tau_2) \to (Y_1, \sigma_1, \sigma_2)$ and $f_2 : (X, \tau_1^*, \tau_2^*) \to (Y_1, \sigma_1^*, \sigma_2^*)$ is a $fpec$ mapping.*

Proof. For convenience, we denote $\lambda = \bigvee_{m,n} (\mu_m \times \nu_n)$, where μ_m's are σ_i-fo sets of Y_1 and $\nu_n' s$ are σ_i^*-fo sets of Y_2. Then λ is a η_i-fo set of $Y_1 \times Y_2$. We have

$$(f_1 \times f_2)^{-1}(\lambda) = \bigvee_{m,n} ((f_1 \times f_2)^{-1}(\mu_m \times \nu_n)) = \bigvee_{m,n} (f_1^{-1}(\mu_m) \times f_2^{-1}(\nu_n)).$$

Since f_1 and f_2 are $fpec$, $f_1^{-1}(\mu_m)'s$ are (τ_i, τ_j)-feo sets of X_1 and $f_2^{-1}(\nu_n)$'s are (τ_1^*, τ_j^*)-feo sets of X_2. Hence $(f_1 \times f_2)^{-1}(\lambda)$ is a (θ_i, θ_j)-feo set of $X_1 \times X_2$. Therefore, $f_1 \times f_2$ is $fpec$. □

Theorem 4.11. *Let (X, τ_1, τ_2), $(X_1, \sigma_1^{(1)}, \sigma_2^{(1)})$ and $(X_2, \sigma_1^{(2)}, \sigma_2^{(2)})$ be fbts's and $\pi_k : (X_1 \times X_2, \theta_1, \theta_2) \to (X_k, \sigma_1^{(k)}, \sigma_2^{(k)})$ $(k = 1, 2)$ be the projections. If $f : X \to X_1 \times X_2$ is a $fpec$ mapping, then so is $\pi_k \circ f$.*

Proof. For a $\sigma_i^{(k)}$-fo set λ of X_k, we have

$$(\pi_k \circ f)^{-1}(\lambda) = f^{-1}(\pi_k^{-1}(\lambda)).$$

Since π_k is fpc and f is $fpec$, $(\pi_k \circ f)^{-1}(\lambda)$ is a (τ_i, τ_j)-feo set of X. hence $\pi_i \circ f$ is $fpec$. □

Theorem 4.12. Let X_1 and X_2 be fbts's such that X_1 is product related to X_2 and let $f : (X, \tau_1, \tau_2) \to (X_2, \tau_1^*, \tau_2^*)$ be a mapping. If the graph mapping $g : (X, \tau_1, \tau_2) \to (X_1 \times X_2, \theta_1, \theta_2)$ of f defined by $g(x) = (x, f(x))$ is a $fpec$ mapping, then f is a $fpec$ mapping.

Proof. Let λ be a τ_i^*-fo set of X_2. Then

$$f^{-1}(\lambda) = 1 \wedge f^{-1}(\lambda) = g^{-1}(1 \times \lambda).$$

Since g is a $fpec$ mapping and $1 \times \lambda$ is a θ_i-fo set of $X_1 \times X_2$, $f^{-1}(\lambda)$ is a (τ_i, τ_j)-feo set of X. Hence f is $fpec$. □

5. Fuzzy pairwise e-open mappings

Definition 5.1. Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then f is called,

(1) a fuzzy pairwise δ-semiopen [6] ($fp\delta s$-open) mapping (respectively fuzzy pairwise δ-semiclosed ($fp\delta s$-closed) mapping) if $f(\lambda)$ is a (τ_i^*, τ_j^*)-$f\delta so$ (respectively (τ_i^*, τ_j^*)-$f\delta sc$) set of Y for each τ_i-fo (respectively τ_i-fc) set λ of X.

(2) a fuzzy pairwise δ-preopen [3] ($fp\delta p$-open) mapping (respectively fuzzy pairwise δ-preclosed ($fp\delta p$-closed) mapping) if $f(\lambda)$ is a (τ_i^*, τ_j^*)-$f\delta po$ (respectively (τ_i^*, τ_j^*)-$f\delta pc$) set of Y for each τ_i-fo (respectively τ_i-fc) set λ of X,

(3) a fuzzy pairwise β-open [7] ($fp\beta$-open) mapping (respectively fuzzy pairwise β-closed ($fp\beta$-closed) mapping) if $f(\lambda)$ is a (τ_i^*, τ_j^*)-$f\beta o$ (respectively (τ_i^*, τ_j^*)-$f\beta c$) set of Y for each τ_i-fo (respectively τ_i-fc) set λ of X.

Definition 5.2. Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then f is called a fuzzy pairwise e-open (fpe-open) mapping (respectively fuzzy pairwise e-closed [fpe-closed] mapping) if $f(\lambda)$ is a (τ_i^*, τ_j^*)-feo (respectively (τ_i^*, τ_j^*)-fec) set of Y for each τ_i-fo (respectively τ_i-fc) set λ of X.

From the above definitions it is clear that every $fp\delta s$-open ($fp\delta s$-closed respectively) is a fpe-open (fpe-closed respectively) mapping and every $fp\delta p$-open ($fp\delta p$-closed respectively) is a fpe-open (fpe-closed respectively) mapping. Also, every fpe-open (fpe-closed respectively) is a $fp\beta$-open ($fp\beta$-closed respectively) mapping. We can easily show that the converses are not true in general.

Example 5.3. Let λ_1, λ_2, η_1, η_2, η_3, η_4 and μ_1 be fuzzy sets of X. Consider fuzzy topology $\tau_1 = \{0,\ 1,\ \lambda_1 = \frac{0.2}{a} + \frac{0.3}{b},\ \lambda_2 = \frac{0.7}{a} + \frac{0.7}{b}\}$, $\tau_2 = \{0,\ 1,\ \eta_1 = \frac{0.7}{a} + \frac{0}{b},$ $\eta_2 = \frac{0.1}{a} + \frac{0.2}{b},\ \eta_3 = \frac{0.7}{a} + \frac{0.2}{b},\ \eta_4 = \frac{0.1}{a} + \frac{0}{b}\}$ $\tau_1^* = \{0,\ 1,\ \mu_1 = \frac{0.7}{a} + \frac{0.8}{b}\}$ and $\tau_2^* = \{0,\ 1\}$. Then μ_1 is fuzzy pairwise β-closed, but μ_1 is not fuzzy pairwise e-closed.

Example 5.4. Let λ_1, λ_2, λ_3, λ_4, η_1, η_2, η_3 and μ_1 be fuzzy sets of X. Consider fuzzy topology $\tau_1 = \{0, 1, \lambda_1 = \frac{0.1}{a} + \frac{0.1}{b}, \lambda_2 = \frac{0.2}{a} + \frac{0.2}{b}, \lambda_3 = \frac{0.3}{a} + \frac{0.3}{b}, \lambda_4 = \frac{0.6}{a} + \frac{0.6}{b}\}$, $\tau_2 = \{0,\ 1,\ \eta_1 = \frac{0.2}{a} + \frac{0.3}{b},\ \eta_2 = \frac{0.3}{a} + \frac{0.4}{b},\ \eta_3 = \frac{0.6}{a} + \frac{0.6}{b}\}$ $\tau_1^* = \{0,\ 1,\ \mu_1 = \frac{0.2}{a} + \frac{0.5}{b}\}$ and $\tau_2^* = \{0,\ 1\}$. Then μ_1 is fuzzy pairwise e-closed, but μ_1 is not fuzzy pairwise δ-semiclosed.

Example 5.5. Let λ_1, λ_2, λ_3, η_1, η_2 and μ_1 be fuzzy sets of X. Consider fuzzy topology $\tau_1 = \{0,\ 1,\ \lambda_1 = \frac{0.7}{a} + \frac{0}{b},\ \lambda_2 = \frac{0}{a} + \frac{0.2}{b},\ \lambda_3 = \frac{0.7}{a} + \frac{0.2}{b}\}$, $\tau_2 = \{0,\ 1,$ $\eta_1 = \frac{0.2}{a} + \frac{0.3}{b},\ \eta_2 = \frac{0.7}{a} + \frac{0.7}{b}\}$ $\tau_1^* = \{0,\ 1,\ \mu_1 = \frac{0.2}{a} + \frac{0.5}{b}\}$ and $\tau_2^* = \{0,\ 1\}$. Then μ_1 is fuzzy pairwise e-closed, but μ_1 is not fuzzy pairwise δ-preclosed.

Theorem 5.6. *Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then the followings are equivalent:*

(1) *f is fpe-open.*
(2) *$f(\tau_i\text{-}Int(\lambda)) \leq (\tau_i^*, \tau_j^*)\text{-}eInt(f(\lambda))$ for each fuzzy set λ of X.*
(3) *$\tau_i\text{-}Int(f^{-1}(\mu)) \leq f^{-1}((\tau_i, \tau_j))\text{-}eInt(\mu))$ for each fuzzy set μ of Y.*

Proof. (1) implies (2) Let λ be any fuzzy set of X. Then τ_i-$Int(\lambda)$ is a τ_i-fo set of X. Then $f(\tau_i\text{-}Int(\lambda))$ is a (τ_i^*, τ_j^*)-feo set of Y. Since $f(\tau_i\text{-}Int(\lambda)) \leq f(\lambda)$,

$$f(\tau_i\text{-}Int(\lambda)) = (\tau_i^*, \tau_j^*)\text{-}eInt(f(\tau_i\text{-}Int(\lambda))) \leq (\tau_i^*, \tau_j^*)\text{-}eInt(f(\lambda))$$

(2) implies (3) Let μ be any fuzzy set of Y. Then $f^{-1}(\mu)$ is a fuzzy set of X. Hence

$$f(\tau_i\text{-}Int(f^{-1}(\mu))) \leq (\tau_i^*, \tau_j^*)\text{-}eInt(f(f^{-1}(\mu))) \leq (\tau_i^*, \tau_j^*)\text{-}eInt(\mu).$$

We have

$$\tau_i\text{-}Int(f^{-1}(\mu))) \leq f^{-1}[f(\tau_i\text{-}Int(f^{-1}(\mu)))] \leq f^{-1}((\tau_i^*, \tau_j^*)\text{-}eInt(\mu)).$$

(3) implies (1) Let λ be any τ_i-fo set of X. Then τ_i-$Int(\lambda) = \lambda$ and $f(\lambda)$ is a fuzzy set of Y. Now,

$$\lambda = \tau_i\text{-}Int(\lambda) \leq \tau_i\text{-}Int(f^{-1}(f(\lambda))) \leq f^{-1}[(\tau_i^*, \tau_j^*)\text{-}eInt(f(\lambda))].$$

So we have

$$f(\lambda) \leq f(f^{-1}[(\tau_i^*, \tau_j^*)\text{-}eInt(f(\lambda))]) \leq (\tau_i^*, \tau_j^*)\text{-}eInt(f(\lambda)).$$

and

$$(\tau_i^*, \tau_j^*)\text{-}eInt(f(\lambda)) \leq f(\lambda).$$

Hence $f(\lambda)$ is a (τ_i^*, τ_j^*)-feo set of Y. Therefore, f is a fpe-open mapping. □

Theorem 5.7. *Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then f is a fpe-closed mapping if and only if for each fuzzy set λ of X, (τ_i^*, τ_j^*)-$eCl(f(\lambda)) \leq f(\tau_i$-$Cl(\lambda))$.*

Proof. Let f be fpe-closed and let λ be any fuzzy set of X. Then $f(\lambda) \leq f(\tau_i$-$Cl(\lambda))$ and $f(\tau_i$-$Cl(\lambda))$ is a (τ_i^*, τ_j^*)-fec set of Y. We have

$$(\tau_i^*, \tau_j^*)\text{-}eCl(f(\lambda)) \leq (\tau_i^*, \tau_j^*)\text{-}eCl(f(\tau_i\text{-}Cl(\lambda))) = f(\tau_i\text{-}Cl(\lambda)).$$

Conversely, let λ be a τ_i-fc set of X. Then

$$f(\lambda) \leq (\tau_i^*, \tau_j^*)\text{-}eCl(f(\lambda)) \leq f(\tau_i\text{-}Cl(\lambda)).$$

and

$$(\tau_i^*, \tau_j^*)\text{-}eCl(f(\lambda)) \leq f(\tau_i\text{-}Cl(\lambda)) = f(\lambda).$$

Hence $f(\lambda)$ is a (τ_i^*, τ_j^*)-fec set of Y. Therefore, f is a fpe-closed mapping. □

Theorem 5.8. *Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a bijection. Then f is a fpe-closed mapping if and only if for each fuzzy set μ of Y, $f^{-1}((\tau_i^*, \tau_j^*)$-$eCl(\mu)) \leq \tau_i$-$Cl(f^{-1}(\mu))$.*

Proof. Let μ be any fuzzy set of Y. Then by Theorem 5.7

$$(\tau_i^*, \tau_j^*)\text{-}eCl(\mu) \leq f(\tau_i\text{-}Cl(f^{-1}(\mu))).$$

Since f is a bijection

$$\begin{aligned} f^{-1}((\tau_i^*, \tau_j^*)\text{-}eCl(\mu)) &= f^{-1}[(\tau_i^*, \tau_j^*)\text{-}eCl(f(f^{-1}(\mu)))] \\ &\leq f^{-1}[f(\tau_i\text{-}Cl(f^{-1}(\mu)))] \\ &= \tau_i\text{-}Cl(f^{-1}(\mu)). \end{aligned}$$

Conversely, let λ be any fuzzy set of X. Since f is a bijection,

$$\begin{aligned} (\tau_i^*, \tau_j^*)\text{-}eCl(f(\lambda)) &= f(f^{-1}[(\tau_i^*, \tau_j^*)\text{-}eCl(f(\lambda))]) \\ &\leq f[\tau_i\text{-}Cl(f^{-1}(f(\lambda)))] \\ &= f(\tau_i\text{-}Cl(\lambda)). \end{aligned}$$

Therefore, by Theorem 5.7, f is a fpe-closed mapping. □

Theorem 5.9. *Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a fpe-closed mapping. If μ is a fuzzy set of Y and ν is a τ_i-fo set of X containing $f^{-1}(\mu)$, then there exists (τ_i^*, τ_j^*)-fe-open set λ of Y containing μ such that $f^{-1}(\lambda) \leq \nu$.*

Proof. Let μ be any fuzzy set of Y and let ν be a τ_i-fo set of X containing $f^{-1}(\mu)$, and let $\lambda = (f(\nu^c))^c$.

$$\text{Since } f^{-1}(\mu) \leq \nu$$
$$(f^{-1}(\mu))^c \geq \nu^c$$
$$f^{-1}(\mu^c) \geq \nu^c$$
$$\mu^c \geq f(\nu^c)$$
$$\text{(i.e.) } f(\nu^c) \leq \mu^c.$$

Since f is fpe-closed, then λ is (τ_i^*, τ_j^*)-fe-open set of Y and $f^{-1}(\lambda) = f^{-1}((f(\nu^c))^c)$. Therefore

$$f^{-1}(\lambda) = f^{-1}(f(\nu^c))^c \leq (\nu^c)^c = \nu$$
$$\Rightarrow f^{-1}(\lambda) \leq \nu.$$

□

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