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On isomorphisms of L-fuzzy graphs

Pramada Ramachandran, K. V. Thomas

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Abstract. In this paper, we study different isomorphisms of an L-Fuzzy Graph (LFG) and show that isomorphism is an equivalence relation on the set of all LFGs. We also define the complement of an L-Fuzzy Graph and show that for a self complementary LFG, L is a Boolean ring with a special sum and product.

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Corresponding Author: Pramada Ramachandran (pramadars@yahoo.com)

1. Introduction

The concept of fuzzy sets and fuzzy relations was first suggested by Zadeh in his papers [27, 28]. The main objective was to create a mathematical framework to deal with the concepts of uncertainty and ambiguity, which arise more often than not in real life. Such uncertainty is a natural phenomenon and does not necessarily imply a loss of accuracy. The crisp set dichotomizes the individuals of the universe into two sharply distinct groups: members and non-members, and this may not be suitable for practical situations. However, the Fuzzy Principle is that membership in a fuzzy set is a matter of degree. This new frame of thought had a tremendous impact in the scientific world. The concept of fuzzy set theory was also introduced into the areas of topology, analysis, algebra, geometry and eventually graph theory too.

In the classical paper [17], Rosenfeld introduced the concept of Fuzzy graphs, as a means to model several real life situations. Ever since then, fuzzy graph theory has witnessed tremendous growth.

Graphs are models of relations between objects. The objects are represented by vertices and relations by edges. When the description of the objects, or relationships, or both happens to possess uncertainty, we design a 'Fuzzy Graph Model'. Yeh and Bang's [26] approach for the study of fuzzy graphs were motivated by its applicability to pattern classification and clustering analysis. They established the fuzzy analogue of Whitney's theorem. The concept of domination in graphs finds widespread applications. Somasundaram and Somasundaram [23] introduced the concepts of domination and total domination in fuzzy graphs. In [8], Sunil Mathew and Sunitha, introduced Menger's theorem for fuzzy graphs and discuss the concepts of strength reducing sets and t-connected fuzzy graphs. Samanta and Pal introduced such concepts as fuzzy threshold graphs, fuzzy tolerance graphs and bipolar fuzzy graphs and studied their properties [18, 19, 20, 21, 22]. Apart from this, many mathematicians applied the concepts of Fuzzy sets in graph theory [1, 3, 4, 12, 13, 14, 15, 16, 23, 25].

Today, fuzzy graph theory finds applications in areas as diverse as computer science, artificial intelligence, decision analysis, pattern recognition, medicine, geography, linguistics and even robotics. An application of fuzzy graph theory to the human cardiac function has been discussed in [24]. More applications of Fuzzy Graph Theory can be found in the books [9, 11].

In this paper, we introduce the concept of L-fuzzy graphs and obtain a few results, as in the studies on L-groups and L-subgroups carried out by Mordeson and Malik [12]. We introduce the concept of L-fuzzy graphs, different types of isomorphisms on L-fuzzy graphs and study their properties. We define the complement of an L-fuzzy graph, self complementary L-fuzzy graphs and obtain results pertaining to these notions.

The seminal book on fuzzy graph theory by Mordeson and Nair [12] is a primary reference. Throughout this paper, 'L' is a finite lattice $(L, \wedge, \vee, 0, 1)$ and '$\leq$' denotes the partial order on the lattice. For all graph theoretic concepts, we follow [5, 6].

2. Preliminaries

In this section, we review some basic definitions that will be needed in the sequel. For details we refer to [10, 12].

Definition 2.1 ([12])**.** A fuzzy relation on a set V is a map $\mu : V \times V \to [0,1]$. A fuzzy graph $G = (V, \sigma, \mu)$ with the underlying set V is a nonempty set V together with a pair of functions $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$ such that $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$, u, v in V.

Definition 2.2 ([10])**.** A fuzzy group of a group V is a mapping $\sigma : V \to [0,1]$ satisfying $\sigma(uv) \geq \sigma(u) \wedge \sigma(v)$, $\forall\ u, v$ in V and $\sigma(v^{-1}) = \sigma(v)$, $\forall v$ in V.

Definition 2.3 ([2])**.** We say a fuzzy group $\sigma : V \to [0,1]$ has an embedding into a fuzzy group $\sigma' : V' \to [0,1]$ if there exists a one-one map $h : V \to V'$ such that $\sigma(x) \leq \sigma'(h(x))$, $\forall x$ in V.

According to Klir and Yuan [7], an L fuzzy set is a fuzzy set in which the range [0, 1] is replaced by a lattice L. Accordingly, we can modify all fuzzy relations to L-Fuzzy relations. For instance, we have:

Definition 2.4 ([10])**.** An L-subgroup of a group V is a mapping $\sigma : V \to L$ satisfying $\sigma(uv) \geq \sigma(u) \wedge \sigma(v)$, $\forall u, v$ in V and $\sigma(v^{-1}) = \sigma(v)$, $\forall v$ in V.

Lemma 2.5 ([10])**.** *Let $L(G)$ be the set of all L-subgroups of G, a group with identity 'e' and $\mu \in L(G)$. Then*

$$\mu(e) \geq \mu(x) \text{ and } \mu(x) = \mu(x^{-1}), \quad \forall x \in G.$$

3. L-Fuzzy Graph

First, we define an L-Fuzzy graph and then the t-cut and partial fuzzy subgraphs of an L-Fuzzy Graph. Also, we define a complete and an ordered L-Fuzzy Graphs. In the process, results in Fuzzy Graph Theory are extended to the L-Fuzzy setting.

Definition 3.1. An L-fuzzy graph (LFG) $G^L = (V, \sigma, \mu)$ with the underlying set V is a nonempty set V together with a pair of functions $\sigma : V \to L$ and $\mu : V \times V \to L$ such that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$, $\forall$ u, v in V.

Definition 3.2. Consider the LFG $G^L = (V, \sigma, \mu)$ and 't' in L. Then ${}^t\sigma = \{u \in V/\sigma(u) \geq t\}$ is called the 't-cut of σ' and ${}^t\mu = \{(u, v) \in V \times V/\mu(u, v) \geq t\}$ is called the 't-cut of μ'.

Proposition 3.3. *Consider the LFG $G^L = (V, \sigma, \mu)$ and $t_1, t_2 \in L$ such that $t_1 \leq t_2$. Then $({}^{t_2}\sigma, {}^{t_2}\mu)$ is a crisp subgraph of $({}^{t_1}\sigma, {}^{t_1}\mu)$.*

Proof. By definition, if $u \in {}^{t_2}\sigma$ then $\sigma(u) \geq t_2$ so that $\sigma(u) \geq t_1$ and hence $u \in {}^{t_1}\sigma$. Similarly, if $u - v \in {}^{t_2}\mu$ then $\mu(u, v) \geq t_2$ so that $\mu(u, v) \geq t_1$ and hence $u - v \in {}^{t_1}\mu$.

We also note that if $t_1, t_2 \in L$ are non comparable with $t_1 \wedge t_2 = t$, then ${}^{t_1}\sigma \cap {}^{t_2}\sigma = {}^{t_t}\sigma$ and ${}^{t_1}\mu \cap {}^{t_2}\mu = {}^{t_t}\mu$. □

Definition 3.4. An LFG $H^L = (V, \nu, \tau)$ is said to be a partial fuzzy subgraph of the LFG $G^L = (V, \sigma, \mu)$ if $\nu(u) \leq \tau(v)$ and $\tau(u, v) \leq \mu(u, v)$, $\forall u, v$ in V.

Proposition 3.5. Let the LFG $H^L = (V, \nu, \tau)$ be a partial fuzzy subgraph of the LFG $G^L = (V, \sigma, \mu)$ and $t \in L$. Then $({}^t\nu, {}^t\tau)$ is a crisp subgraph of $({}^t\sigma, {}^t\mu)$.

Proof. If $u \in {}^t\nu$ then $\nu(u) \geq t$ so that $\sigma(u) \geq t$ and hence $u \in {}^t\sigma$. Similarly, if $uv \in {}^t\tau$ then $\tau(u, v) \geq t$ so that $\mu(u, v) \geq t$ and hence $uv \in {}^t\mu$. □

Definition 3.6. The LFG $G^L = (V, \sigma, \mu)$ is said to be complete if it satisfies the condition

$$\mu(u, v) = \sigma(u) \wedge \sigma(v), \quad \forall u, v \text{ in } V.$$

Example 3.7. Let 'L' be the power set of $\{a, b, c\}$ with set inclusion as the partial order. Let $V = \{v_1, v_2, v_3, v_4\}$. Define $G^L = (V, \sigma, \mu)$ as follows:
$\sigma(v_1) = \{a, b\}$, $\sigma(v_2) = \{b, c\}$, $\sigma(v_3) = \{a, c\}$, $\sigma(v_4) = \{a, b, c\}$, $\mu(v_1, v_2) = \{b\}$. Obviously, G^L is complete.

Definition 3.8. The LFG $G^L = (V, \sigma, \mu)$ is said to be ordered if it satisfies the following condition:
$\mu(u_1, v_1) \leq \mu(u_2, v_2)$, $\forall u_1, u_2, v_1, v_2$ in V such that $\sigma(u)_1 \leq \sigma(u_2)$ and $\sigma(v_1) \leq \sigma(v_2)$.

We note that all complete LFGs are ordered but not vice versa. Hence there are ordered LFGs that are not complete.

4. Automorphisms of LFGs

In this section, we define the concept of homomorphisms on LFGs and find several interesting results.

Definition 4.1. By a homomorphism of LFGs $G_1^{L_1} = (V_1, \sigma_1, \mu_1)$ and $G_2^{L_2} = (V_2, \sigma_2, \mu_2)$, we mean a mapping $h : V_1 \to V_2$ together with a mapping $l : L_1 \to L_2$ such that

$$l[\sigma_1(u)] \leq \sigma_2[h(u)]$$

and

$$l[\mu_1(u, v)] \leq \mu_2[h(u), h(v)], \quad \forall u,\, v \text{ in } V_1.$$

We note that when $L_1 \cong L_2$, l is simply the identity map.

Definition 4.2. An endomorphism of an LFG $G^L = (V, \sigma, \mu)$ is a homomorphism of the LFG onto itself.

Definition 4.3. By an isomomorphism of LFGs $G_1^{L_1} = (V_1, \sigma_1, \mu_1)$ and $G_1^{L_2} = (V_2, \sigma_2, \mu_2)$, we mean a bijective mapping $h : V_1 \to V_2$ together with a bijective mapping $l : L_1 \to L_2$ such that

$$l[\sigma_1(u)] = \sigma_2[h(u)]$$

and

$$l[\mu_1(u, v)] = \mu_2[h(u), h(v)], \quad \forall\ u, v \text{ in } V_1.$$

Symbolically, we write $G_1^{L_1} \cong G_2^{L_2}$.

Definition 4.4. An automorphism of an LFG $G^L = (V, \sigma, \mu)$ is a isomomorphism of the LFG onto itself.

Theorem 4.5. *A bijective endomorphism of an LFG $G^L = (V, \sigma, \mu)$ is an automorphism of the LFG G^L.*

Proof. Consider the bijective endomorphism of an LFG $G^L = (V, \sigma, \mu)$. Then by the definition, $\exists$ a bijective mapping $h : V \to V$ such that

$$\sigma(u) \leq \sigma[h(u)] \text{ and } \mu(u, v) \leq \mu[h(u), h(v)], \quad \forall\ u, v \text{ in } V.$$

Since h is a bijection,

$$\sigma(u) \leq \sigma(h(u)) \leq \sigma(h^2(u)) \leq \cdots \leq \sigma(h^n(u))$$

for any u in V.

However, h is a bijective map from V to itself. Therefore $h^n(u) = h(u)$ for some n.

So $\sigma(u) \leq \sigma(h(u)) \leq \sigma(u)$. Hence $\sigma(u) = \sigma(h(u))$.

Similarly, $\mu(u, v) \leq \mu[h(u), h(v)] \leq \mu(u, v)$ which implies $\mu(u, v) = \mu[h(u), h(v)]$. Thus a bijective endomorphism of an LFG $G^L = (V, \sigma, \mu)$ is an automorphism of the LFG G^L. □

Theorem 4.6. *The collection G_*^L of all automorphisms of an LFG $G^L = (V, \sigma, \mu)$ is a group under set theoretic product of maps as a binary operation.*

Proof. Let $h_1, h_2 \in G_*^L$. Then $\sigma[h_1 \circ h_2(u)] = \sigma[h_1(h_2(u))] = \sigma[h_2(u))] = \sigma(u)$. This establishes closure property. Associativity follows from associativity of homomorphisms and identity element is the identity map, which is a homomorphism. Now, if $h : V \to V$ is an automorphism, then it is bijective and so $h^{-1} : V \to V$ exists and is a bijection. Let $h^{-1}(u) = u'$. Then $\sigma[h^{-1}(u)] = \sigma[u'] = \sigma[u]$ by Theorem 4.5.

Similarly, $\mu[h^{-1}(u), h^{-1}(v)] = \mu[u', v'] = \mu(u, v)$ by the same theorem. Thus G_*^L is a group. □

In the following theorems, we associate LFGs with L-Fuzzy groups.

Theorem 4.7. *Consider the LFG $G^L = (V, \sigma, \mu)$ and the collection G_*^L of all automorphisms of $G^L = (V, \sigma, \mu)$. Define $\tau : G_*^L \to L$ by $\tau(h) = \vee\{\mu(h(u), h(v))/(u, v) \in V \times V\}$. Then τ is an L-fuzzy group.*

Proof. Let $h_1, h_2 \in G_*^L$. Then

$$\begin{aligned}\tau[h_1 \circ h_2] &= \vee\{\mu(h_1 \circ h_2(u), h_1 \circ h_2(v))/(u, v) \in V \times V\}\\ &= \vee\{\mu(u, v)/(u, v) \in V \times V\}\end{aligned}$$

by the property of homomorphisms of LFGs.

Again

$$\begin{aligned}\tau[h_1] &= \vee\{\mu(h_1(u), h_1(v))/u \in V\}\\ &= \vee\{\mu(u, v)/(u, v) \in V \times V\}.\end{aligned}$$

Similarly

$$\begin{aligned}\tau[h_2] &= \vee\{\mu(h_2(u), h_2(v))/(u \in V)\}\\ &= \vee\{\mu(u, v)/(u, v) \in V \times V\}\end{aligned}$$

so that

$$\begin{aligned}\tau[h_1] \wedge \tau[h_2] &= \vee\{\mu(u, v)/(u, v) \in V \times V\}\\ &= \tau[h_1 \circ h_2]\end{aligned}$$

Again

$$\begin{aligned}\tau[h^{-1}] &= \vee\{\mu(h^{-1}(u), h^{-1}(v))/(u, v) \in V \times V\}\\ &= \vee\{\mu(u, v)/(u, v) \in V \times V\}\\ &\quad \text{by the property of homomorphisms of LFGs}\\ &= \tau[h]\end{aligned}$$

Since it satisfies the required axioms, τ is an L-fuzzy group. ☐

Definition 4.8. Consider the group $V, \sigma : V \to L$ is an L-fuzzy group and we define $\mu_\sigma : V \times V \to L$ by $\mu_\sigma(u, v) = \sigma(u) \wedge \sigma(v)$, $\forall u, v$ in V. Then $G^L = (V, \sigma, \mu_\sigma)$ is an LFG.

Theorem 4.9. *Every L-Fuzzy group $\sigma : G \to L$ has an embedding into the L-Fuzzy group of the group of automorphisms of some LFG.*

Proof. Let $G^L = (V, \sigma, \mu_\sigma)$ be the LFG corresponding to the L-Fuzzy group $\sigma : G \to L$, G being a group with identity 'e'. Let G_*^L be the group of all automorphisms of G^L. To this, we associate the L-Fuzzy group $\tau : G_*^L \to L$ as in Theorem 4.7. On G^L, we define the automorphism $h_u : G^L \to G^L$ as $h_u(a) = u^{-1}a$. Then

$$\begin{aligned}\tau[h_u] &= \vee\{\mu_\sigma(h_u(a), h_u(b))/a, b \in V\}\\ &= \vee\{\sigma(h_u(a)) \wedge \sigma(h_u(b))/a, b \in V\}\\ &= \vee\{\sigma(a) \wedge \sigma(b)/a, b \in V\}\\ &= \sigma(e)\end{aligned}$$

Again, by lemma 2.5, $\sigma(u) \leq \sigma(e)$. Hence $\sigma(u) \leq \tau[h_u]$, $\forall u$ in G. Thus $\sigma : G \to L$ has an embedding in $\tau : G_*^L \to L$. □

5. Properties of isomorphic LFGs

In this section, we study some interesting results that rise due to the inherent characteristics of lattices.

Theorem 5.1. *Let L_1 and L_2 be any two sublattices of the lattice L. Consider the two LFGs $G_1^{L_1} = (V_1, \sigma_1, \mu_1)$ and $G_2^{L_2} = (V_2, \sigma_2, \mu_2)$. If there exists a bijective mapping $h : V_1 \to V_2$ such that $\sigma_1(u) = \sigma_2[h(u)]$ and $\mu_1(u, v) = \mu_2[h(u), h(v)]$, $\forall u$, v in V_1 then L_1 and L_2 are isomorphic.*

Proof. Consider u and v in V_1 and the identity map 'I' on L. Then,

$$\begin{aligned} I[\sigma_1(u) \vee \sigma_2(v)] &= \sigma_1(u) \vee \sigma_2(v) \\ &= \sigma_2[h(u)] \vee \sigma_2[h(v)] \\ &= I\{\sigma_2[h(u)]\} \vee I\{\sigma_2[h(v)]\} \end{aligned}$$

Similarly,

$$\begin{aligned} I[\sigma_1(u) \wedge \sigma_2(v)] &= \sigma_1(u) \wedge \sigma_2(v) \\ &= \sigma_2[h(u)] \wedge \sigma_2[h(v)] \\ &= I\{\sigma_2[h(u)]\} \wedge I\{\sigma_2[h(v)]\} \end{aligned}$$

Thus the identity map is the isomorphic mapping between the two lattices. □

Theorem 5.2. *Let L_1 and L_2 be any two isomorphic lattices. Then it is possible to construct the LFGs $G_1^{L_1} = (V_1, \sigma_1, \mu_1)$ and $G_2^{L_2} = (V_2, \sigma_2, \mu_2)$, such that $G_1^{L_1} \cong G_2^{L_2}$, provided V_1 and V_2 are of the same order.*

Proof. The construction is as follows:
Since L_1 and L_2 are isomorphic lattices, there exists a bijective mapping $f : L_1 \to L_2$ such that $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$, $\forall a, b$ in L.
To choose 'h': Consider $u_1 \in V_1$. Let $\sigma_i(u_i) = a_i$ and $f(a_i) = b_i$, $i = 1, 2$. We now choose $\sigma_2[h(u_1)] = b_1$.

Let $u_1 - u_2$ be an edge in $G_1^{L_1} = (V_1, \sigma_1, \mu_1)$ with weight $\mu_1(u_1, u_2) = a_{12}$, $a_{12} \leq a_1 \wedge a_2$. Suppose $h(u_i) = v_i$.
Choose $\mu_2(v_1, v_2) = f[\mu_1(u_1, u_2)] = b_{12}$, say.
Then, $\mu_2(v_1, v_2) = b_{12} = f[\mu_1(u_1, u_2)] = f(a_{12}) \leq f(a_1 \wedge a_2) = f(a_1) \wedge f(a_2) = b_1 \wedge b_2 = \sigma_2(v_1) \wedge \sigma_2(v_2)$.
Thus, $\mu_2(v_1, v_2) \leq \sigma_2(v_1) \wedge \sigma_2(v_2)$. Thus $G_2^{L_2} = (V_2, \sigma_2, \mu_2)$ is an LFG.
Further, $h : V_1 \to V_2$ is such that $\sigma_2[h(u_i)] = b_i = f(a_i) = f[\sigma_1(u_i)]$. Thus $G_1^{L_1} \cong G_2^{L_2}$. □

Theorem 5.3. *Consider the LFGs $G_1^L = (V_1, \sigma_1, \mu_1)$ and $G_2^L = (V_2, \sigma_2, \mu_2)$, such that $G_1^L \cong G_2^L$. Then this isomorphism is an equivalence relation on the set of all LFGs on the lattice L.*

Proof. We are required to prove that the isomorphism relation is reflexive, symmetric and transitive.
Reflexivity: The mapping 'l' as required in the definition of isomorphism of LFGs is the identity map and we choose 'h' as the identity map itself. Then every LFG is isomorphic to itself.
Symmetry: If $G_1^L \cong G_2^L$ with the bijective mapping $h : V_1 \to V_2$, then $G_1^L \cong G_2^L$ with $h^{-1} : V_2 \to V_1$.
Transivity: Next, let $G_1^L \cong G_2^L$ with the bijective mapping $h : V_1 \to V_2$ and $G_1^L \cong G_3^L$ with the bijective mapping $k : V_2 \to V_3$. Then $k \circ h : V_1 \to V_3$ is a bijection. Further, $\sigma_1(u) = \sigma_2[h(u)] = \sigma_3\{k[h(u)]\} = \sigma_3[k \circ h(u)]$ and $\mu_1(u, v) = \mu_2[h(u), h(v)] = \mu_3[k \circ h(u), k \circ h(v)]$. Thus $G_1^L \cong G_3^L$.

The isomorphism relation is thus an equivalence relation on the set of all LFGs on the lattice L. □

Definition 5.4. Consider the LFG $G^L = (V, \sigma, \mu)$, where L is complete. We define the order 'p' and size 'q' of G^L as $p = \vee_{x \in V} \sigma(x)$ and $q = \vee_{x,y \in V} \mu(x, y)$.

Definition 5.5. Consider the LFG $G^L = (V, \sigma, \mu)$ where L is complete. Then we define the degree '$d(u)$' of a vertex 'u' in G^L as $d(u) = \vee_{v \neq u, v \in V} \mu(u, v)$.

Theorem 5.6. *If $G_1^L \cong G_2^L$, where L is complete, then $p_1 = p_2$ and $q_1 = q_2$, where p_i and q_i are the order and size of G_i^L, $i = 1, 2$ respectively. Also, the degree of each vertex is preserved under the isomorphism.*

Proof. By definition of isomorphism of LFGs,

$$p_1 = \vee_{x \in V_1} \sigma_1(x) = \vee_{x \in V_2} \sigma_2[h(x)] = \vee_{x \in V_2} \sigma_2(y) = p_2.$$

Also,

$$q_1 = \vee_{x,y \in V_1} \mu_1(x, y) = \vee_{x,y \in V_1} \mu_2[h(x), h(y)] = \vee_{x,y \in V_2} \mu_2(a, b) = p_2.$$

Further, for any vertex 'u' in the LFG, G_1^L,

$$d(u) = \vee_{v \neq u, v \in V_1} \mu_1(u, v) = \vee_{h(v) \neq h(u), h(v) \in V_2} \mu_2[h(u), h(v)] = d[h(u)].$$

□

Remark. The converse of the above theorem need not be true, *i.e.* if the order and size of two LFGS are the same, they need not be isomorphic. For example, let L be the following lattice:

Consider

Both have the same size and order but are not isomorphic.

Similarly, the following LFGs are such that the degrees of the corresponding vertices are the same but they are non isomorphic.

Definition 5.7. By a weak isomorphism of LFGs $G_1^{L_1} = (V_1, \sigma_1, \mu_1)$ and $G_2^{L_2} = (V_2, \sigma_2, \mu_2)$, we mean a bijective mapping $h : V_1 \to V_2$ together with a bijective mapping $l : L_1 \to L_2$ such that

$$l[\sigma_1(u)] = \sigma_2[h(u)], \quad \forall u \text{ in } V_1$$

and

$$l[\mu_1(u, v)] \leq \mu_2[h(u), h(v)], \quad \forall u \text{ and } v \text{ in } V_1$$

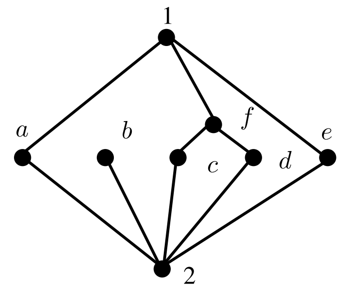

FIGURE 1.

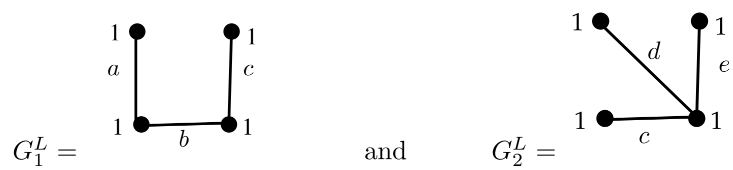

FIGURE 2.

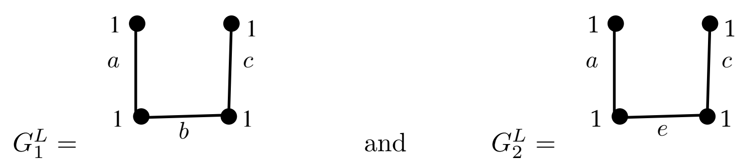

FIGURE 3.

Thus a weak isomorphism of LFGs preserves the weight of the vertices.

A weak isomorphism of LFGs need not be an isomorphism of LFGs, as is illustrated by the following:

Example 5.8. Let $L_1 = L_2$ be as in the following figure:
Let $G_1^{L_1} = (V_1, \sigma_1, \mu_1)$, $G_2^{L_2} = (V_2, \sigma_2, \mu_2)$ and the weak isomorphism 'h' of these LFGS be defined as follows:

$$V_1 = \{u_1, u_2, u3\} \text{ and } V_2 = \{v_1, v_2, v_3\},$$
$$\sigma_1(u_1) = a,\ \sigma_1(u_2) = b,\ \sigma_1(u_3) = c;\ \mu_1(u_1, u_2) = d,\ \mu_1(u_2, u_3) = f$$
$$\sigma_2(v_1) = c,\ \sigma_2(v_2) = a,\ \sigma_2(v_3) = b;\ \mu_2(v_1, v_2) = f,\ \mu_2(v_2, v_3) = g$$
$$h(u_1) = v_2,\ h(u_2) = v_3,\ h(u3) = v_1$$

It may be verified that 'h' is a weak isomorphism, but $\mu_1(u_1, u_2) = d \neq g = \mu_2(v_2, v_3)$.

Theorem 5.9. *Weak isomorphism of LFGs is a partially ordered relation on the set of all LFGs defined on the lattice L.*

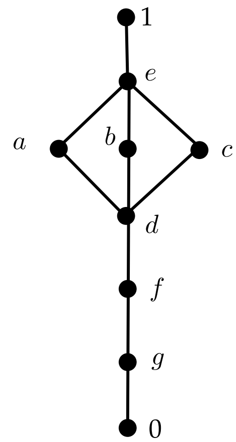

FIGURE 4.

Proof. By a weak isomorphism of LFGs $G_1^{L_1} = (V_1, \sigma_1, \mu_1)$ and $G_1^L = (V_2, \sigma_2, \mu_2)$, we mean a bijective mapping $h : V_1 \to V_2$ such that

$$\sigma_1(u) = \sigma_2[h(u)]$$

and

$$\mu_1(u, v) \leq \mu_2[h(u), h(v)], \quad \forall\ u, v \text{ in } V_1.$$

We have to show that this relation is reflexive, anti-symmetric and transitive.
Reflexivity: Consider the LFG $G^L = (V, \sigma, \mu)$. Choose 'h' and 'l' to be the identity map itself. Then obviously, $\mu(u, v) = \mu[h(u), h(v)]$, $\forall u, v$ in V.
Anti-symmetry: Let $G_1^L = (V_1, \sigma_1, \mu_1)$ and $G_1^L = (V_2, \sigma_2, \mu_2)$ be weak isomorphic. Then, $\exists$ a bijective mapping $h : V_1 \to V_2$ such that

$$\sigma_1(u) = \sigma_2[h(u)]$$

and

$$\mu_1(u, v) \leq \mu_2[h(u), h(v)], \quad \forall u,\ v \text{ in } V_1.$$

Again, let $G_1^L = (V_2, \sigma_2, \mu_2)$ and $G_1^L = (V_1, \sigma_1, \mu_1)$ be weak isomorphic. Then $\exists$ a bijective mapping $h' : V_2 \to V_1$ such that

$$\sigma_2(u) = \sigma_1[h'(u)]$$

and

$$\mu_2(u, v) \leq \mu_1[h'(u), h'(v)], \quad \forall\ u, v \text{ in } V_2.$$

The two inequalities can be satisfied simultaneously only when $V_1 = V_2$, $\sigma_1 = \sigma_2$ and $\mu_1 = \mu_2$. Hence the two LFGs are the same.
Transitivity: Next, let $G_1^L \wedge G_1^L$ be weak isomorphic with the bijective mapping $h : V_1 \to V_2$ and be weak isomorphic with the bijective mapping $k : V_2 \to V_3$. Consider $k \circ h : V_1 \to V_3$. It is also a bijection. Further,

$$\sigma_1(u) = \sigma_2[h(u)] = \sigma_3\{k[h(u)]\} = \sigma 3[k \circ h(u)]$$

and

$$\mu_1(u,v) \leq \mu_2[h(u),h(v)] \leq \mu 3[k \circ h(u), k \circ h(v)].$$

Thus $G_1^L \wedge G_3^L$ are weak isomorphic.

Hence weak isomorphism of LFGs is a partially ordered relation on the set of all LFGs defined on the lattice L. □

6. Complement of an LFG

Here, we firstly define the complement of an LFG and show that the complement of a strong LFG is also a strong LFG. We also obtain a condition on the lattice on which a self complementary LFG is defined.

Throughout this section, we take L to be a Boolean Algebra.

Definition 6.1. Consider the LFG $G^L = (V, \sigma, \mu)$ where L is a Boolean Algebra. We define its complement as the LFG $G^L = (V, \sigma^C, \mu^C)$ with

$$\sigma^C(u) = \sigma(u), \quad \forall u \in V \text{ and } \mu^C(u,v) = \sigma(u) \wedge \sigma(v) \wedge \mu(u,v)^C, \quad \forall \text{ edges } u\text{-}v \in G$$

Example 6.2. Let L be the following Boolean Algebra:

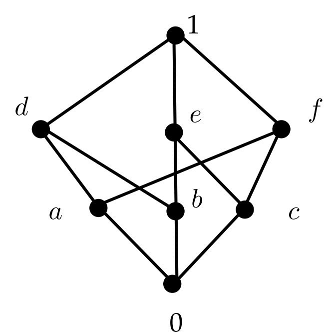

Figure 5.

Consider the LFG in Fig. 6. Then its complement is the LFG in Fig 7.

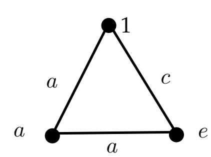

Figure 6.

Lemma 6.3. $(G^{LC}) = G^L$.

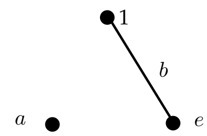

FIGURE 7.

Proof. Consider the LFG $G^L = (V, \sigma, \mu)$. Then,

$$\begin{aligned}(\mu^C)^C(u,v) &= \sigma(u) \wedge \sigma(v) \wedge [\mu^C(u,v)]^C \\ &= \sigma(u) \wedge \sigma(v) \wedge [(\sigma(u) \wedge \sigma(v) \vee \mu(u,v)^C]^C \\ &= \sigma(u) \wedge \sigma(v) \wedge [(\sigma(u) \wedge \sigma(v))^C \vee \mu(u,v)] \\ &= \sigma(u) \wedge \sigma(v) \wedge \mu(u,v) \\ &= \mu(u,v), \quad \forall u, v \in \mu^*\end{aligned}$$

Hence $(G^{LC})^C = G^L$. □

Theorem 6.4. *The complement of a strong LFG is also strong.*

Proof. Let the LFG $G^L = (V, \sigma, \mu)$ be strong. Then $\mu(u,v) = \sigma(u) \wedge \sigma(v)$, $\forall u, v \in \mu^*$ by definition.

When $\mu(u,v) = 0$, $\mu^C(u,v) = \sigma(u) \wedge \sigma(v) \wedge \mu(u,v)^C = \sigma(u) \wedge \sigma(v) \wedge 1 = \sigma(u) \wedge \sigma(v)$.
When $\mu(u,v) > 0$, $\mu^C(u,v) = \sigma(u) \wedge \sigma(v) \wedge \mu(u,v)^C = \sigma(u) \wedge \sigma(v) \wedge [\sigma(u) \wedge \sigma(v)]^C = 0$.
Hence $\mu^C(u,v) = \sigma(u) \wedge \sigma(v)$, $\forall u, v \in \mu^{C^*}$.
Hence the complement of a strong LFG is also strong. □

Definition 6.5. The LFG $G^L = (V, \sigma, \mu)$ is self complementary if $\exists$ an isomorphism between $G^L = (V, \sigma, \mu)$ and $G^{L^C}(V, \sigma^C, \mu^C)$.

Example 6.6. Let L be as in Fig. 5. Then Fig 8 shows a self complementary LFG and its complement. All the vertices in these graphs have membership degree 1.

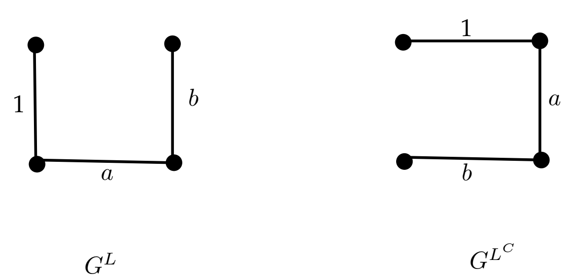

FIGURE 8.

Theorem 6.7. *If $G^L = (V, \sigma, \mu)$ is self complementary, then the Boolean Algebra L is a Boolean Ring with $a + b = a \vee b$ and $a \cdot b = 0$, $\forall a, b \in L$.*

Proof. Since $G^L = (V, \sigma, \mu)$ is self complementary, $\exists$ a bijective mapping $h : V \to V$ such that

$$\sigma^C[h(u)] = \sigma(u), \quad \forall u \in V \tag{6.1}$$

and

$$\mu^C[h(u), h(v)] = \mu(u, v), \quad \forall u, v \text{ in } V \tag{6.2}$$

Then by definition,

$$\mu^C[h(u), h(v)] = \sigma(u) \wedge \sigma(v) \wedge [\mu(h(u), h(v))]^C \text{ so that by (6.2)},$$

$$\mu(u, v) = \sigma(u) \wedge \sigma(v) \wedge [\mu(h(u), h(v))]^C, \quad \forall u, v \in V \tag{6.3}$$

We have $a + b = (a \wedge b^C) \vee (a^C \wedge b)$. Hence

$$\begin{aligned}\mu(u, v) + \mu[h(u), h(v)] &= \{\mu(u, v) \wedge [\mu(h(u), h(v))]^C\} \vee \{[\mu(u, v)]^C \wedge \mu[h(u), h(v)]\} \\ &= \mu(u, v) \vee \{[(\sigma(u) \wedge \sigma(v))^C \vee \mu(h(u), h(v))] \wedge \mu[h(u), h(v)]\} \\ &= \mu(u, v) \vee \mu[h(u), h(v)] \text{ by absorption law.}\end{aligned}$$

Thus $a + b = a \vee b$, $\forall a, b \in L$.
Next, $a \cdot b = a \wedge b$.
But $\mu(u, v) \cdot \mu[h(u), h(v)] = \mu(u, v) \wedge \mu[h(u), h(v)] = 0$ by (6.3). Thus $a \cdot b = 0$, $\forall a, b \in L$.
Hence L is a Boolean Ring with addition and multiplication as defined above. □

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Pramada Ramachandran (pramadars@yahoo.com)
P. G. Department of Mathematics, St. Paul's College, Kalamassery, Kochi-683 503, Kerala, India

K. V. Thomas (kvtbmc@gmail.com)
P. G. Department of Mathematics and Research Centre
, Bharata Mata College, Thrikkakara, Kochi-682 021, Kerala, India