**A**nnals of **F**uzzy **M**athematics and **I**nformatics
Volume 11, No. 2, (February 2016), pp. 293–300
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

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# Hahn Banach theorem on fuzzy normed linear spaces

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Received 14 March 2015; Revised 6 August 2015; Accepted 28 August 2015

Abstract. In the present paper, we consider fuzzy normed linear space with min t-norm and define equivalent fuzzy norms. Moreover, we show that all fuzzy norms on a finite dimensional vector space are equivalent. At the end we verify Hahn Banach theorem on fuzzy normed linear spaces.

2010 AMS Classification: 47s40

Keywords: Fuzzy normed linear space, Finite dimensional vector space, Equivalent fuzzy norms, Hahn Banach Theoren.

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## 1. Introduction

Felbin [4] has offered in 1992 an alternative definition of a fuzzy norm on a linear space with an associated metric of the Kaleva and Seikkala type [6]. He has shown that every finite dimensional normed linear space has a completion. Then Xiao and Zhu [9] have modified the definition of this fuzzy norm and studied the topological properties of fuzzy normed linear spaces. Another fuzzy norm is defined by Bag and Samanta [1].

Bag and Samanta [3] have defined concepts of weakly fuzzy boundedness, strongly fuzzy boundedness, fuzzy continuity, strongly fuzzy continuity, weakly fuzzy continuity, sequentially fuzzy continuity and fuzzy norm of linear operators with an associated fuzzy norm defined in [1]. In [5] the authors have defined a norm of operator with an associated fuzzy norm defined by Felbin [4], and studied some of their properties. This concept has been used in developing fuzzy functional analysis and its applications and a large number of papers by different authors have been published (see [7, 8]).

In the this paper, using fuzzy normed linear space with min t-norm, we show that on a finite dimensional vector space $X$, every pair of fuzzy norms $N_1$ and $N_2$ are equivalent. Moreover, we prove the Hahn Banach theorem on fuzzy normed linear spaces.

## 2. Preliminaries

We start our work with the following definitions.

**Definition 2.1** ([2]). A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a t-norm if it satisfies the following conditions:
(i) $*$ is associative and commutative,
(ii) $a * 1 = a$, for all $a \in [0,1]$,
(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.
If $*$ is continuous then it is called continuous t-norm.

**Definition 2.2** ([2]). The 3-tuple $(X, N, *)$ is said to be a fuzzy normed space if X is a vector space over $\mathbb{R}$ (real number), $*$ is a continuous t-norm and $N$ is a fuzzy set on $X \times \mathbb{R}$ satisfying the following conditions for every $x, y \in X$ and $s, t \in \mathbb{R}$
$(N1)$ $N(x,t) = 0$, for all $t \leq 0$,
$(N2)$ $x = 0$ if and only if $N(x,t) = 1$, for all $t > 0$,
$(N3)$ If $c \neq 0$ then $N(cx,t) = N(x, t/|c|)$, for all $t \in \mathbb{R}$,
$(N4)$ $N(x+u, s+t) \geq N(x,s) * N(u,t)$, for all $s, t \in \mathbb{R}$,
$(N5)$ $N(x,.)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

We assume that
$(N6)$ $N(x,t) > 0$, for all $t > 0$ implies $x = 0$,
$(N7)$ For $x \neq 0$, $N(x,.)$ is a continuous function of $\mathbb{R}$ and strictly increasing on the subset $\{t : 0 < N(x,t) < 1\}$ of $\mathbb{R}$.

In the sequel we fix $s * t = min(s,t)$ for all $s, t \in [0,1]$ and we write $(X, N)$ when $*$ is as indicated above.

**Example 2.3.** Let $(X, \|.\|)$ be a normed space. We define

$$N(x,t) = \begin{cases} t/(t + \|x\|) & , \quad t > 0,\ x \in X \\ 0 & , \quad t \leq 0,\ x \in X. \end{cases}$$

Then $(X, N)$ is a fuzzy normed linear space such that $N$ satisfying $(N7)$.

**Definition 2.4** ([3]). Let $(X, N)$ be a fuzzy normed linear space.
i) A sequence $\{x_n\} \subseteq X$ is said to converge to $x \in X$ ($\lim\limits_{n\to\infty} x_n = x$), if $\lim\limits_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$.
ii) A sequence $\{x_n\} \subseteq X$ is called Cauchy, if $\lim\limits_{n,m\to\infty} N(x_n - x_m, t) = 1$, for all $t > 0$.

**Definition 2.5.** If $X$ is a vector space over $R$, a seminorm is a function $p : X \longrightarrow [0, \infty)$ having the properties:
$(i)$ $p(c(x)) = |c| p(x)$ for all $c \in R$ and $x \in X$.
$(ii)$ $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

## 3. Hahn Banach Theorem

**Theorem 3.1.** *Let $(X, N)$ be a fuzzy normed linear space. Define $\|x\|_\alpha = \inf\{t > 0 : N(x,t) \geq \alpha\}$, $\alpha \in (0,1)$. Then $\{\|.\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of seminorms on $X$ and they are called $\alpha$-seminorms on $X$ corresponding to the fuzzy norm $N$ on $X$.*

*Proof.* (*i*) Let $x \in X$, $c \in R$ and $\alpha \in (0,1)$, we have

$$\begin{aligned} \|cx\|_\alpha &= \wedge\{t > 0 : N(cx,t) \geq \alpha\} \\ &= \wedge\{t > 0 : N(x, t/|c|) \geq \alpha\} \\ &= \wedge\{|c|t > 0 : N(x,t) \geq \alpha\} \\ &= |c|\|x\|_\alpha. \end{aligned}$$

(*ii*) Let $x, y \in X$ and $\alpha \in (0,1)$, we obtain that $N(x+y, \|x\|_\alpha + \|y\|_\alpha + \epsilon) \geq min\{N(x, \|x\|_\alpha + \epsilon/2), N(y, \|y\|_\alpha + \epsilon/2)\} \geq \alpha$, hence $\|x+y\|_\alpha \leq \|x\|_\alpha + \|y\|_\alpha + \epsilon$, as $\epsilon \longrightarrow 0$ then $\|x+y\|_\alpha \leq \|x\|_\alpha + \|y\|_\alpha$. □

**Definition 3.2.** A fuzzy norm $N_1$ on a vector space $X$ is said to be equivalent to a fuzzy norm $N_2$ on $X$ if there are families $\{m_\alpha\}_{\alpha \in (0,1)} \subseteq (0, +\infty)$ and $\{M_\alpha\}_{\alpha \in (0,1)} \subseteq (0, +\infty)$ such that for all $x \in X$, $t \in \mathbb{R}$ and $\alpha \in (0,1)$

$$N_1(x,t) \geq \alpha \text{ implies that } N_2(x, m_\alpha t) \geq \alpha,$$

and

$$N_2(x,t) \geq \alpha \text{ implies that } N_1(x, M_\alpha t) \geq \alpha.$$

**Example 3.3.** Let $X$ be a finite dimensional vector space and $\{e_1, ..., e_n\}$ be a basis for $X$. Define a fuzzy norm $N_0 : X \times \mathbb{R} \longrightarrow [0,1]$ as follows:

$$N_0(x,t) = \begin{cases} 1 & , \quad \sum_{i=1}^n |\lambda_i| \leq t \\ 0 & , \quad t < \sum_{i=1}^n |\lambda_i|, \end{cases}$$

where $x = \sum_{i=1}^{n} \lambda_i e_i$. It is clear that $\|x\|_{0\alpha} = \sum_{i=1}^{n} |\lambda_i|$, for all $\alpha \in (0,1)$.

**Theorem 3.4.** *Let $(X, N)$ be a finite dimensional fuzzy normed linear space such that $N$ satisfying $(N7)$. Then the fuzzy norm $N$ is equivalent to the fuzzy norm $N_0$ defined in Example 3.3.*

*Proof.* Let $\dim X = n$ and $\{e_1, ..., e_n\}$ be a basis for $X$. suppose that $\|.\|_\alpha$ is a $\alpha$-seminorms on $X$ corresponding to the fuzzy norm $N$ and $\|.\|_{0\alpha}$ ia a $\alpha$-seminorms on $X$ corresponding to the fuzzy norm $N_0$.
Now we show that for all $\alpha \in (0,1)$, there exists $M_\alpha > 0$ such that $\|x\|_\alpha \leq M_\alpha \|x\|_{0\alpha}$, for all $x \in X$.
Let $\alpha \in (0,1)$. Since $\|.\|_\alpha$ is a seminorm on $X$,

$$\|x\|_\alpha \leq \sum_{i=1}^{n} |\lambda_i| \|e_i\|_\alpha, \text{ for all } x = \sum_{i=1}^{n} \lambda_i e_i \in X.$$

Let $M_\alpha = \max\{\|e_i\|_\alpha : 1 \leq i \leq n\}$. So $\|x\|_\alpha \leq M_\alpha \sum_{i=1}^{n} |\lambda_i|$, for all $x = \sum_{i=1}^{n} \lambda_i e_i \in X$.

Since $\|x\|_{0\alpha} = \sum_{i=1}^{n} |\lambda_i|$ it follows that $\|x\|_\alpha \leq M_\alpha \|x\|_{0\alpha}$.
Now we show that for all $\alpha \in (0,1)$, there exists $m_\alpha > 0$ such that

$$\|x\|_\alpha \geq m_\alpha, \text{ for all } x = \sum_{i=1}^{n} \lambda_i e_i \in X \text{ with } \sum_{i=1}^{n} \lambda_i = 1.$$

Suppose that this is false. Then there exist $\alpha_0 \in (0,1)$ and sequence $\{x_m\} \subseteq X$ such that $x_m = \sum_{i=1}^{n} \lambda_i^m e_i$, $\sum_{i=1}^{n} \lambda_i^m = 1$, for all $m \in \mathbb{N}$ and $\lim_{m \longrightarrow +\infty} \|x_m\|_{\alpha_0} = 0$. Since $\sum_{i=1}^{n} \lambda_i^m = 1$, for all $m \in \mathbb{N}$, we have $|\lambda_1^m| \leq 1$, for all $m \in \mathbb{N}$. Hence the Sequence $\{\lambda_1^m\}$ is bounded. Consequently, by the Bolzano-Weierstrass theorem, $\{\lambda_1^m\}$ has a convergent subsequence. Let $\lambda_1^0$ denote the limit of that subsequence, and let $\{x_{1m}\}$ denote the corresponding subsequence of $\{x_m\}$. By the same argument, $\{x_{1m}\}$ has a subsequence $\{x_{2m}\}$ for which the corresponding subsequence of scalars $\lambda_2^m$ converges to $\lambda_2^0$. Continuing in this way, after n steps we obtain a subsequence $\{x_{nm}\}$ of $\{x_m\}$ such that $x_{nm} = \sum_{i=1}^{n} \gamma_i^m e_i$, $\sum_{i=1}^{n} \gamma_i^m = 1$ and $\lim_{m \longrightarrow +\infty} \gamma_i^m = \lambda_i^0$, for all $1 \leq i \leq n$. Hence $\|x_{nm} - x_0\|_{\alpha_0} \longrightarrow 0$ where $x_0 = \sum_{i=1}^{n} \lambda_i^0 e_i$. Since $\sum_{i=1}^{n} \lambda_i^0 = 1$ it follows that $x_0 \neq 0$. We have $\|x_{nm}\|_{\alpha_0} \longrightarrow \|x_0\|_{\alpha_0}$. So $\|x_0\|_{\alpha_0} = 0$. Hence $\inf\{t > 0 : N(x_0, t) \geq \alpha_0\} = 0$. Therefore $N(x_0, t) \geq \alpha_0$, for all $t > 0$. By $(N7)$, we obtain that $0 = N(x_0, 0) = \lim_{t \to 0^+} N(x_0, t) \geq \alpha_0$. This is a cotradiction.
Hence for all $\alpha \in (0,1)$, there exists $m_\alpha > 0$ such that

$$\|x\|_\alpha \geq m_\alpha, \text{ for all } x = \sum_{i=1}^{n} \lambda_i e_i \in X \text{ with } \sum_{i=1}^{n} \lambda_i = 1.$$

This implies that

$$\|x\|_\alpha \geq m_\alpha \sum_{i=1}^{n} |\lambda_i|, \text{ for all } x = \sum_{i=1}^{n} \lambda_i e_i \in X.$$

So

$$\|x\|_\alpha \geq m_\alpha \|x\|_{0\alpha}, \text{ for all } x \in X.$$

Now we show that $N$ and $N_0$ are equivalent.
Let $x \in X$, $t \in \mathbb{R}$ and $\alpha \in (0,1)$ and $N(x,t) \geq \alpha$. Hence $\|x\|_\alpha \leq t$. Therefore $m_\alpha \|x\|_{0\alpha} \leq \|x\|_\alpha \leq t$. Thus $\|x\|_{0\alpha} \leq t/m_\alpha$. So $\inf\{t > 0 : N_0(x,t) \geq \alpha\} \leq t/m_\alpha$. Then $N_0(x, t/m_\alpha) \geq \alpha$.
Let $x \in X$, $t \in \mathbb{R}$ and $\alpha \in (0,1)$ and $N_0(x,t) \geq \alpha$. Thus $\|x\|_{0\alpha} \leq t$. So $\|x\|_\alpha / M_\alpha \leq \|x\|_{0\alpha} \leq t$. Hence $\|x\|_\alpha \leq M_\alpha t$. If $N(x, tM_\alpha) < \alpha$. By $(N7)$, there exists $s > tM_\alpha$ such that $N(x,s) < \alpha$. Hence $s \leq \|x\|_\alpha$. This is a contradiction. Therefore $N(x, tM_\alpha) \geq \alpha$.
Therefore $N$ is equivalent to $N_0$. □

**Corollary 3.5.** *Let $X$ be a finite dimensional vector space and $N_1$, $N_2$ be fuzzy norms on $X$ satisfying $(N7)$. Then the fuzzy norm $N_1$ is equivalent to the fuzzy norm $N_2$.*

**Theorem 3.6.** ***(Hahn Banach Theorem)*** *Let $(X, N)$ be a fuzzy normed linear space and $(\mathbb{R}, N_0)$ be a fuzzy normed linear space defined in Example 3.3. Moreover,* *let $f : Z \longrightarrow \mathbb{R}$ be a linear functional which is defined on a subspace $Z$ of $X$ such that*

$$N(x,t) \geq \alpha \text{ implies that } N_0(f(x), \eta_\alpha^- t) \geq \alpha \text{ for all } x \in Z, \text{ and all } t \in \mathbb{R},$$

*for some increasing family $\{\eta_\alpha\}_{\alpha\in(0,1)} \subseteq (0,+\infty)$. Then $f$ has a linear extension $\widetilde{f}$ from $Z$ to $X$ satisfying*

$$N(x,t) \geq \alpha \text{ implies that } N_0(\widetilde{f}(x), \eta_\alpha^- t) \geq \alpha \text{ for all } x \in X, \text{ and all } t \in \mathbb{R}.$$

*Proof.* Let $A$ be a set of all linear extensions $g : Y \longrightarrow \mathbb{R}$ of $f$ satisfying

$$N(x,t) \geq \alpha, \text{ implies that } N_0(g(x), \eta_\alpha^- t) \geq \alpha \text{ for all } x \in Y, \text{ and all } t \in \mathbb{R}.$$

We define a partial ordering on $A$ by $g_1 \leq g_2$ if and only if $Y_1 \subseteq Y_2$ and $g_1(x) = g_2(x)$, for all $x \in Y_1$. Hence Zorn's Lemma yields a maximal element $\widetilde{f}$ of $A$. Now, we show that $D(\widetilde{f})$ is all of $X$. Suppose that this is false. Let $x_1 \in X - D(\widetilde{f})$ and consider the subspace $Y$ of $X$ spanned by $D(\widetilde{f})$ and $x_1$. We have $|\widetilde{f}(x)| \leq \eta_\alpha^- \|x\|_\alpha$, for all $x \in D(\widetilde{f})$ and all $\alpha \in (0,1]$. Hence

$$\begin{aligned} |\widetilde{f}(y) - \widetilde{f}(z)| = |\widetilde{f}(y-z)| &\leq \eta_\alpha^- \|y - z\|_\alpha \\ &\leq \eta_\alpha^- \|y + x_1\|_\alpha + \eta_\alpha^- \|x_1 + z\|_\alpha, \end{aligned}$$

for all $y, z \in D(\widetilde{f})$ and all $\alpha \in (0,1]$. Therefore

$$-\eta_\alpha^- \|x_1 + z\|_\alpha - \widetilde{f}(z) \leq \eta_\alpha^- \|y + x_1\|_\alpha - \widetilde{f}(y),$$

for all $y, z \in D(\widetilde{f})$ and all $\alpha \in (0,1]$. Suppose that $\alpha_n = 1/(n+1)$. Thus there exists $c_n \in \mathbb{R}$ such that

$$\sup_{z\in D(\widetilde{f})} (-\eta_{\alpha_n}^- \|x_1 + z\|_{\alpha_n} - \widetilde{f}(z)) \leq c_n \leq \inf_{y\in D(\widetilde{f})} (\eta_{\alpha_n}^- \|y + x_1\|_{\alpha_n} - \widetilde{f}(y)),$$

for all $n \in \mathbb{N}$. Hence there is a subsequence $\{c_{n_k}\}$ of $\{c_n\}$ such that $c_{n_k} \longrightarrow c$. Let $\alpha \in (0,1)$, then there exists $N_1 > 0$ such that $1/(n_k + 1) \leq \alpha$, for all $n_k > N_1$. Thus

$$\sup_{z\in D(\widetilde{f})} (-\eta_\alpha^- \|x_1 + z\|_\alpha - \widetilde{f}(z)) \leq c_{n_k} \leq \inf_{y\in D(\widetilde{f})} (\eta_\alpha^- \|x_1 + y\|_\alpha - \widetilde{f}(y)), \text{ for all } n_k > N_1.$$

Hence

$$\sup_{z\in D(\widetilde{f})} (-\eta_\alpha^- \|x_1 + z\|_\alpha - \widetilde{f}(z)) \leq c \leq \inf_{y\in D(\widetilde{f})} (\eta_\alpha^- \|x_1 + y\|_\alpha - \widetilde{f}(y)), \text{ for all } \alpha \in (0,1).$$

Therefore $-\eta_\alpha^- \|x_1 + z\|_\alpha - \widetilde{f}(z) \leq c \leq \eta_\alpha^- \|x_1 + y\|_\alpha - \widetilde{f}(y)$, for all $z, y \in D(\widetilde{f})$, and all $\alpha \in (0,1)$. Hence $|\widetilde{f}(y) + c| \leq \eta_\alpha^- \|x_1 + y\|_\alpha$, for all $y \in D(\widetilde{f})$, and all $\alpha \in (0,1)$. Thus $|\widetilde{f}(y) + \lambda c| \leq \eta_\alpha^- \|\lambda x_1 + y\|_\alpha$, for all $y \in D(\widetilde{f})$, and all $\alpha \in (0,1)$. Now we define a linear functional $\widetilde{g} : Y \longrightarrow \mathbb{R}$ by $\widetilde{g}(x + \lambda x_1) = \widetilde{f}(x) + \lambda c$, for all $x \in D(\widetilde{f})$. Hence

$$|\widetilde{g}(x + \lambda x_1)| = |\widetilde{f}(x) + \lambda c| \leq \eta_\alpha^- \|\lambda x_1 + x\|_\alpha, \text{ for all } \alpha \in (0,1].$$

Let $x \in D(\widetilde{f})$, $t \in \mathbb{R}$ and $N(x + \lambda x_1, t) \geq \alpha$. So $\eta_\alpha^- \|\lambda x_1 + x\|_\alpha \leq \eta_\alpha^- t$. Thus $|\widetilde{g}(x + \lambda x_1)| \leq \eta_\alpha^- \|\lambda x_1 + x\|_\alpha \leq \eta_\alpha^- t$ Hence $N(\widetilde{g}(x + \lambda x_1), \eta_\alpha^- t) = 1 \geq \alpha$. Therefore $\widetilde{g} \in A$. This ia a contradiction. So $D(\widetilde{f}) = X$. □

**Corollary 3.7.** *Let $(X, N)$ be a fuzzy normed linear space, $(\mathbb{R}, N_1)$ be a fuzzy normed linear space such that $N_1$ satisfying (N7) and $(\mathbb{R}, N_0)$ be a fuzzy normed linear space defined in Example 3.3. And let $f : Z \longrightarrow \mathbb{R}$ be a linear functional which is defined on a subspace $Z$ of $X$ such that*

$$N(x,t) \geq \alpha \text{ implies that } N_1(f(x), \eta_\alpha^- t) \geq \alpha \text{ for all } x \in Z, \text{ and all } t \in \mathbb{R},$$

*for some increasing family $\{\eta_\alpha\}_{\alpha\in(0,1)} \subseteq (0, +\infty)$. Moreover, let $\{\eta_\alpha m_\alpha\}_{\alpha\in(0,1)}$ be a increasing family of positive real number where*

$$N_1(x,t) \geq \alpha \text{ implies that } N_0(x, m_\alpha t) \geq \alpha,$$

*and*

$$N_0(x,t) \geq \alpha \text{ implies that } N_1(x, M_\alpha t) \geq \alpha,$$

*for all $x \in \mathbb{R}$, $t \in \mathbb{R}$ and $\alpha \in (0,1)$. Then $f$ has a linear extension $\widetilde{f}$ from $Z$ to $X$ satisfying*

$$N(x,t) \geq \alpha \text{ implies that } N_0(\widetilde{f}(x), m_\alpha M_\alpha \eta_\alpha^- t) \geq \alpha \text{ for all } x \in X, \text{ and all } t \in \mathbb{R}.$$

*Proof.* Let $x \in X$, $t \in \mathbb{R}$, $\alpha \in (0,1)$ and $N(x,t) \geq \alpha$. Hence $N_1(f(x), \eta_\alpha^- t) \geq \alpha$. So $N_0(f(x), m_\alpha \eta_\alpha t) \geq \alpha$. Thus

$$N(x,t) \geq \alpha \text{ implies that } N_0(f(x), m_\alpha \eta_\alpha^- t) \geq \alpha \text{ for all } x \in Z, \text{ and all } t \in \mathbb{R}.$$

Since $\{\eta_\alpha m_\alpha\}_{\alpha\in(0,1)}$ is a increasing family of positive real number, by Theorem 3.6, $f$ has a linear extension $\widetilde{f}$ from $Z$ to $X$ satisfying

$$N(x,t) \geq \alpha \text{ implies that } N_0(\widetilde{f}(x), m_\alpha \eta_\alpha^- t) \geq \alpha \text{ for all } x \in X, \text{ and all } t \in \mathbb{R}.$$

Let $x \in X$, $t \in \mathbb{R}$, $\alpha \in (0,1)$ and $N(x,t) \geq \alpha$. Hence $N_0(\widetilde{f}(x), m_\alpha \eta_\alpha^- t) \geq \alpha$. So $N_1(\widetilde{f}(x), M_\alpha m_\alpha \eta_\alpha t) \geq \alpha$. Thus

$$N(x,t) \geq \alpha \text{ implies that } N_1(\widetilde{f}(x), M_\alpha m_\alpha \eta_\alpha t) \geq \alpha \text{ for all } x \in X, \text{ and all } t \in \mathbb{R}.$$

□

Now we give two Examples which satisfy in Corollary 3.7.

**Example 3.8.** Let $(X, N)$ be a fuzzy normed linear space and $\mathbb{R}$ be a real number set. Define a fuzzy norm $N_1 : \mathbb{R} \times \mathbb{R} \longrightarrow [0, 1]$ as follows:

$$N_1(x,t) = \begin{cases} t/|x| & , \quad 0 < t \leq |x| \\ 1 & , \quad |x| \leq t \\ 0 & , \quad t \leq 0. \end{cases}$$

It is clear that $\|x\|_{1\alpha} = \alpha|x|$, for all $\alpha \in (0,1)$.
Moreover, let $f : Z \longrightarrow \mathbb{R}$ be a linear functional which is defined on a subspace $Z$ of $X$ such that

$$N(x,t) \geq \alpha \text{ implies that } N_1(f(x), \eta_\alpha^- t) \geq \alpha, \text{ for all } x \in Z, \ t \in \mathbb{R},$$

and for some increasing family $\{\eta_\alpha\}_{\alpha\in(0,1)} \subseteq (0, +\infty)$. And let $\{\eta_\alpha/\alpha\}_{\alpha\in(0,1)}$ be a increasing family of positive real number.
Suppose that $(\mathbb{R}, N_0)$ is a fuzzy normed linear space defined in Example 3.3. We have

$$N_1(x,t) \geq \alpha \text{ implies that } N_0(x, t/\alpha) \geq \alpha,$$

and

$$N_0(x,t) \geq \alpha \text{ implies that } N_1(x, \alpha t) \geq \alpha,$$

for all $x \in \mathbb{R}$, $t \in \mathbb{R}$ and $\alpha \in (0,1)$. Since $N_1$ satifies in $(N7)$ and $\{\eta_\alpha/\alpha\}_{\alpha\in(0,1)}$ is a increasing family of positive real number, by Corollary 3.7, $f$ has a linear extension $\widetilde{f}$ from $Z$ to $X$ satisfying

$$N(x,t) \geq \alpha \text{ implies that } N_1(\widetilde{f}(x), \eta_\alpha t) \geq \alpha \text{ for all } x \in X, \text{ and all } t \in \mathbb{R}.$$

**Example 3.9.** Let $(X, N)$ be a fuzzy normed linear space and $\mathbb{R}$ be a real number set. Define a fuzzy norm $N_2 : \mathbb{R} \times \mathbb{R} \longrightarrow [0,1]$ as follows:

$$N_1(x,t) = \begin{cases} t/(t+|x|) & , \quad t > 0 \\ 0 \quad , & \quad t \leq 0. \end{cases}$$

It is clear that $\|x\|_{2\alpha} = (\alpha/(1-\alpha))|x|$, for all $\alpha \in (0,1)$.
Moreover, let $f : Z \longrightarrow \mathbb{R}$ be a linear functional which is defined on a subspace $Z$ of $X$ such that

$$N(x,t) \geq \alpha \text{ implies that } N_1(f(x), \eta_\alpha^- t) \geq \alpha \text{ for all } x \in Z, \ t \in \mathbb{R},$$

and for some increasing family $\{\eta_\alpha\}_{\alpha\in(0,1)} \subseteq (0, +\infty)$. And let $\{\eta_\alpha((1-\alpha)/\alpha)\}_{\alpha\in(0,1)}$ be a increasing family of positive real number.
Suppose that $(\mathbb{R}, N_0)$ is a fuzzy normed linear space defined in Example 3.3. We have

$$N_1(x,t) \geq \alpha \text{ implies that } N_0(x, t/((1-\alpha)\alpha)) \geq \alpha,$$

and

$$N_0(x,t) \geq \alpha \text{ implies that } N_1(x, (\alpha/(1-\alpha))t) \geq \alpha,$$

for all $x \in \mathbb{R}$, $t \in \mathbb{R}$ and $\alpha \in (0,1)$. Since $N_1$ satifies in $(N7)$ and $\{\eta_\alpha((1-\alpha/\alpha)\}_{\alpha\in(0,1)}$ is a increasing family of positive real number, by Corollary 3.7, $f$ has a linear extension $\widetilde{f}$ from $Z$ to $X$ satisfying

$$N(x,t) \geq \alpha \text{ implies that } N_1(\widetilde{f}(x), \eta_\alpha t) \geq \alpha \text{ for all } x \in X, \text{ and all } t \in \mathbb{R}.$$

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