Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 11, No. 2, (February 2016), pp. 283–292
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI

On falling fuzzy ideals in BCI-algebra

Hossein Naraghi, Bahman Taherkhani

Received 2 June 2015; Revised 14 July 2015; Accepted 28 August 2015

Abstract. As a generalization of a fuzzy associative ideal of a BCI-algebra, the notion of a falling fuzzy associative ideal and falling fuzzy p-ideal of a BCI-algebra is introduced by using the theory of a falling shadow. Relations between falling fuzzy ideals and falling fuzzy associative ideals and falling fuzzy quasi associative ideals and falling fuzzy p-ideals are given. Conditions for a falling fuzzy ideal to be a falling fuzzy associative ideal are provided.

2010 AMS Classification: Primary 06F35; Secondary 03G25, 08A72

Keywords: Falling shadow, (Fuzzy) associative (Quasi associative) ideal, (Fuzzy) p-ideal, Falling fuzzy ideal, Falling fuzzy (quasi) associative ideal, Falling fuzzy p-ideal

Corresponding Author: Hossein Naraghi (ho.naraghi@pnu.ac.ir)

1. Introduction

In the study of a unified treatment of uncertainty modelled by means of combining probability and fuzzy set theory, Goodman [1] pointed out the equivalence of a fuzzy set and a class of random sets. Wang and Sanchez [16] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. Falling shadow representation theory shows us the way of selection relaid on the joint degrees distributions. It is reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. The mathematical structure of the theory of falling shadows is formulated in [13]. Tan et al. [14] established a theoretical approach to define a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows. Jun and Park [5] discussed the notion of a falling fuzzy subalgebra/ideal of a BCK/BCI-algebra. Jun and Park [6] studied the falling fuzzy positive implicative ideals of a BCK-algebra. In [8], Jun and Song studied the falling fuzzy quasi associative ideals of a BCI-algebra. Jun and Kang [4] considered the fuzzification of generalized Tarski filters of generalized Tarski algebras, and investigated related properties. They established characterizations of a fuzzy generalized Tarski filter, and introduced the notion of falling fuzzy generalized Tarski filters in generalized Tarski algebras based on the theory of falling shadows. They provided relations between fuzzy generalized Tarski filters and falling fuzzy generalized Tarski filters, and established a characterization of a falling fuzzy generalized Tarski filter. Jun and et al., [3], [7], provided some conditions for a falling fuzzy ideal to be a falling fuzzy positive implicative ideal and established a characterization of a falling fuzzy positive implicative ideals in BCK-algebras. In this paper, we use the theory of falling shadows to establish a falling fuzzy associative ideal and falling fuzzy p-ideal in a BCI-algebra as a generalization of a fuzzy associative ideal and fuzzy p-ideal in BCI-algebras. We provide relations between falling fuzzy associative ideals, falling fuzzy p-ideals and falling fuzzy ideals. We also consider relations between fuzzy associative ideals and falling fuzzy associative ideals and also relations between fuzzy p-ideals and falling fuzzy p-ideals. We give conditions for a falling fuzzy ideal to be a falling fuzzy associative ideal and falling fuzzy p-ideal. Finally, we consider some relation between falling fuzzy associative ideal, falling fuzzy quasi associative ideal and falling fuzzy p-ideal.

2. Preliminaries

By a BCI-algebra we mean an algebra $(X;*,0)$ of type $(2,0)$ satisfying the following axioms:

(1) $((x*y)*(x*z))*(z*y)=0$,
(2) $(x*(x*y))*y)=0$,
(3) $x*x=0$,
(4) $x*y=0$ and $y*x=0$ imply $x=y$.

for all $x,y,z \in X$. We can define a partial ordering $'' \leq ''$ on X by $x \leq y$ if and only if $x*y=0$.

The following statements are true in any BCI-algebra X:

(1.1) $(x*y)*z=(x*z)*y$,
(1.2) $x*0=x$,
(1.3) $(x*z)*(y*z) \leq x*y$,
(1.4) $x \leq y$ implies $x*z \leq y*z$ and $z*y \leq z*x$,
(1.5) $0*(x*y)=(0*x)*(0*y)$,
(1.6) $x*(x*(x*y))=x*y$.

Definition 2.1. A non empty subset I of X is called an ideal of X if it satisfies:

(I_1) $0 \in I$,
(I_2) $x*y \in I$ and $y \in I$ imply $x \in I$.

Definition 2.2 ([18])**.** A nonempty subset I of X is called an p-ideal of X if it satisfies condition (I_1) and
(I_3) $(x*z)*(y*z) \in I$ and $y \in I$ imply $x \in I$.
Putting $z=0$ in (I_3), we can see that every p-ideal is an ideal.

Definition 2.3 ([17])**.** A nonempty subset I of X is called an quasi associative ideal of X if it satisfies condition (I_1) and
(I_4) $x*(y*z) \in I$ and $y \in I$ imply $x*z \in I$.
Putting $z=0$ in (I_4), we can see that every quasi associative ideal is an ideal.

Definition 2.4 ([11]). A nonempty subset I of X is called an associative ideal of X if it satisfies condition (I_1) and
(I_5) $(x * z) * (0 * y) \in I$ and $z \in I$ imply $y * x \in I$.

In [11], Y. L. Liu and et.al. shown that the relation between p-ideals, quasi associative ideals and associative ideals and they proved some properties of them.

Definition 2.5. A fuzzy set μ of BCI-algebra X is called fuzzy ideal of X if it satisfies
(FI_1) $\mu(0) \geq \mu(x)$
(FI_2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$.

Definition 2.6 ([5, 10]). A fuzzy set μ of BCI-algebra X is called fuzzy p-ideal of X if it satisfies (FI_1) and
(FI_3) $\mu(x) \geq \min\{\mu((x * z) * (y * z)), \mu(y)\}$.

Definition 2.7 ([2]). A fuzzy set μ of BCI-algebra X is called fuzzy quasi associative ideal of X if it satisfies (FI_1) and
(FI_4) $\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\}$.

Definition 2.8 ([12]). A fuzzy set μ of BCI-algebra X is called fuzzy associative ideal of X if it satisfies (FI_1) and
(FI_4) $\mu(y * x) \geq \min\{\mu((x * z) * (0 * y)), \mu(z)\}$.

Proposition 2.9 ([9, 11]). *Let μ be a fuzzy set in a BCI-algebra X. Then μ is a fuzzy associative (resp., quasi associative,p-ideal) ideal of X if and only if for all $t \in [0, 1]$,*
$\mu_t \neq \varnothing \Rightarrow \mu_t$ *is an associative (resp., quasi associative,p-ideal) ideal of X, where $\mu_t = \{x \in X | \mu(x) \geq t\}$.*

We now display the basic theory on falling shadows. For getting the more information we refer to the papers [1, 13, 15, 16] regarding the theory of falling shadows. Given a universe of discourse $\mathfrak{U}$, let $\mathcal{P}(\mathfrak{U})$ denote the power set of $\mathfrak{U}$: For each $u \in \mathfrak{U}$, let

$$\dot{u} := \{E | u \in E \text{ and } E \subseteq \mathfrak{U}\}.$$

and for all $E \in \mathcal{P}(\mathfrak{U})$, let

$$\dot{E} := \{\dot{u} | u \in E\}.$$

An ordered pair $(\mathcal{P}(\mathfrak{U}), \mathfrak{B})$ is said to be a hyper-measurable structure on $\mathfrak{U}$ if $\mathfrak{B}$ is a σ-field in $\mathcal{P}(\mathfrak{U})$ and $\dot{\mathfrak{U}} \subseteq \mathfrak{B}$. Given a probability space $(\Omega, \mathfrak{A}, \mathcal{P})$ and a hyper-measurable structure $(\mathcal{P}(\mathfrak{U}), \mathfrak{B})$ on $\mathfrak{U}$, a random set on $\mathfrak{U}$ is defined to be a mapping $\zeta : \Omega \to \mathcal{P}(\mathfrak{U})$ which is $\mathfrak{A} - \mathfrak{B}$ measurable, that is, for all $C \in \mathfrak{B}$,

$$\zeta^{-1}(C) = \{\omega | \omega \in \Omega, \zeta(\omega) \in C\} \in \mathfrak{A}.$$

Suppose that ζ is a random set on $\mathfrak{U}$. Let $\hat{H}(u) := P(\omega | u \in \zeta(\omega))$ for each $u \in \mathfrak{U}$. Then $\hat{H}$ is a kind of fuzzy set in $\mathfrak{U}$. We call $\hat{H}$ a falling shadow of the random set ζ and ζ is called a cloud of $\hat{H}$.

3. Major section

In this paper we consider that X is a BCI-algebra unless otherwise.

Definition 3.1 ([6])**.** Let $(\Omega, \mathfrak{A}, \mathcal{P})$ be a probability space, and let $\zeta : \Omega \to \mathcal{P}(X)$ be a random set. If $\zeta(\omega)$ is an ideal (resp. a subalgebra) of X for any $\omega \in \Omega$, then the falling shadow $\hat{H}$ of the random set ζ, i.e., $\hat{H}(x) = P(\omega | x \in \zeta(\omega))$ is called a falling fuzzy ideal (resp. falling fuzzy subalgebra) of X.

Let $(\Omega, \mathfrak{A}, \mathcal{P})$ be a probability space and let $F(X) := \{\phi | \phi : \Omega \to X \; is\; a\; mapping\}$, where X is a BCI-algebra. For all $\phi, \psi \in F(X)$, we define an operation $\circledcirc$ on $F(X)$ by

$$\forall \omega \in \Omega, (\phi \circledcirc \psi)(\omega) = \phi(\omega) * \psi(\omega).$$

Let for all $\omega \in \Omega$ and for $o \in F(X)$ be defined $o(\omega) = 0$. In [6] checked that $(F(X), \circledcirc, o)$ is a BCI-algebra.

Definition 3.2. Let $(\Omega, \mathfrak{A}, \mathcal{P})$ be a probability space and let $\zeta : \Omega \to \mathcal{P}(X)$ be a random set. If $\zeta(\omega)$ is a p-ideal of X for any $\omega \in \Omega$ then the falling shadow $\hat{H}$ of the random set ζ, i.e., $\hat{H}(x) = P(\omega | x \in \zeta(\omega))$ is called a falling fuzzy p-ideal of X.

For any subset L of X and $\phi \in F(X)$, let $L_\phi := \{\omega | \phi(\omega) \in L\}$ and $\zeta : \Omega \to P(F(X)), \zeta(\omega) = \{\phi \in F(X) | \phi(\omega) \in L\}$.

Example 3.3. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Take $(\Omega, \mathcal{A}, P) = ([0,1]; \mathcal{A}, m)$ as a probability space and define a random set $\zeta : \Omega \to P(X)$ as follows:

$$\zeta(t) = \begin{cases} \{0, a\} & \text{if } \; t \in [0, 0.6) \\ X & \text{if } \; t \in [0.6, 1], \end{cases}$$

Then the falling shadow $\hat{H}$ of ζ is represented as follows:

$$\hat{H}(x) = \begin{cases} 1 & \text{if } \; x \in \{0, a\} \\ 0.4 & \text{if } \; x \in \{b, c\}, \end{cases}$$

and $\hat{H}$ is a falling fuzzy p-ideal of X.

Theorem 3.4. *If L is a p-ideal of X, then $\zeta(\omega) = \{\phi \in F(X) | \phi(\omega) \in L\}$ is a p-ideal of $F(X)$.*

Proof. Assume that L is a p-ideal of X and let $\omega \in \Omega$. Since $o(\omega) = 0 \in L$, by (I_1), we have o is in $\zeta(\omega)$. Let $f_1, f_2, f_3 \in F(X)$ be such that $f_2 \in \zeta(\omega)$ and $(f_1 \circledcirc f_3) \circledcirc (f_2 \circledcirc f_3) \in \zeta(\omega)$. Then $f_2(\omega) \in L$ and $(f_1(\omega) * f_3(\omega)) * (f_2(\omega) * f_3(\omega)) = [(f_1 \circledcirc f_3) \circledcirc (f_2 \circledcirc f_3)](\omega) \in L$. Since L is a p-ideal of X, we get $f_1(\omega) \in L$, i.e., $f_1 \in \zeta(\omega)$. Therefore $\zeta(\omega)$ is a p-ideal of $F(X)$. □

Since $\zeta^{-1}(\phi) = \{\omega \in \Omega | \phi \in \zeta(\omega)\} = \{\omega \in \Omega | \phi(\omega) \in L\} = L_\phi \in \mathfrak{A}$, we see that X is a random set on $F(X)$. Let $\hat{H} = P(\omega|\phi(\omega) \in L)$. Then $\hat{H}$ is a falling fuzzy p-ideal of $F(X)$.

Theorem 3.5. *Every fuzzy p-ideal of X is a falling fuzzy p-ideal of X.*

Proof. Consider the probability space $(\Omega, \mathfrak{A}, \mathcal{P}) = ([0,1], \mathfrak{A}, m)$, where $\mathfrak{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesque measure. Let μ be a fuzzy p-ideal of X. Then by Proposition 2.9, for all $t \in [0,1]$, μ_t is a p-ideal of X. Let $\zeta : [0,1] \to P(X)$ be a random set and $\zeta(t) = \mu_t$ for every $t \in [0,1]$. Then μ is a falling fuzzy p-ideal of X.

□

Theorem 3.6. *Every falling fuzzy p-ideal is both a falling fuzzy ideal and a falling fuzzy subalgebra.*

Proof. Let $\hat{H}$ be a falling fuzzy p-ideal of X. Then $\zeta(\omega)$ is a p-ideal of X and hence it is both an ideal and a subalgebra of X. Thus $\hat{H}$ is both a falling fuzzy ideal and a falling fuzzy subalgebra of X.

□

The converse of Theorem 3.6, is not true in general as shown by the following examples:

Example 3.7. Consider a BCI-algebra $X = \{0, a, b\}$ with the following Cayley table:

*	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

Take $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ as a probability space and define a random set $\zeta : \Omega \to P(X)$ as follows:

$$\zeta(t) = \begin{cases} \{0\} & \text{if } \ t \in [0, 0.4) \\ X & \text{if } \ t \in [0.4, 1]. \end{cases}$$

Then $\zeta(t)$ is both a subalgebra and an ideal of X for all $t \in [0,1]$. But if $t \in [0, 0.4)$ then $\zeta(t)$ is not a p-ideal of X since $(a * a) * (0 * a) \in \zeta(t)$ and $0 \in \zeta(t)$ but $a \notin \zeta(t)$.

Let $(\Omega, \mathfrak{A}, \mathcal{P})$ be a probability space and $\hat{H}$ a falling shadow of a random set $\zeta : \Omega \to P(X)$. For any $x \in X$, let $\Omega(x; \zeta) := \{\omega \in \Omega | x \in \zeta(\omega)\}$. Then $\Omega(x; \zeta) \in \mathfrak{A}$.

Theorem 3.8. *Let $\hat{H}$ be a falling shadow of a random set $\zeta : \Omega \to P(X)$. If $\hat{H}$ is a falling fuzzy p-ideal of X, then for all $x, y, z \in X$ we have*

$$\Omega((x * z) * (y * z); \zeta) \cap \Omega(y; \zeta) \subseteq \Omega(x; \zeta).$$

Proof. Assume that $\omega \in \Omega((x * z) * (y * z); \zeta) \cap \Omega(y; \zeta)$. Then $(x * z) * (y * z) \in \zeta(\omega)$ and $y \in \zeta(\omega)$. Since $\zeta(\omega)$ is a p-ideal of X then $x \in \zeta(\omega)$. Therefore $\omega \in \Omega(x; \zeta)$.

□

Corollary 3.9. *Let $\hat{H}$ be a falling shadow of a random set $\zeta : \Omega \to P(X)$. If $\hat{H}$ is a falling fuzzy p-ideal of X, then for all $x, y \in X$ we have*
(i) $\Omega(x * y; \zeta) \cap \Omega(y; \zeta) \subseteq (x; \zeta)$,
(ii) $\Omega(0 * (0 * x); \zeta) \subseteq \Omega(x; \zeta)$.

Proof. It's straightforward.

□

Theorem 3.10. *Let $\hat{H}$ be a falling shadow of a random set $\zeta : \Omega \to P(X)$ be a falling fuzzy ideal of X. Then $\hat{H}$ is a falling fuzzy p-ideal of X if and only if for all $x \in X$, $\Omega(0 * (0 * x); \zeta) \subseteq \Omega(x; \zeta)$.*

Proof. If $\hat{H}$ is a falling fuzzy p-ideal of X then by Corollary 3.9 we have, $\Omega(0 * (0 * x); \zeta) \subseteq \Omega(x; \zeta)$. Conversely, for all $x, y, z \in X$ we have $(x * z) * (y * z) \in \zeta(\omega)$ and $y \in \zeta(\omega)$, then we have

$$\begin{aligned}((0 * (0 * x)) * y) * ((x * z) * (y * z)) &= ((0 * ((x * z) * (y * z))) * (0 * x)) * y \\ &= ([((0 * x) * (0 * z)) * ((0 * y) * (0 * z))] * (0 * x)) * y \\ &\leq (((0 * x) * (0 * y)) * (0 * x)) * y \\ &= (0 * (0 * y)) * y = 0,\end{aligned}$$

since $0 \in \zeta(\omega)$ and $(x * z) * (y * z) \in \zeta(\omega)$ and $y \in \zeta(\omega)$ then $0 * (0 * x) \in \zeta(\omega)$, i.e., $\omega \in \Omega(0 * (0 * x); \zeta) \subseteq \Omega(x; \zeta)$, so $x \in \zeta(\omega)$. Hence $\zeta(\omega)$ is a p-ideal, therefore $\hat{H}$ is a falling fuzzy p-ideal of X.

□

Definition 3.11. Let $(\Omega, \mathfrak{A}, \mathcal{P})$ be a probability space and let $\zeta : \Omega \to \mathcal{P}(X)$ be a random set. If $\zeta(\omega)$ is a associative ideal of X for any $\omega \in \Omega$ then the falling shadow $\hat{H}$ of the random set ζ, i.e., $\hat{H}(x) = P(\omega | x \in \zeta(\omega))$ is called a falling fuzzy associative ideal of X.

Example 3.12. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

*	0	a	b	c
0	0	a	b	a
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Take $(\Omega, \mathcal{A}, P) = ([0, 1]; \mathcal{A}, m)$ as a probability space and define a random set $\zeta : \Omega \to P(X)$ as follows:

$$\zeta(t) = \begin{cases} \{0, a\} & \text{if } \ t \in [0, 0.6) \\ X & \text{if } \ t \in [0.6, 1], \end{cases}$$

Then the falling shadow $\hat{H}$ of ζ is represented as follows:

$$\hat{H}(x) = \begin{cases} 1 & \text{if } \ x \in \{0, a\} \\ 0.4 & \text{if } \ x \in \{b, c\}, \end{cases}$$

and $\hat{H}$ is a falling fuzzy associative ideal of X.

Theorem 3.13. *If L is an associative ideal of X, then $\zeta(\omega) = \{\phi \in F(X) | \phi(\omega) \in L\}$ is an associative ideal of $F(X)$.*

Proof. Assume that L is an associative ideal of X and let $\omega \in \Omega$. Since $o(\omega) = 0 \in L$, by (I_1), we have o is in $\zeta(\omega)$. Let $f_1, f_2, f_3 \in F(X)$ be such that $f_3 \in \zeta(\omega)$ and $(f_1 \circledcirc f_3) \circledcirc (o \circledcirc f_2) \in \zeta(\omega)$. Then $f_3(\omega) \in L$ and $(f_1(\omega) * f_3(\omega)) * (0 * f_2(\omega)) = [(f_1 \circledcirc f_3) \circledcirc (o \circledcirc f_2)](\omega) \in L$. Since L is an associative ideal of X, we get $f_2(\omega) * f_1(\omega) \in L$, i.e., $f_2 \circledcirc f_1 \in \zeta(\omega)$. Therefore $\zeta(\omega)$ is an associative ideal of $F(X)$.

□

Since $\zeta^{-1}(\phi) = \{\omega \in \Omega | \phi \in \zeta(\omega)\} = \{\omega \in \Omega | \phi(\omega) \in L\} = L_\phi \in \mathfrak{A}$, we see that X is a random set on $F(X)$. Let $\hat{H} = P(\omega | \phi(\omega) \in L)$. Then $\hat{H}$ is a falling fuzzy associative ideal of $F(X)$.

Theorem 3.14. *Every fuzzy associative ideal of X is a falling fuzzy associative ideal of X.*

Proof. Consider the probability space $(\Omega, \mathfrak{A}, \mathcal{P}) = ([0,1], \mathfrak{A}, m)$, where $\mathfrak{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesque measure. Let μ be a fuzzy associative ideal of X. Then by Proposition 2.9, for all $t \in [0,1]$, μ_t is an associative ideal of X. Let $\zeta : [0,1] \to P(X)$ be a random set and $\zeta(t) = \mu_t$ for every $t \in [0,1]$. Then μ is a falling fuzzy associative ideal of X.

□

Theorem 3.15. *Every falling fuzzy associative ideal is both a falling fuzzy ideal and a falling fuzzy subalgebra.*

Proof. Let $\hat{H}$ be a falling fuzzy associative ideal of X. Then $\zeta(\omega)$ is an associative ideal of X and hence it is both an ideal and a subalgebra of X. Thus $\hat{H}$ is both a falling fuzzy ideal and a falling fuzzy subalgebra of X.

□

The converse of Theorem 3.15, is not true in general as shown by the following examples:

Example 3.16. In example 3.7, $\zeta(t)$ is both a subalgebra and an ideal of X for all $t \in [0,1]$. But if $t \in [0, 0.4)$ then $\zeta(t)$ is not a associative ideal of X since $(0 * 0) * (0 * a) \in \zeta(t)$ and $0 \in \zeta(t)$ but $a * 0 = a \notin \zeta(t)$.

Theorem 3.17. *Let $\hat{H}$ be a falling shadow of a random set $\zeta : \Omega \to P(X)$ be a falling fuzzy ideal of X. Then the following statements are equivalent:*
(i) $\hat{H}$ is a falling fuzzy associative ideal of X,
(ii) for all $x, y, z \in X$, $\Omega((x * z) * (0 * y); \zeta) \subseteq \Omega(y * (x * z); \zeta)$,
(iii) for all $x, y \in X$, $\Omega(x * (0 * y); \zeta) \subseteq \Omega(y * x; \zeta)$.

Proof. $(i) \Rightarrow (ii)$ Assume that $\hat{H}$ is a falling fuzzy associative ideal of X. Then for all $\omega \in \Omega((x * z) * (0 * y); \zeta)$ we have $((x * z) * 0) * (0 * y) \in \zeta(\omega)$ and by (I_5) we have $y * (x * z) \in \zeta(\omega)$, i.e., $\omega \in \Omega(y * (x * z); \zeta)$.
$(ii) \Rightarrow (iii)$ Taking $z = 0$ in (ii), induces (iii).
$(iii) \Rightarrow (i)$ Assume that $x, y, z \in X$ such that $(x * z) * (0 * y) \in \zeta(\omega)$ and $z \in \zeta(\omega)$ since $(x * (0 * y)) * ((x * z) * (0 * y)) \leq x * (x * z) \leq z$ we have $x * (0 * y) \in \zeta(\omega)$. Now, by (iii), $y * x \in \zeta(\omega)$. Hence $\zeta(\omega)$ is a associative ideal of X and therefore $\hat{H}$ is a falling fuzzy associative ideal of X.

□

Theorem 3.18. *Let $\hat{H}$ be a falling shadow of a random set $\zeta : \Omega \to P(X)$ be a falling fuzzy ideal of X such that for all $x, y, z \in X$, $\Omega((x * y) * z; \zeta) = \Omega(x * (y * z); \zeta)$. Then $\hat{H}$ is a falling fuzzy associative ideal of X.*

Proof. For all $x \in X$ we have $\Omega(0*x;\zeta) = \Omega((x*x)*x;\zeta) = \Omega(x*(x*x);\zeta) = \Omega(x;\zeta)$, i.e., $\Omega(0 * x; \zeta) = \Omega(x; \zeta)$. Now, for all $x, y \in X$ we have

$$\begin{aligned}
\Omega(x * (0 * y); \zeta) &= \Omega(x * y; \zeta) \\
&= \Omega(0 * (x * y); \zeta) \\
&= \Omega((0 * x) * y; \zeta) \\
&= \Omega((0 * y) * x; \zeta) \\
&= \Omega(0 * (y * x); \zeta) \\
&= \Omega(0 * (y * x); \zeta) \\
&= \Omega(y * x; \zeta).
\end{aligned}$$

Hence by 3.17(iii), $\hat{H}$ is a falling fuzzy associative ideal of X. □

Theorem 3.19. *Let $\hat{H}$ be a falling shadow of a random set $\zeta : \Omega \to P(X)$ be a falling fuzzy ideal of X. If $\hat{H}$ is a falling fuzzy associative ideal of X, then the set $\Pi = \{x \in X | \Omega(x; \zeta) = \Omega(0; \zeta)\}$ is an associative ideal of X.*

Proof. Obviously $0 \in \Pi$. Let $x, y, z \in X$ such that $(x * z) * (0 * y) \in \Pi$ and $z \in \Pi$, then we have

$$\Omega(0; \zeta) \supseteq \Omega(y * x; \zeta) \supseteq \Omega((x * z) * (0 * y); \zeta) \cap \Omega(z; \zeta) = \Omega(0; \zeta).$$

Hence $y * x \in \Pi$, therefore Π is associative ideal of X. □

Theorem 3.20. *Let $\hat{H}$ be a falling shadow of a random set $\zeta : \Omega \to P(X)$ be a falling fuzzy ideal of X. If $\hat{H}$ is a falling fuzzy associative ideal of X, then it is falling fuzzy p-ideal of X.*

Proof. By setting $x = z = 0$ in Theorem 3.17(ii), we have

$$\Omega((0 * 0) * (0 * y); \zeta) \subseteq \Omega(y * (0 * 0); \zeta),$$

hence for all $y \in X$, $\Omega(0 * (0 * y); \zeta) \subseteq \Omega(y; \zeta)$, and by Theorem 3.10, $\hat{H}$ is a falling fuzzy p-ideal of X. □

Theorem 3.21. *Let $\hat{H}$ be a falling shadow of a random set $\zeta : \Omega \to P(X)$ be a falling fuzzy ideal of X. If $\hat{H}$ is a falling fuzzy associative ideal of X, then it is falling fuzzy quasi associative ideal of X.*

Proof. For all $x, y \in X$ we have

$$(0 * (0 * (y * (0 * x)))) * (x * (0 * y)) = ((0 * (0 * y)) * (0 * x)) * (x * (0 * y)) = 0,$$

so for all $x, y \in X$, we have

$$\Omega(y * (0 * x); \zeta) \subseteq \Omega(0 * (0 * (y * (0 * x))); \zeta) \subseteq \Omega(x * (0 * y); \zeta)$$

Therefore for all $x, y \in X$, $\Omega(y * (0 * x); \zeta) \subseteq \Omega(x * (0 * y); \zeta)$ and since $\hat{H}$ is a falling fuzzy associative ideal of X, by Theorem 3.17(iii), $\Omega(x * (0 * y); \zeta) \subseteq \Omega(y * x)$. Hence for all $x, y \in X$, $\Omega(y*(0*x);\zeta) \subseteq \Omega(y*x)$, i.e., $\hat{H}$ is a falling fuzzy quasi associative ideal of X, by [Theorem 3.14,[8]].

$\square$

Theorem 3.22. *Let $\hat{H}$ be a falling shadow of a random set $\zeta : \Omega \to P(X)$ be a falling fuzzy ideal of X. Then $\hat{H}$ is a falling fuzzy associative ideal of X if and only if $\hat{H}$ is falling fuzzy quasi associative ideal and falling fuzzy p-ideal of X.*

Proof. It is enough to show that for all $x, y \in X$, $\Omega(x*(0*y);\zeta) \subseteq \Omega(y*x;\zeta)$. For all $x, y \in X$ we have,

$$(0*(y*x))*(x*y) = ((0*y)*(0*x))*(x*y) = 0,$$

we obtain $0*(y*x) \leq x*y$. Thus $\Omega(x*y;\zeta) \subseteq \Omega(0*(y*x);\zeta)$. Since $\hat{H}$ is falling fuzzy subalgebra, so we have

$$\Omega(x*y;\zeta) \subseteq \Omega(0*(y*x);\zeta) \subseteq \Omega(0*(0*(y*x));\zeta) \subseteq \Omega(y*x;\zeta).$$

Now, it follows from Theorem 3.14(ii) of [8] that $\Omega(x*(0*y);\zeta) \subseteq \Omega(x*y;\zeta) \subseteq \Omega(y*x;\zeta)$. Hence by Theorem 3.17 (iii), $\hat{H}$ is a falling fuzzy associative ideal of X. Conversely, it follows from Theorem 3.20 and Theorem 3.18.

$\square$

4. Conclusions

We introduced the notion of a falling fuzzy associative ideal and falling fuzzy p-ideal of a BCI-algebra as a generalization of a fuzzy associative ideal and fuzzy p-ideal of a BCI-algebra, respectively, by using the theory of a falling shadow. Relations between falling fuzzy ideals and falling fuzzy associative ideals and falling fuzzy quasi associative ideals and falling fuzzy p-ideals are given. Conditions for a falling fuzzy ideal to be a falling fuzzy associative ideal are provided.

Acknowledgements. The authors are thankful to the referee for a careful reading of the paper and for valuable comments and suggestions.

References

[1] I. R. Goodman, Fuzzy sets as equivalence classes of random sets, in ecent Developments in Fuzzy Sets and Possibility TheoryR. Yager, Ed.), Pergamon, New York 198.
[2] Y. B. Jun, Fuzzy quasi-associative ideals in BCI-algebras, J. Fuzzy Math. 4 (3) (1996) 567–581.
[3] Y. B. Jun and M. S. Kang, Fuzzy positive implicative ideals of BCK-algebras based on the theory of falling shadows, Comput. Math. Appl. 61 (1) (2011) 62–67.
[4] Y. B. Jun and M. S. Kang, Fuzzifications of generalized Tarski filters in Tarski algebras, Comput. Math. Appl. 61 (1) (2011) 1–7.
[5] Y.B. Jun and J. Meng, Fuzzy p -ideals in BCI-algebra, Math. Japon. 2 (1994) 271–282.
[6] Y. B. Jun and C. H. Park, Falling shadows applied to subalgebras and ideals of BCK/BCI-algebras, Honam Mathematical J. 34 (2) (2012) 135–144.
[7] Y. B. Jun, M. A. Öztürk and A. Firat, Characterizations of a falling fuzzy positive implicative ideals in BCK-algebras, Annals of Fuzzy Mathematics and Informatics, Volume 7 (2) (2014) 197–204.
[8] Y.B. Jun and S. Z. Song, Falling fuzzy quasi-associative ideals of BCI-algebras, Filomat 26 (4) (2012) 649–656.
[9] A. Kordi and A. Moussavi, On fuzzy ideals of BCI-algebras, PU.M.A 18 (34) (2007) 301–310.

[10] A. Kordi and A. Moussavi and A. Ahmadi, Algorithms and computations for (m, n)-fold p-ideals in BCI-algebras, Journal of Applied Logic 8 (2010) 22–32 22-32.

[11] Y. Y. Liu, J. Meng, X. H. Zhang and Z. C. Yue, q-ideals and a-ideals in BCI-algebras, Southeast Asian Bull. Math. 24 (2) (2000) 243–253.

[12] Y. L. Liu and X. H. Zhang, Fuzzy a-ideals in BCI-algebras, Adv. in Math. (China) 31 (2002) 65–73.

[13] S. K. Tan, P. Z. Wang and E. S. Lee, Fuzzy set operations based on the theory of falling shadows, J. Math. Anal. Appl. 174 (1993) 242–255.

[14] S. K. Tan, P. Z. Wang and X. Z. Zhang, Fuzzy inference relation based on the theory of falling shadows, Fuzzy Sets and Systems 53 (1993) 179–188.

[15] P. Z. Wang, Fuzzy Sets and Falling Shadows of Random Sets, Beijing Normal Univ. Press, People Republic of China 1985. [In Chinese]

[16] P. Z. Wang and E. Sanchez, Treating a fuzzy subset as a projectable random set, in: M. M. Gupta, E. Sanchez, Eds., uzzy Information and Decision(Pergamon, New York (1982) 212–219.

[17] Z. Yue and X. H. Zhang, Quasi-associative ideals of BCI-algebras, Selected Papers on BCK and BCI-algebras (in P.R. China) 1 (1992) 85–86.

[18] X. H. Zhang, J. Hao and S. A. Bhatti, On p-ideals of a BCI-algebra, Punjab Univ. J. Math. 27 (1994) 121–128.

Hossein Naraghi (ho.naraghi@pnu.ac.ir)
Department of Mathematics, Payame Noor University, P. O. Box: 19395-3697, Tehran, Iran

Bahman Taherkhani (bahman.taherkhani@pnu.ac.ir)
Department of Mathematics, Payame Noor University, P. O. Box: 19395-3697, Tehran, Iran