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β-connectedness in fuzzy soft topological spaces

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Abstract.
The purpose of this paper is to introduce the concepts of fuzzy soft β-open sets, fuzzy soft β-continuous functions, fuzzy soft β-connectedness, fuzzy soft β-strongly connectedness, fuzzy soft β-C_5 connectedness. Some interesting properties of these notions are studied. In this connection, interrelations are discussed. Examples are provided wherever necessary.

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [8]. Molodstov [4] introduced the concept of soft sets. The definition of fuzzy soft sets was introduced by Roy, Biswas and Maji [3]. The notion of fuzzy soft topological space was introduced by Roy and Samanta [5]. The definition of fuzzy β-open set was given by Balasubramanian [1]. In this paper, the concepts of fuzzy soft β-open sets, fuzzy soft β-connectedness are introduced and studied. Some interesting characterizations are discussed. In this connection, interrelations are investigated. Examples are provided wherever necessary.

2. Preliminaries

Throughout this papers, U refers to an initial universe, E is the set of all parameters for U and I^U denotes the set of all fuzzy subset of U and also by "(U, E)", we mean "the universal set U and the parameter E".

Definition 2.1 ([3]). Let $A \subseteq E$. Then the mapping $F_A : E \to I^U$ defined by $F_A(e) = \mu^e_{F_A}$ (a fuzzy subset of U) is called fuzzy soft set over (U, E), where $\mu^e_{F_A} = \bar{0}$ if $e \in E - A$ and $\mu^e_{F_A} \neq \bar{0}$ if $e \in A$, where $\bar{0}(x) = 0$ for each $x \in U$.

The set of all fuzzy soft set over (U, E) is denoted by $FS(U, E)$.

Definition 2.2 ([5]). The fuzzy soft set $F_\Phi \in FS(U, E)$ is called null fuzzy soft set and it is denoted by $\tilde{\Phi}$. Here $F_\Phi(e) = \bar{0}$ for every $e \in E$.

Definition 2.3 ([5]). Let $F_E \in FS(U, E)$ and $F_E(e) = \bar{1}$ for all $e \in E$, where $\bar{1}(x) = 1$ for each $x \in U$. Then F_E is called absolute fuzzy soft set. It is denoted by $\tilde{E}$.

Definition 2.4 ([5]). Let $F_A, G_B \in FS(U, E)$. If $F_A(e) \subseteq G_B(e)$ for all $e \in E$, i.e., if $\mu^e_{F_A}(x) \subseteq \mu^e_{G_B}(x)$ for all $x \in U$, then F_A is said to be fuzzy soft subset of G_B, denoted by $F_A \tilde{\subseteq} G_B$.

Definition 2.5 ([5]). Let $F_A, G_B \in FS(U, E)$. Then the union of F_A and G_B is also a fuzzy soft set H_C, defined by $H_C(e) = \mu^e_{H_C} = \mu^e_{F_A} \cup \mu^e_{G_B}$ for all $e \in E$, where $C = A \cup B$. Here we write $H_C = F_A \tilde{\cup} G_B$.

Definition 2.6 ([5]). Let $F_A, G_B \in FS(U, E)$. Then the intersection of F_A and G_B is also a fuzzy soft set H_C, defined by $H_C(e) = \mu^e_{H_C} = \mu^e_{F_A} \cap \mu^e_{G_B}$ for all $e \in E$, where $C = A \cap B$. Here we write $H_C = F_A \tilde{\cap} G_B$.

Definition 2.7 ([7]). Let $F_A \in FS(U, E)$. The complement of F_A is denoted by ${F_A}^c$ and is defined by ${F_A}^c : E \to I^U$ is a mapping given by ${F_A}^c(e) = (F_A(e))^c$, for every $e \in E$.

Proposition 2.8 ([7]). *Let $F_A, G_B, H_C \in FS(U, E)$. It can be verified that the following hold according to our notion of fuzzy soft sets.*

(i) $\tilde{\Phi} \tilde{\subseteq} F_A \tilde{\subseteq} \tilde{E}$.
(ii) $F_A \tilde{\cup} G_B = G_B \tilde{\cup} F_A$ *and* $F_A \tilde{\cap} G_B = G_B \tilde{\cap} F_A$.
(iii) $F_A \tilde{\cup} (G_B \tilde{\cup} H_C) = (F_A \tilde{\cup} G_B) \tilde{\cup} H_C$, $F_A \tilde{\cap} (G_B \tilde{\cap} H_C) = (F_A \tilde{\cap} G_B) \tilde{\cap} H_C$.
(iv) $F_A \tilde{\cup} (G_B \tilde{\cap} H_C) = (F_A \tilde{\cup} G_B) \tilde{\cap} (F_A \tilde{\cup} H_C)$, $F_A \tilde{\cap} (G_B \tilde{\cup} H_C) = (F_A \tilde{\cap} G_B) \tilde{\cup}$
$(F_A \tilde{\cap} H_C)$.
(v) $F_A, G_B \tilde{\subseteq} F_A \tilde{\cup} G_B$, $F_A \tilde{\cap} G_B \tilde{\subseteq} F_A, G_B$.
(vi) $F_A \tilde{\subseteq} G_B \Rightarrow {G_B}^c \tilde{\subseteq} {F_A}^c$.
(vii) $\tilde{\Phi}^c = \tilde{E}$, $\tilde{E}^c = \tilde{\Phi}$.
(viii) $({F_A}^c)^c = F_A$.
(ix) $(\cap_j(F^j_{A_j}))^c = \cup_j({F^j_{A_j}}^c)$, $j \in J$ *be an index set.*
(x) $(\cup_j(F^j_{A_j}))^c = \cap_j({F^j_{A_j}}^c)$, $j \in J$ *be an index set.*

Definition 2.9 ([5]). A fuzzy soft topology $\mathfrak{J}$ on (U, E) is a family of fuzzy soft sets over (U, E) satisfying the following properties.

(i) $\tilde{\Phi}, \tilde{E} \in \mathfrak{J}$,
(ii) If $F_A, G_B \in \mathfrak{J}$ then $F_A \tilde{\cap} G_B \in \mathfrak{J}$,
(iii) If $F^j_{A_j} \in \mathfrak{J}$ for all $j \in J$, an index set, then $\cup_{j \in J} F^j_{A_j} \in \mathfrak{J}$.

Definition 2.10 ([5]). If $\mathfrak{J}$ is a fuzzy soft topology on (U, E), the triple $(U, E, \mathfrak{J})$ is said to be a fuzzy soft topological space. Also each member of $\mathfrak{J}$ is called a fuzzy soft open set in $(U, E, \mathfrak{J})$.

Definition 2.11 ([5]). A fuzzy soft set F_A over (U, E) is called a fuzzy soft closed set in $(U, E, \mathfrak{J})$ if its complement ${F_A}^c$ is fuzzy soft open set in $(U, E, \mathfrak{J})$.

Definition 2.12 ([7]). Let $(U, E, \mathfrak{J})$ be a fuzzy soft topological space. Let F_A be a fuzzy soft set over (U, E). The fuzzy soft closure of F_A is defined as the intersection of all fuzzy soft closed sets which contains F_A and is denoted by $\overline{F_A}$. We write

$$\overline{F_A} = \widetilde{\cap}\,\{G_B : G_B \text{ is fuzzy soft closed and } F_A \,\widetilde{\subseteq}\, G_B\}$$

It is obvious that

(i) $\overline{F_A}$ is fuzzy soft closed.
(ii) $F_A \;\widetilde{\subseteq}\; \overline{F_A}$.

Definition 2.13 ([7]). Let $(U, E, \mathfrak{J})$ be a fuzzy soft topological space. Let F_A be a fuzzy soft set over (U, E). The fuzzy soft interior of F_A is defined as the union of all fuzzy soft open sets contained in F_A and is denoted by ${F_A}^\circ$. We write

$${F_A}^\circ = \widetilde{\cup}\,\{G_B : G_B \text{ is fuzzy soft open and } G_B \,\widetilde{\subseteq}\, F_A\}$$

It is obvious that

(i) ${F_A}^\circ$ is fuzzy soft open.
(ii) ${F_A}^\circ \,\widetilde{\subseteq}\, F_A$.
(iii) ${F_A}^\circ$ is the largest fuzzy soft open set contained in F_A.

Proposition 2.14 ([7]). *Let $(U, E, \mathfrak{J})$ be a fuzzy soft topological space. Let F_A, G_B be two fuzzy soft sets over (U, E). Then*

(i) $\overline{\tilde{\Phi}} = \tilde{\Phi}$;
(ii) $F_A \,\widetilde{\subseteq}\, G_B \Rightarrow \overline{F_A} \,\widetilde{\subseteq}\, \overline{G_B}$;
(iii) $\overline{F_A \,\widetilde{\cup}\, G_B} = \overline{F_A} \,\widetilde{\cup}\, \overline{G_B}$;
(iv) $\overline{F_A \,\widetilde{\cap}\, G_B} \,\widetilde{\subseteq}\, \overline{F_A} \,\widetilde{\cap}\, \overline{G_B}$;
(v) $(F_A \,\widetilde{\cap}\, G_B)^\circ = (F_A)^\circ \,\widetilde{\cap}\, (G_B)^\circ$;
(vi) $(F_A \,\widetilde{\cup}\, G_B)^\circ \,\widetilde{\supseteq}\, (F_A)^\circ \,\widetilde{\cup}\, (G_B)^\circ$;
(iii) $\overline{\overline{F_A}} = \overline{F_A}$.

Definition 2.15 ([2]). Let $FS(X, E)$ and $FS(Y, K)$ be the families of all fuzzy soft sets over X and Y, respectively. Let $u : X \to Y$ and $p : E \to K$ be two functions. Then the pair f_{up} is called a fuzzy soft mapping X to Y and denoted by $f_{up} : FS(X, E) \to FS(Y, K)$.

(i) Let $f_A \in FS(X, E)$, then the image of f_A under the fuzzy soft mapping f_{up} is the fuzzy soft set over Y defined by $f_{up}(f_A)$, where

$$f_{up}(f_A)(k)(y) = \begin{cases} \bigvee_{x \in u^{-1}(y)} (\bigvee_{e \in p^{-1}(k) \cap A} f_A(e))(x), & \text{if } u^{-1}(y) \neq \varnothing, \\ & p^{-1}(k) \cap A \neq \varnothing\ ; \\ 0_Y, & \text{otherwise.} \end{cases}$$

(ii) Let $g_B \in FS(Y, K)$, then the preimage of g_B under the fuzzy soft mapping f_{up} is the fuzzy soft set over X defined by $f_{up}^{-1}(g_B)$, where

$$f_{up}^{-1}(g_B)(e)(x) = \begin{cases} g_B(p(e))(u(x)), & \text{for } p(e) \in B; \\ 0_X, & \text{otherwise.} \end{cases}$$

Proposition 2.16 ([2])**.** *Let X and Y crisp sets, $f_A, f_{A_i} \in \mathfrak{F}(X, E)$ and $g_B, g_{B_i} \in \mathfrak{F}(Y, K)$, $\forall i \in J$, where J is an index set.*

(i) *If $f_{A_1} \widetilde{\subseteq} f_{A_2}$, then $f_{up}(f_{A_1}) \widetilde{\subseteq} f_{up}(f_{A_2})$.*
(ii) *If $g_{B_1} \widetilde{\subseteq} g_{B_2}$, then $f_{up}^{-1}(g_{B_1}) \widetilde{\subseteq} f_{up}^{-1}(g_{B_2})$.*
(iii) *$f_A \widetilde{\subseteq} f_{up}^{-1}(f_{up}(f_A))$, the equality holds if f_{up} is injective.*
(iv) *$f_{up}(f_{up}^{-1}(g_B)) \widetilde{\subseteq} g_B$, the equality holds if f_{up} is surjective.*
(v) *$f_{up}(\widetilde{\bigcup}_{i \in J}(f_{A_j})) = \widetilde{\bigcup}_{i \in J} f_{up}(f_{A_i})$.*
(vi) *$f_{up}(\widetilde{\bigcap}_{i \in J}(f_{A_i})) \widetilde{\subseteq} \widetilde{\bigcap}_{i \in J} f_{up}(f_{A_i})$, the equality holds if f_{up} is injective.*
(vii) *$f_{up}^{-1}(\widetilde{\bigcup}_{i \in J}(g_{B_i})) = \widetilde{\bigcup}_{i \in J} f_{up}^{-1}(g_{B_i})$.*
(viii) *$f_{up}^{-1}(\widetilde{\bigcap}_{i \in J}(g_{B_i})) = \widetilde{\bigcap}_{i \in J} f_{up}^{-1}(g_{B_i})$.*
(ix) *$f_{up}^{-1}(\tilde{E}) = \tilde{E}$, $f_{up}^{-1}(\tilde{\Phi}) = \tilde{\Phi}$.*
(x) *$f_{up}(\tilde{E}) = \tilde{E}$, f_{up} is surjective.*
(xi) *$f_{up}(\tilde{\Phi}) = \tilde{\Phi}$.*

Definition 2.17 ([6])**.** Let (X, τ_1) and (Y, τ_2) be two fuzzy soft topological spaces. A fuzzy soft mapping $f_{up} : (X, \tau_1) \to (Y, \tau_2)$ is called fuzzy soft continuous if $f_{up}^{-1}(g_B) \in \tau_1$, for each $g_B \in \tau_2$.

Definition 2.18 ([6])**.** Let $f_A, g_B \in FS(X, E)$. f_A is said to be fuzzy soft quasi coincident with g_B, denoted by $f_A q g_B$, if there exist $e \in E$ and $x \in X$ such that $f_A(e)(x) + g_B(e)(x) > 1$.

If f_A is not fuzzy soft quasi coincident with g_B, then we write $f_A \not{q} g_B$.

Proposition 2.19 ([6])**.** *Let $f_A, g_B \in FS(X, E)$. Then the followings are true.*

(i) *$f_A \widetilde{\subseteq} g_B \Leftrightarrow f_A \not{q} {g_B}^c$.*
(ii) *$f_A \widetilde{\subseteq} {g_B}^c \Leftrightarrow f_A \not{q} g_B$.*
(iii) *$f_A q g_B \Rightarrow f_A \widetilde{\cap} g_B \neq \tilde{\Phi}$.*

3. Fuzzy soft β-open sets

In this section, the concepts of fuzzy soft β-open sets, fuzzy soft β-interiors, fuzzy soft β-closures, fuzzy soft β-continuous functions, fuzzy soft β-irresolute functions are introduced and some of their properties are studied.

Notation 3.1. Fuzzy soft interior and fuzzy soft closure denote $\widetilde{FSint}$ and $\widetilde{FScl}$ respectively.

Definition 3.2. A fuzzy soft set F_A in a fuzzy soft topological space $(U, E, \mathfrak{I})$ is said to be:

(i) fuzzy soft β-open set if $F_A \widetilde{\subseteq} \widetilde{FScl}(\widetilde{FSint}(\widetilde{FScl}(F_A)))$.
(ii) fuzzy soft β-closed set if $F_A \widetilde{\supseteq} \widetilde{FSint}(\widetilde{FScl}(\widetilde{FSint}(F_A)))$.

Example 3.3. Let $U = \{a, b\}$ and $E = \{e_1, e_2, e_3\}$, $A = \{e_1\} \subseteq E$, $B = \{e_2, e_3\} \subseteq E$. Let

$F_A = \{F(e_1) = \{(a, 0.2), (b, 0.3)\},$
$\quad F(e_2) = \{(a, 0), (b, 0)\},$
$\quad F(e_3) = \{(a, 0), (b, 0)\}\},$
$G_B = \{G(e_1) = \{(a, 0), (b, 0)\},$
$\quad G(e_2) = \{(a, 0.3), (b, 0.3)\},$
$\quad G(e_3) = \{(a, 0.4), (b, 0.3)\}\}$
be fuzzy soft sets over (U, E).
$F_A \,\widetilde{\cup}\, G_B = H_C, C = A \cup B = \{H(e_1) = \{(a, 0.2), (b, 0.3)\},$
$\quad H(e_2) = \{(a, 0.3), (b, 0.3)\},$
$\quad H(e_3) = \{(a, 0.4), (b, 0.3)\}\}.$
Then the family $\mathfrak{J} = \{\tilde{\Phi}, \tilde{E}, F_A, G_B, H_C\}$ is a fuzzy soft topology on (U, E). Clearly, $(U, E, \mathfrak{J})$ is a fuzzy soft topological space. Consider the fuzzy soft set
$I_E = \{I(e_1) = \{(a, 0.8), (b, 0.5)\},$
$\quad I(e_2) = \{(a, 0.7), (b, 0.5)\},$
$\quad I(e_3) = \{(a, 0.5), (b, 0.2)\}\}.$
Hence I_E is a fuzzy soft β-open set over (U, E), since $I_E \,\widetilde{\subseteq}\, \widetilde{FS}cl(\widetilde{FS}int(\widetilde{FS}cl(I_E)))$.

Remark 3.4. Every fuzzy soft open (resp., closed) set is a fuzzy soft β-open (resp., β-closed) set.

Proposition 3.5. (i) Arbitrary union of fuzzy soft β-open sets is a fuzzy soft β-open set.
(ii) Arbitrary intersection of fuzzy soft β-closed sets is a fuzzy soft β-closed set.

Proof. **(i)** Let $\{F_{A_j} : j \in J\}$ be a collection of fuzzy soft β-open sets of a fuzzy soft topological space $(U, E, \mathfrak{J})$. Then for each j, $F_{A_j} \,\widetilde{\subseteq}\, \widetilde{FS}cl(\widetilde{FS}int(\widetilde{FS}cl(F_{A_j})))$. By definition 3.2. and Proposition 2.14.,

$$\begin{aligned}\widetilde{\cup}\,(F_{A_j}) &\widetilde{\subseteq}\, \widetilde{\cup}\,(\widetilde{FS}cl(\widetilde{FS}int(\widetilde{FS}cl(F_{A_j})))) \\ &\widetilde{\subseteq}\, (\widetilde{FS}cl(\widetilde{\cup}\,\widetilde{FS}int(\widetilde{FS}cl(F_{A_j})))) \\ &\widetilde{\subseteq}\, (\widetilde{FS}cl(\widetilde{FS}int(\widetilde{\cup}\,\widetilde{FS}cl(F_{A_j})))) \\ &\widetilde{\subseteq}\, (\widetilde{FS}cl(\widetilde{FS}int(\widetilde{FS}cl(\widetilde{\cup}\,F_{A_j})))).\end{aligned}$$

Therefore, $\widetilde{\cup}\,(F_{A_j}) = \widetilde{\subseteq}\, (\widetilde{FS}cl(\widetilde{FS}int(\widetilde{FS}cl(\widetilde{\cup}\,F_{A_j}))))$.
(ii) The proof is similar by (i). □

Definition 3.6. Let $(U, E, \mathfrak{J})$ be a fuzzy soft topological space and F_A be a fuzzy soft open set over (U, E). Then the fuzzy soft β interior is denoted by $\widetilde{FS}\beta\text{-}int$ and is defined by

$$\widetilde{FS}\beta\text{-}int(F_A) = \widetilde{\cup}\,\{G_B : G_B \text{ is a fuzzy soft } \beta\text{-open set in } (U, E) \text{ and } G_B \subseteq F_A\}.$$

Definition 3.7. Let $(U, E, \mathfrak{J})$ be a fuzzy soft topological space and F_A be a fuzzy soft closed set over (U, E). Then the fuzzy soft β closure is denoted by $\widetilde{FS}\beta\text{-}cl$ and is defined by

$\widetilde{FS}\beta\text{-}cl(F_A) = \widetilde{\cap}\,\{G_B : G_B$ is a fuzzy soft β-closed set in (U, E) and $G_B \supseteq F_A\}$.

Remark 3.8. (i) $\widetilde{FS}\beta\text{-}int(\tilde{E}) = \tilde{E}$.

(ii) If $F_A \in FS(U, E)$, then $((\widetilde{FS}\beta\text{-}cl(F_A))^c)^c = \widetilde{FS}\beta\text{-}cl(F_A)$.

Proposition 3.9. Let $(U, E, \mathfrak{J})$ be a fuzzy soft topological space and $F_A, G_B \in FS(U, E)$. Then

(i) $(\widetilde{FS}\beta\text{-}cl(F_A))^c \,\widetilde{\subseteq}\, \widetilde{FS}\beta\text{-}int({F_A}^c)$.

(i) $(\widetilde{FS}\beta\text{-}int(F_A))^c \,\widetilde{\subseteq}\, \widetilde{FS}\beta\text{-}cl({F_A}^c)$.

Proof.

(i) $$(\widetilde{FS}\beta\text{-}cl(F_A))^c = (\widetilde{\cap}\,\{G_B : G_B \text{ is fuzzy soft } \beta\text{-closed set over } (U,E) \mathit{and}\ F_A \,\widetilde{\subseteq}\, G_B\})^c$$
$$= \widetilde{\cup}\,\{{G_B}^c : G_B \text{ is fuzzy soft } \beta\text{-closed set over } (U,E) \text{ and } F_A \,\widetilde{\subseteq}\, G_B\}$$
$$= \widetilde{\cup}\,\{{G_B}^c : {G_B}^c \text{ is fuzzy soft } \beta\text{-open set over } (U,E) \text{ and } F_A \,\widetilde{\supseteq}\, G_B\}$$
$$= \widetilde{FS}\beta\text{-}int({F_A}^c).$$

(ii) The proof is similar by (i). □

Notation 3.10. (φ, ψ) denotes fuzzy soft function from U to U'.

Definition 3.11. Let $(U, E, \mathfrak{J}_1)$ and $(U', E', \mathfrak{J}_2)$ be two fuzzy soft topological spaces. A fuzzy soft function $(\varphi, \psi) : (U, E, \mathfrak{J}_1) \to (U', E', \mathfrak{J}_2)$ is called fuzzy soft β-continuous if $(\varphi, \psi)^{-1}(G_B)$ is a fuzzy soft β-open (resp., β-closed) set in $\mathfrak{J}_1$, for each fuzzy soft open (resp., closed) set G_B in $\mathfrak{J}_2$.

Definition 3.12. Let $(U, E, \mathfrak{J}_1)$ and $(U', E', \mathfrak{J}_2)$ be two fuzzy soft topological spaces. A fuzzy soft function $(\varphi, \psi) : (U, E, \mathfrak{J}_1) \to (U', E', \mathfrak{J}_2)$ is called fuzzy soft β-irresolute if $(\varphi, \psi)^{-1}(G_B)$ is a fuzzy soft β-open (resp., β-closed) set in $\mathfrak{J}_1$, for each fuzzy soft β-open (resp., β-closed) set G_B in $\mathfrak{J}_2$.

4. Types of fuzzy soft β-connectedness in fuzzy soft topological spaces

In this section, the concepts of fuzzy soft β-connectedness, fuzzy soft β-clopen sets,fuzzy soft β-strongly connectedness, fuzzy soft β-C_5 connectedness, fuzzy soft β-C_S connectedness, fuzzy soft β-C_M connectedness are introduced and studied. Some interesting properties are discussed.

Definition 4.1. Let $(U, E, \mathfrak{J})$ be a fuzzy soft topological space. A fuzzy soft β-separation of $\tilde{E}$ is a pair F_A, G_B of no-null fuzzy soft β-open sets such that
$$\tilde{E} = F_A \,\widetilde{\cup}\, G_B \text{ and } \tilde{\Phi} = F_A \,\widetilde{\cap}\, G_B.$$

Definition 4.2. A fuzzy soft topological space $(U, E, \mathfrak{J})$ is said to be fuzzy soft β-connected if there does not exist a fuzzy soft β-separation of $\tilde{E}$. Otherwise, $(U, E, \mathfrak{J})$ is said to be fuzzy soft β-disconnected.

Example 4.3. Let $U = \{a, b\}$ and $E = \{e_1, e_2, e_3\}$, $A = \{e_1\} \subseteq E$, $B = \{e_1, e_2\} \subseteq E$. Let
$F_A = \{F(e_1) = \{(a, 0.3), (b, 0.5)\},$

$F(e_2) = \{(a, 0), (b, 0)\},$
$F(e_3) = \{(a, 0), (b, 0)\}\},$
$G_B = \{G(e_1) = \{(a, 0.3), (b, 0.5)\},$
$G(e_2) = \{(a, 0.4), (b, 0.5)\},$
$G(e_3) = \{(a, 0), (b, 0)\}\}$
be fuzzy soft sets over (U, E). Then the family $\mathfrak{J} = \{\tilde{\Phi}, \tilde{E}, F_A, G_B\}$ is a fuzzy soft topology on (U, E). Clearly, $(U, E, \mathfrak{J})$ is a fuzzy soft topological space. Consider the fuzzy soft set
$I_E = \{I(e_1) = \{(a, 0.7), (b, 0.5)\},$
$I(e_2) = \{(a, 0.6), (b, 0.5)\},$
$I(e_3) = \{(a, 1), (b, 1)\}\},$
$H_B = \{H(e_1) = \{(a, 0.6), (b, 0.5)\},$
$H(e_2) = \{(a, 0.5), (b, 0.5)\},$
$H(e_3) = \{(a, 0), (b, 0)\}\},$
I_E and H_B are fuzzy soft β-open sets in (U, E), $I_E \neq \tilde{\Phi}$, $H_B \neq \tilde{\Phi}$ and $I_E \,\widetilde{\cap}\, H_B = H_B \neq \tilde{\Phi}$, $I_E \,\widetilde{\cup}\, H_B = I_E \neq \tilde{E}$. Hence $(U, E, \mathfrak{J})$ is fuzzy soft β-connected.

Example 4.4. Let $U = \{a, b\}$ and $E = \{e_1, e_2, e_3\}$, $A = \{e_1, e_2\} \subseteq E$. Let
$F_A = \{F(e_1) = \{(a, 0.4), (b, 0.2)\},$
$F(e_2) = \{(a, 0.3), (b, 0.5)\},$
$F(e_3) = \{(a, 0), (b, 0)\}\}$
be a fuzzy soft set over (U, E). Then the family $\mathfrak{J} = \{\tilde{\Phi}, \tilde{E}, F_A\}$ is a fuzzy soft topology on (U, E). Clearly, $(U, E, \mathfrak{J})$ is a fuzzy soft topological space. Consider the fuzzy soft sets
$G_E = \{G(e_1) = \{(a, 1), (b, 0)\},$
$G(e_2) = \{(a, 0), (b, 1)\},$
$G(e_3) = \{(a, 1), (b, 0)\}\},$
$H_E = \{H(e_1) = \{(a, 0), (b, 1)\},$
$H(e_2) = \{(a, 1), (b, 0)\},$
$H(e_3) = \{(a, 0), (b, 1)\}\},$
G_E and H_E are fuzzy soft β-open sets in (U, E), $G_E \neq \tilde{\Phi}$, $H_E \neq \tilde{\Phi}$ and $G_E \,\widetilde{\cap}\, H_E = \tilde{\Phi}$, $G_E \,\widetilde{\cup}\, H_E = \tilde{E}$. Hence $(U, E, \mathfrak{J})$ is fuzzy soft β-disconnected.

Proposition 4.5. A fuzzy soft topological space $(U, E, \mathfrak{J})$ is a fuzzy soft β-connected space if and only if there exist no non-zero fuzzy soft β-open sets F_A and G_B in $(U, E, \mathfrak{J})$ such that $F_A = {G_B}^c$.

Proof. **Necessity.**
Let F_A and G_B be two fuzzy soft β-open sets in $(U, E, \mathfrak{J})$ such that $F_A \neq \tilde{\Phi}$, $\tilde{\Phi} \neq {G_B}^c$ and $F_A = {G_B}^c$. Therefore ${G_B}^c$ is a fuzzy soft β-closed set. Since $F_A \neq \tilde{\Phi}$, $G_B \neq \tilde{E}$. This implies that G_B is a proper fuzzy soft set which is both fuzzy soft β-open and fuzzy soft β-closed in $(U, E, \mathfrak{J})$. Hence $(U, E, \mathfrak{J})$ is not a fuzzy soft β-connected space. But this is a contradiction to our hypothesis. Thus there exist no non-zero fuzzy soft β-open sets F_A and G_B in $(U, E, \mathfrak{J})$ such that $F_A = {G_B}^c$.
Sufficiency.
Let F_A be both fuzzy soft β-open and fuzzy soft β-closed in $(U, E, \mathfrak{J})$ such that $\tilde{\Phi} \neq F_A$, $F_A \neq \tilde{E}$. Let ${F_A}^c = G_B$. Then G_B is a fuzzy soft β-open set and ${G_B}^c \neq \tilde{E}$.

This implies that $G_B = {F_A}^c \neq \tilde{\Phi}$, which is a contradiction to our hypothesis. Hence $(U, E, \mathfrak{J})$ is a fuzzy soft β-connected space. □

Proposition 4.6. A fuzzy soft topological space $(U, E, \mathfrak{J})$ is a fuzzy soft β-connected space if and only if there exist no non-zero fuzzy soft β-open sets F_A and G_B in $(U, E, \mathfrak{J})$ such that $F_A = {G_B}^c$, $G_B = (\widetilde{FS}\beta\text{-}cl(F_A))^c$ and $F_A = (\widetilde{FS}\beta\text{-}cl(G_B))^c$.

Proof. **Necessity.**

Assume that there exists fuzzy soft sets F_A and G_B such that $F_A \neq \tilde{\Phi}$, $\tilde{\Phi} \neq {G_B}^c$, $F_A = {G_B}^c$, $G_B = (\widetilde{FS}\beta\text{-}cl(F_A))^c$ and $F_A = (\widetilde{FS}\beta\text{-}cl(G_B))^c$. Since $(\widetilde{FS}\beta\text{-}cl(F_A))^c$ and $(\widetilde{FS}\beta\text{-}cl(G_B))^c$ are fuzzy soft β-open sets in $(U, E, \mathfrak{J})$, F_A and G_B are fuzzy soft β-open sets in $(U, E, \mathfrak{J})$. This implies $(U, E, \mathfrak{J})$ is not a fuzzy soft β-connected space, which is a contradiction. Thus there exist no non-zero fuzzy soft β-open sets F_A and G_B in $(U, E, \mathfrak{J})$ such that $F_A = {G_B}^c$, $G_B = (\widetilde{FS}\beta\text{-}cl(F_A))^c$ and $F_A = (\widetilde{FS}\beta\text{-}cl(G_B))^c$.

Sufficiency.

Let F_A be both fuzzy soft β-open and fuzzy soft β-closed in $(U, E, \mathfrak{J})$ such that $\tilde{\Phi} \neq F_A$, $F_A \neq \tilde{E}$. Now by taking ${F_A}^c = G_B$, we obtain a contradiction to our hypothesis. Hence $(U, E, \mathfrak{J})$ is a fuzzy soft β-connected space. □

Definition 4.7. A fuzzy soft topological space $(U, E, \mathfrak{J})$ is said to be fuzzy soft C_5-disconnected if there exists fuzzy soft sets F_A over (U, E), which is both fuzzy soft open set and fuzzy soft closed set such that $F_A \neq \tilde{\Phi}$ and $F_A \neq \tilde{E}$. If $(U, E, \mathfrak{J})$ is not fuzzy soft C_5-disconnected then it is said to be fuzzy soft C_5-connected.

Proposition 4.8. Let $(U, E, \mathfrak{J}_1)$ and $(U', E', \mathfrak{J}_2)$ be two fuzzy soft topological spaces. Let $(\varphi, \psi) : (U, E, \mathfrak{J}_1) \to (U', E', \mathfrak{J}_2)$ is a fuzzy soft β-continuous and surjective function. If $(U, E, \mathfrak{J}_1)$ is a fuzzy soft β-connected space, then $(U', E', \mathfrak{J}_2)$ is a fuzzy soft C_5-connected space.

Proof. Let $(U, E, \mathfrak{J}_1)$ is a fuzzy soft β-connected space. Suppose $(U', E', \mathfrak{J}_2)$ is not a fuzzy soft C_5-connected space, then there exists a proper fuzzy soft set F_A which is both fuzzy soft open and fuzzy soft closed in $(U', E', \mathfrak{J}_2)$. Since (φ, ψ) is a fuzzy soft β-continuous function, $(\varphi, \psi)^{-1}(F_A)$ is both fuzzy soft β-open and fuzzy soft β-closed in $(U, E, \mathfrak{J}_1)$. But this is a contradiction to hypothesis. Hence $(U', E', \mathfrak{J}_2)$ is a fuzzy soft C_5-connected space. □

Definition 4.9. A fuzzy soft topological space $(U, E, \mathfrak{J})$ is said to be fuzzy soft βC_5-disconnected if there exists fuzzy soft sets F_A over (U, E), which is both fuzzy soft β-open set and fuzzy soft β-closed set such that $F_A \neq \tilde{\Phi}$ and $F_A \neq \tilde{E}$. If $(U, E, \mathfrak{J})$ is not fuzzy soft β-C_5 disconnected then it is said to be fuzzy soft β-C_5 connected.

Definition 4.10. A fuzzy soft topological space $(U, E, \mathfrak{J})$ is said to be fuzzy soft β-clopen set, which is both fuzzy soft β-open set and fuzzy soft β-closed set.

Example 4.11. Let $U = \{a, b\}$ and $E = \{e_1, e_2, e_3\}$, $A = \{e_1, e_2\} \subseteq E$. Let
$F_A = \{F(e_1) = \{(a, 0.5), (b, 0.5)\},$
$F(e_2) = \{(a, 0.5), (b, 0.5)\},$
$F(e_3) = \{(a, 0.5), (b, 0.5)\}\}$

be a fuzzy soft set over (U,E). Then the family $\mathfrak{J} = \{\tilde{\Phi}, \tilde{E}, F_A\}$ is a fuzzy soft topology on (U,E). Clearly, $(U,E,\mathfrak{J})$ is a fuzzy soft topological space. Clearly, F_A is both fuzzy soft β-open set and fuzzy soft β-closed set.

Proposition 4.12. Fuzzy soft β-C_5 connectedness implies fuzzy soft β-connectedness.

Proof. Suppose that there exists non-empty fuzzy soft β-open sets F_A and G_B such that $F_A \widetilde{\cup} G_B = \tilde{E}$ and $F_A \widetilde{\cap} G_B = \tilde{\Phi}$ (fuzzy soft β-disconnected), then $F_E(e) = F_A(e) \cup G_B(e)$ and $F_\Phi(e) = F_A(e) \cap G_B(e)$, for all $e \in E$. In other words, $F_A = {G_B}^c$. Hence F_A is a fuzzy soft β-clopen set which implies that $(U,E,\mathfrak{J})$ is fuzzy soft β-C_5 disconnected. □

Remark 4.13. The converse of the above Proposition need not be true as shown by the following example.

Example 4.14. Let $U = \{a,b\}$ and $E = \{e_1,e_2,e_3\}$, $A = \{e_1,e_2\} \subseteq E$. Let
$F_A = \{F(e_1) = \{(a,0.5),(b,0.5)\},$
$F(e_2) = \{(a,0.5),(b,0.5)\},$
$F(e_3) = \{(a,0),(b,0)\}\}$
be a fuzzy soft set over (U,E). Then the family $\mathfrak{J} = \{\tilde{\Phi}, \tilde{E}, F_A\}$ is a fuzzy soft topology on (U,E). Clearly, $(U,E,\mathfrak{J})$ is a fuzzy soft topological space. Consider the fuzzy soft set
$G_E = \{G(e_1) = \{(a,0.4),(b,0.2)\},$
$G(e_2) = \{(a,0.3),(b,0.5)\},$
$G(e_3) = \{(a,1),(b,1)\}\}.$
Hence F_A and G_E are fuzzy soft β-open sets over (U,E). Also, $F_A \widetilde{\cup} G_E = F_A \neq \tilde{E}$, $F_A \widetilde{\cap} G_E = G_E \neq \tilde{\Phi}$. Hence $(U,E,\mathfrak{J})$ is fuzzy soft β-connected. Since F_A is both fuzzy soft β-open set and fuzzy soft β-closed set over (U,E), $(U,E,\mathfrak{J})$ is fuzzy soft β-C_5 disconnected.

Proposition 4.15. Let $(U,E,\mathfrak{J}_1)$ and $(U',E',\mathfrak{J}_2)$ be two fuzzy soft topological spaces. Let $(\varphi,\psi) : (U,E,\mathfrak{J}_1) \to (U',E',\mathfrak{J}_2)$ be a fuzzy soft β-irresolute and surjective function. If $(U,E,\mathfrak{J}_1)$ is fuzzy soft β-connected, then $(U',E',\mathfrak{J}_2)$ is fuzzy soft β-connected.

Proof. Assume that $(U',E',\mathfrak{J}_2)$ is not fuzzy soft β-connected. Thus there exists non-empty fuzzy soft β-open sets F_A and G_B in $(U',E',\mathfrak{J}_2)$ such that $F_A \widetilde{\cup} G_B = \tilde{E}$ and $F_A \widetilde{\cap} G_B = \tilde{\Phi}$. Since (φ,ψ) is fuzzy soft β-irresolute function, $H_C = (\varphi,\psi)^{-1}(F_A)$, $I_D = (\varphi,\psi)^{-1}(G_B)$ are fuzzy soft β-open sets over (U,E). From $F_A \neq \tilde{\Phi}$, we get $H_C = (\varphi,\psi)^{-1}(F_A) \neq \tilde{\Phi}$. (If $(\varphi,\psi)^{-1}(F_A) \neq \tilde{\Phi}$, then $F_A = (\varphi,\psi)((\varphi,\psi)^{-1}(F_A)) =$

$(\varphi, \psi)(\tilde{\Phi}) = \tilde{\Phi}$, which is a contradiction.) Similarly we obtain $I_D \neq \tilde{\Phi}$. Now,

$$F_A \mathbin{\widetilde{\cup}} G_B = \tilde{E}$$
$$(\varphi, \psi)^{-1}(F_A) \mathbin{\widetilde{\cup}} (\varphi, \psi)^{-1}(G_B) = (\varphi, \psi)^{-1}(\tilde{E})$$
$$H_C \mathbin{\widetilde{\cup}} I_D = \tilde{E}$$
$$F_A \mathbin{\widetilde{\cap}} G_B = \tilde{\Phi}$$
$$(\varphi, \psi)^{-1}(F_A) \mathbin{\widetilde{\cap}} (\varphi, \psi)^{-1}(G_B) = (\varphi, \psi)^{-1}(\tilde{\Phi})$$
$$H_C \mathbin{\widetilde{\cap}} I_D = \tilde{\Phi}.$$

This implies that $H_C \mathbin{\widetilde{\cup}} I_D = \tilde{E}$ and $H_C \mathbin{\widetilde{\cap}} I_D = \tilde{\Phi}$. Thus $(U, E, \mathfrak{J}_1)$ is fuzzy soft β-connected, which is a contradiction to our hypothesis. Hence $(U', E', \mathfrak{J}_2)$ is fuzzy soft β-connected. □

Proposition 4.16. $(U, E, \mathfrak{J})$ is fuzzy soft β-C_5 connected if and only if there exists no non-empty fuzzy soft β-open sets F_A and G_B over (U, E) such that $F_A = {G_B}^c$.

Proof. Suppose that F_A and G_B are fuzzy soft β-open sets over (U, E) such that $F_A \neq \tilde{\Phi}$, $\tilde{\Phi} \neq G_B$ and $F_A = {G_B}^c$. Since $F_A = {G_B}^c$, ${G_B}^c$ is a fuzzy soft β-open set and G_B is a fuzzy soft β-closed set. And $F_A \neq \tilde{\Phi}$ implies $G_B \neq \tilde{E}$. But this is a contradiction to the fact that $(U, E, \mathfrak{J})$ is fuzzy soft β-C_5connected.

Conversely,

Let F_A be both fuzzy soft β-open set and fuzzy soft β-closed set over (U, E) such that $\tilde{\Phi} \neq F_A$, $F_A \neq \tilde{E}$. Now take $G_B = {F_A}^c$. In this case G_B is a fuzzy soft β-open set and $F_A \neq \tilde{E}$ which implies $G_B = {F_A}^c \neq \tilde{\Phi}$, which is a contradiction. □

Proposition 4.17. $(U, E, \mathfrak{J})$ is fuzzy soft β-C_5 connected if and only if there exists no non-empty fuzzy soft sets over (U, E) such that ${F_A}^c = G_B$, $G_B = (\widetilde{FS}\beta\text{-}cl(F_A))^c$, $F_A = (\widetilde{FS}\beta\text{-}cl(G_B))^c$.

Proof. Assume that there exist fuzzy soft sets F_A and G_B such that $F_A \neq \tilde{\Phi}$, $\tilde{\Phi} \neq G_B$, ${F_A}^c = G_B$, $G_B = (\widetilde{FS}\beta\text{-}cl(F_A))^c$, $F_A = (\widetilde{FS}\beta\text{-}cl(G_B))^c$. Since $(\widetilde{FS}\beta\text{-}cl(F_A))^c$ and $(\widetilde{FS}\beta\text{-}cl(G_B))^c$ are fuzzy soft β-open sets over (U, E), F_A and G_B are fuzzy soft β-open sets over (U, E), which is a contradiction.

Conversely,

Let F_A be both fuzzy soft β-open set and fuzzy soft β-closed set over (U, E) such that $\tilde{\Phi} \neq F_A$, $F_A \neq \tilde{E}$. Taking $G_B = {F_A}^c$, we obtain a contradiction. □

Definition 4.18. A fuzzy soft topological space $(U, E, \mathfrak{J})$ is said to be fuzzy soft β-strongly connected if there exists no non-empty fuzzy soft β-closed sets F_A and G_B over (U, E) such that $F_A + G_B \mathbin{\widetilde{\subseteq}} \tilde{E}$.

In otherwords, a fuzzy soft topological space $(U, E, \mathfrak{J})$ is said to be fuzzy soft β-strongly connected if there exists no non-empty fuzzy soft β-closed sets F_A and G_B over (U, E) such that $F_A \mathbin{\widetilde{\cap}} G_B = \tilde{E}$.

Proposition 4.19. $(U, E, \mathfrak{J})$ is fuzzy soft β-strongly connected if and only if there exists no non-empty fuzzy soft β-open sets F_A and G_B over (U, E) such that $F_A \neq \tilde{E} \neq G_B$ and $F_A + G_B \mathbin{\widetilde{\supseteq}} \tilde{E}$.

Proof. **Necessity.**
Let F_A and G_B are fuzzy soft β-open sets over (U, E) such that $F_A \neq \tilde{E} \neq G_B$ and $F_A + G_B \widetilde{\supseteq} \tilde{E}$. If we take $H_C = {F_A}^c$ and $I_D = {G_B}^c$, then H_C and I_D become fuzzy soft β-closed sets over (U, E) and $H_C \neq \tilde{\Phi} \neq I_D$ and $H_C + I_D \widetilde{\subseteq} \tilde{E}$. Which is a contradiction. Hence $(U, E, \mathfrak{J})$ is fuzzy soft β-strongly connected.
Sufficiency.
Let F_A and G_B be non-empty fuzzy soft β-closed sets over (U, E) such that $F_A + G_B \widetilde{\subseteq} \tilde{E}$. If $H_C = {F_A}^c$ and $I_D = {G_B}^c$, then H_C and I_D become fuzzy soft β-open sets over (U, E) and $H_C \neq \tilde{E}$, $\tilde{E} \neq I_D$ and $H_C + I_D \widetilde{\supseteq} \tilde{E}$. Which is a contradiction. Thus there exists no non-empty fuzzy soft β-open sets F_A and G_B over (U, E) such that $F_A \neq \tilde{E}$, $\tilde{E} \neq G_B$ and $F_A + G_B \widetilde{\supseteq} \tilde{E}$. □

Proposition 4.20. Let $(U, E, \mathfrak{J}_1)$ and $(U', E', \mathfrak{J}_2)$ be two fuzzy soft topological spaces. Let $(\varphi, \psi) : (U, E, \mathfrak{J}_1) \to (U', E', \mathfrak{J}_2)$ be a fuzzy soft β-irresolute and surjective function. If $(U, E, \mathfrak{J}_1)$ is fuzzy soft β-strongly connected, then $(U', E', \mathfrak{J}_2)$ is fuzzy soft β-strongly connected.

Proof. Suppose that $(U', E', \mathfrak{J}_2)$ is not fuzzy soft β-strongly connected. Then there exists non-empty fuzzy soft β-closed sets H_A and I_B in $(U', E', \mathfrak{J}_2)$ such that $H_A \neq \tilde{\Phi}$, $\tilde{\Phi} \neq I_B$, $H_A + I_B \widetilde{\subseteq} \tilde{\Phi}$. Since (φ, ψ) is fuzzy soft β-irresolute function, $(\varphi, \psi)^{-1}(H_A)$, $(\varphi, \psi)^{-1}(I_B)$ are fuzzy soft β-closed sets over (U, E) and $(\varphi, \psi)^{-1}(H_A) \widetilde{\cap} (\varphi, \psi)^{-1}(I_B) = \tilde{\Phi}$, $(\varphi, \psi)^{-1}$
$(H_A) \neq \tilde{\Phi}$, $(\varphi, \psi)^{-1}(I_B) \neq \tilde{\Phi}$. (If $(\varphi, \psi)^{-1}(H_A) = \tilde{\Phi}$, then $(\varphi, \psi)((\varphi, \psi)^{-1}(H_A)) = H_A$ which implies $(\varphi, \psi)(\tilde{\Phi}) = H_A$. So $\tilde{\Phi} = H_A$ a contradiction.) Hence $(U, E, \mathfrak{J}_1)$ is fuzzy soft β-strongly connected, a contradiction to our hypothesis. Thus $(U', E', \mathfrak{J}_2)$ is fuzzy soft β-strongly connected. □

Remark 4.21. Fuzzy soft β-strongly connected does not imply fuzzy soft β-C_5 connected.

Example 4.22. Let $U = \{a, b\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2, e_3\} \subseteq E$. Let
$F_A = \{F(e_1) = \{(a, 0.3), (b, 0.5)\},$
$F(e_2) = \{(a, 0.4), (b, 0.3)\},$
$F(e_3) = \{(a, 0.5), (b, 0.5)\},$
$F(e_4) = \{(a, 0), (b, 0)\}\}$
be a fuzzy soft set over (U, E).Then the family $\mathfrak{J} = \{\tilde{\Phi}, \tilde{E}, F_A\}$ is a fuzzy soft topology on (U, E). Clearly, $(U, E, \mathfrak{J})$ is a fuzzy soft topological space. Consider the fuzzy soft set
$G_E = \{G(e_1) = \{(a, 0.4), (b, 0.5)\},$
$G(e_2) = \{(a, 0.4), (b, 0.5)\},$
$G(e_3) = \{(a, 0.5), (b, 0.5)\},$
$G(e_4) = \{(a, 1), (b, 1)\}\}.$
Hence F_A and G_E are fuzzy soft β-open sets over (U, E). Also,
$F_A + G_E = H_E = \{H(e_1) = \{(a, 0.7), (b, 1)\},$
$H(e_2) = \{(a, 0.8), (b, 0.8)\},$
$H(e_3) = \{(a, 1), (b, 1)\},$
$H(e_4) = \{(a, 1), (b, 1)\}\},$
$F_A + G_E \widetilde{\subseteq} \tilde{E}$. Hence $(U, E, \mathfrak{J})$ is fuzzy soft β-strongly connected. But $(U, E, \mathfrak{J})$ is

not fuzzy soft β-C_5 connected, since F_A is both fuzzy soft β-open set and fuzzy soft β-closed set over (U, E).

Remark 4.23. Fuzzy soft β-C_5 connected does not imply fuzzy soft β-strongly connected.

Example 4.24. Let $U = \{a, b\}$ and $E = \{e_1, e_2, e_3\}$, $A = \{e_1, e_2\} \subseteq E$. Let
$F_A = \{F(e_1) = \{(a, 0.4), (b, 0.3)\},$
$F(e_2) = \{(a, 0.3), (b, 0.5)\},$
$F(e_3) = \{(a, 0), (b, 0)\}\}$
be a fuzzy soft set over (U, E). Then the family $\mathfrak{J} = \{\tilde{\Phi}, \tilde{E}, F_A\}$ is a fuzzy soft topology on (U, E). Clearly, $(U, E, \mathfrak{J})$ is a fuzzy soft topological space. Consider the fuzzy soft set
$G_E = \{G(e_1) = \{(a, 0.7), (b, 0.8)\},$
$G(e_2) = \{(a, 0.5), (b, 0.5)\},$
$G(e_3) = \{(a, 1), (b, 1)\}\}.$
Hence G_E is a fuzzy soft β-open set over (U, E), since $G_E \,\tilde{\subseteq}\, \widetilde{FS}cl(\widetilde{FS}int(\widetilde{FS}cl(G_E)))$. But G_E is not fuzzy soft β-closed set over (U, E). Also, $\tilde{\Phi} \neq G_E \neq \tilde{E}$. Thus $(U, E, \mathfrak{J})$ is fuzzy soft β-C_5connected. But $(U, E, \mathfrak{J})$ is not fuzzy soft β-strongly connected, since F_A and G_E are fuzzy soft β-open sets over (U, E) such that $F_A + G_E \,\tilde{\supseteq}\, \tilde{E}$.

Note 4.25. (i) If $F_A \,\tilde{\cap}\, G_B = \tilde{\Phi}$, then $F_A \,\tilde{\subseteq}\, {G_B}^c$.
(ii) If $F_A \,\tilde{\nsubseteq}\, {G_B}^c$, then $F_A \,\tilde{\cap}\, G_B \neq \tilde{\Phi}$.

Definition 4.26. If F_A and G_B are non-zero fuzzy soft sets over (U, E). Then F_A and G_B are said to be

(i) fuzzy soft β-weakly separated if $\widetilde{FS}\beta$-$cl(F_A) \,\tilde{\subseteq}\, {G_B}^c$ and $\widetilde{FS}\beta$-$cl(G_B) \,\tilde{\subseteq}\, {F_A}^c$.
(ii) fuzzy soft β-q-separated if $\widetilde{FS}\beta$-$cl(F_A) \,\tilde{\cap}\, G_B = \tilde{\Phi} = F_A \,\tilde{\cap}\, \widetilde{FS}\beta$-$cl(G_B)$.

Definition 4.27. A fuzzy soft topological space $(U, E, \mathfrak{J})$ is said to be fuzzy soft β-C_S disconnected if there exists fuzzy soft β-weakly separated non-zero fuzzy soft sets F_A and G_B over (U, E) such that $F_A \,\tilde{\cup}\, G_B = \tilde{E}$.

Example 4.28. Let $U = \{a, b\}$ and $E = \{e_1, e_2, e_3\}$, $A = \{e_1, e_2\} \subseteq E$. Let
$F_A = \{F(e_1) = \{(a, 0.4), (b, 0.2)\},$
$F(e_2) = \{(a, 0.3), (b, 0.5)\},$
$F(e_3) = \{(a, 0), (b, 0)\}\}$
be a fuzzy soft set over (U, E). Then the family $\mathfrak{J} = \{\tilde{\Phi}, \tilde{E}, F_A\}$ is a fuzzy soft topology on (U, E). Clearly, $(U, E, \mathfrak{J})$ is a fuzzy soft topological space. Consider the fuzzy soft sets
$G_E = \{G(e_1) = \{(a, 1), (b, 0)\},$
$G(e_2) = \{(a, 0), (b, 1)\},$
$G(e_3) = \{(a, 1), (b, 0)\}\}$ and
$H_E = \{H(e_1) = \{(a, 0), (b, 1)\},$
$H(e_2) = \{(a, 1), (b, 0)\},$
$H(e_3) = \{(a, 0), (b, 1)\}\}$ over (U, E).
Hence G_E and H_E are fuzzy soft β-open sets over (U, E), $\widetilde{FS}\beta$-$cl(G_E) \,\tilde{\subseteq}\, {H_E}^c$ and

$\widetilde{FS}\beta$-$cl(H_E) \mathrel{\tilde{\subseteq}} {G_E}^c$. Hence G_E and H_E are fuzzy soft β-weakly separated and $G_E \mathbin{\tilde{\cup}} H_E = \tilde{E}$. Thus $(U, E, \mathfrak{J})$ is fuzzy soft β-C_S disconnected.

Definition 4.29. A fuzzy soft topological space $(U, E, \mathfrak{J})$ is said to be fuzzy soft β-C_M disconnected if there exists fuzzy soft β-q-separated non-zero fuzzy soft sets F_A and G_B over (U, E) such that $F_A \mathbin{\tilde{\cup}} G_B = \tilde{E}$.

Example 4.30. Let $U = \{a, b\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2, e_3\} \subseteq E$. Let
$F_A = \{F(e_1) = \{(a, 0.4), (b, 0.2)\},$
$F(e_2) = \{(a, 0.3), (b, 0.5)\},$
$F(e_3) = \{(a, 0.1), (b, 0.5)\},$
$F(e_4) = \{(a, 0), (b, 0)\}\}$
be a fuzzy soft set over (U, E). Then the family $\mathfrak{J} = \{\tilde{\Phi}, \tilde{E}, F_A\}$ is a fuzzy soft topology on (U, E). Clearly, $(U, E, \mathfrak{J})$ is a fuzzy soft topological space. Consider the fuzzy soft sets
$G_E = \{G(e_1) = \{(a, 1), (b, 0)\},$
$G(e_2) = \{(a, 0), (b, 1)\},$
$G(e_3) = \{(a, 1), (b, 0)\},$
$G(e_4) = \{(a, 0), (b, 1)\}\}$ and
$H_E = \{H(e_1) = \{(a, 0), (b, 1)\},$
$H(e_2) = \{(a, 1), (b, 0)\},$
$H(e_3) = \{(a, 0), (b, 1)\},$
$H(e_4) = \{(a, 1), (b, 0)\}\}$ over (U, E).
Hence G_E and H_E are fuzzy soft β-open sets over (U, E), $\widetilde{FS}\beta$-$cl(G_E) \mathbin{\tilde{\cap}} H_E = \tilde{\Phi}$ and $\widetilde{FS}\beta$-$cl(H_E) \mathrel{\tilde{\subseteq}} G_E = \tilde{\Phi}$. Hence G_E and H_E are fuzzy soft β-q-separated and $G_E \mathbin{\tilde{\cup}} H_E = \tilde{E}$. Thus $(U, E, \mathfrak{J})$ is fuzzy soft β-C_M disconnected.

Remark 4.31. A fuzzy soft topological space $(U, E, \mathfrak{J})$ is said to be fuzzy soft β-C_S connected if and only if $(U, E, \mathfrak{J})$ is fuzzy soft β-C_M connected.

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References

[1] G. Balasubramanian, On fuzzy β-compact spaces and fuzzy β-extremally disconnected spaces, Kybernetika 33 (1997) 271–277.

[2] A. Kharal and B. Ahmad, Mappings on fuzzy soft classes, Advances in Fuzzy Systems 2009 (2009) 1–6.

[3] P. K. Maji, R. Biswas and A. R. Roy, Fuzzy soft sets, J. Fuzzy Math. 9 (3) (2001) 589–602.

[4] D. A. Molodstov, Soft set theory - First Result, Comput. Math. Appl. 3 (1999) 19–31.

[5] Sanjay Roy and T. K. Samanta, A note on fuzzy soft topological spaces, Ann. Fuzzy Math.Inform. 3 (2) (2012) 305- 311.

[6] Serkan Atmaca and Idris Zorlutuna, On fuzzy soft topological spaces, Ann. Fuzzy Math.Inform. 5 (2) (2013) 377–386.

[7] Tridiv Jyoti Neog, Dusmanta Kumar Sut and G. C. Hazarika, Fuzzy soft topological spaces, International Journal of Latest Trends Mathematics 2 (1) (2012) 54–67.

[8] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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