Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 11, No. 2, (February 2016), pp. 223–239
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Fuzzy soft separation axioms based on fuzzy β-open soft sets

A. M. Abd El-latif

Received 6 June 2015; Revised 23 July 2015; Accepted 11 August 2015

Abstract. Many scientists have studied and improved the soft set theory, which is initiated by Molodtsov [34] and easily applied to many problems having uncertainties from social life. In the present paper, we continue the study on fuzzy soft topological spaces and investigate the properties of fuzzy β-open (closed) soft sets, fuzzy β-soft interior (closure), fuzzy β-continuous (open) soft functions and fuzzy β-separation axioms which are important for further research on fuzzy soft topology. In particular we study the relationship between fuzzy β-soft interior fuzzy β-soft closure. Moreover, we study the properties of fuzzy soft β-regular spaces and fuzzy soft β-normal spaces. We hope that the findings in this paper will help researcher enhance and promote the further study on fuzzy soft topology to carry out a general framework for their applications in practical life.

2010 AMS Classification: 06B05, 06D72, 54A10, 54A40

Keywords: Soft set, Fuzzy soft set, Fuzzy soft topological space, Fuzzy β-soft interior, Fuzzy β-soft closure, Fuzzy β-open soft, Fuzzy β-closed soft, Fuzzy β-continuous soft functions, Fuzzy soft β-separation axioms, Fuzzy soft β-T_i-spaces $(i = 1, 2, 3, 4)$, Fuzzy soft β-regular, Fuzzy soft β-normal.

Corresponding Author: Alaa Mohamed Abd El-latif (alaa_8560@yahoo.com; alaa8560@hotmail.com)

1. Introduction

In real life situation, the problems in economics, engineering, social sciences, medical science etc. do not always involve crisp data. So, we cannot successfully use the traditional classical methods because of various types of uncertainties presented in these problems. To exceed these uncertainties, some kinds of theories were given like theory of fuzzy set, intuitionistic fuzzy set, rough set, bipolar fuzzy set, i.e. which we can use as mathematical tools for dealings with uncertainties. But, all these theories have their inherent difficulties. The reason for these difficulties

Molodtsov [34] initiated the concept of soft set theory as a new mathematical tool for dealing with uncertainties which is free from the above difficulties. In [34, 35], Molodtsov successfully applied the soft theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement, and so on. After presentation of the operations of soft sets [32], the properties and applications of soft set theory have been studied increasingly [6, 28, 35]. Xiao et al.[44] and Pei and Miao [38] discussed the relationship between soft sets and information systems. They showed that soft sets are a class of special information systems. In recent years, many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets [4, 5, 7, 9, 16, 26, 30, 31, 32, 33, 35, 36, 47]. To develop soft set theory, the operations of the soft sets are redefined and a uni-int decision making method was constructed by using these new operations [10].
Recently, in 2011, Shabir and Naz [41] initiated the study of soft topological spaces. They defined soft topology as a collection τ of soft sets over X. Consequently, they defined basic notions of soft topological spaces such as open soft and closed soft sets, soft subspace, soft closure, soft nbd of a point, soft separation axioms, soft regular spaces and soft normal spaces and established their several properties. Min in [43] investigate some properties of these soft separation axioms. In [17], Kandil et. al. introduced some soft operations such as semi open soft, pre open soft, α-open soft and β-open soft and investigated their properties in detail. Kandil et al. [25] introduced the notion of soft semi separation axioms, which extended in [13]. The notion of soft ideal was initiated for the first time by Kandil et al.[21]. They also introduced the concept of soft local function. These concepts are discussed with a view to find new soft topologies from the original one, called soft topological spaces with soft ideal $(X, \tau, E, \tilde{I})$. Applications to various fields were further investigated by Kandil et al. [18, 20, 22, 23, 24, 27]. The notion of supra soft topological spaces was initiated for the first time by El-sheikh and Abd El-latif [14]. They also introduced new different types of subsets of supra soft topological spaces and study the relations between them in detail. The notion of b-open soft sets was initiated for the first time by El-sheikh and Abd El-latif [12], which is extended in [1, 19]. Maji et. al. [30] initiated the study involving both fuzzy sets and soft sets. In [8] the notion of fuzzy soft set was introduced as a fuzzy generalization of soft sets and some basic properties of fuzzy soft sets are discussed in detail. Then, many scientists such as X. Yang et. al. [45], improved the concept of fuzziness of soft sets. In [2, 3] Karal and Ahmed defined the notion of a mapping on classes of (fuzzy) soft sets, which is a fundamental important in (fuzzy) soft set theory, to improve this work and they studied properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets. Chang [11] introduced the concept of fuzzy topology on a set X by axiomatizing a collection $\mathfrak{T}$ of fuzzy subsets of X. Tanay et.al. [42] introduced the definition of fuzzy soft topology over a subset of the initial universe set while Roy and Samanta [39] gave the definition f fuzzy soft topology over the initial universe set. Some fuzzy soft topological properties based on fuzzy semi open soft sets namely, fuzzy semi open soft sets, fuzzy semi closed soft sets, fuzzy semi soft interior, fuzzy semi soft closure fuzzy semi separation axioms and fuzzy soft semi connectedness, were introduced by Kandil et. al. in [16, 26].

The purpose of our paper, is to introduce some new concepts in fuzzy soft topological spaces, such as fuzzy β-open soft sets, fuzzy β-closed soft sets, fuzzy β-soft interior, fuzzy β-soft closure and fuzzy β-separation axioms. To facilitate our discussion, we first review some background on soft set and fuzzy soft set. In section 3, we gave the notion of fuzzy β-open soft sets, fuzzy β-closed soft sets, fuzzy β-soft interior, fuzzy β-soft closure and study various properties and notions related to these structures. In section 4, we introduce the notion of fuzzy β-soft function in fuzzy soft topological spaces and study its basic properties. In section 5, we introduce the notions of fuzzy soft β-separation axioms in fuzzy soft topological spaces and study some of its basic properties. Finally, some conclusions are pointed out in Section 6.

2. Preliminaries

From the literature, we recall the following definitions and results for the development of soft set theory and fuzzy soft set theory, which will be needed in this paper.

Definition 2.1 ([46]). A fuzzy set A in a non-empty set X is characterized by a membership function $\mu_A : X \longrightarrow [0,1] = I$ whose value $\mu_A(x)$ represents the "degree of membership" of x in A for $x \in X$.
Let I^X denotes the family of all fuzzy sets on X. If $A, B \in I^X$. Then some basic set operations for fuzzy sets are given by Zadeh [46], as follows :

(1) $A \leq B \ \Leftrightarrow\ \mu_A(x) \leq\ \mu_B(x)\ \forall\ x \in X$.
(2) $A = B \ \Leftrightarrow\ \mu_A(x) =\ \mu_B(x)\ \forall\ x \in X$.
(3) $C = A \vee B \ \Leftrightarrow\ \mu_C(x) = \mu_A(x) \vee \mu_B(x)\ \forall\ x \in X$.
(4) $D = A \wedge B \ \Leftrightarrow\ \mu_D(x) = \mu_A(x) \wedge \mu_B(x)\ \forall\ x \in X$.
(5) $M = A' \ \Leftrightarrow\ \mu_M(x) = 1 - \mu_A(x)\ \forall\ x \in X$.

Definition 2.2 ([34]). Let X be an initial universe and E be a set of parameters. Let $P(X)$ denote the power set of X and A be a non-empty subset of E. A pair (F, A) denoted by F_A is called a soft set over X , where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parametrized family of subsets of the universe X. For a particular $e \in A$, $F(e)$ may be considered the set of e-approximate elements of the soft set (F, A) and if $e \notin A$, then $F(e) = \phi$ i.e
$F_A = \{F(e) : e \in A \subseteq E,\ F : A \to P(X)\}$. The family of all these soft sets over X denoted by $SS(X)_A$.

Definition 2.3 ([30]). Let $A \subseteq E$. A pair (f, A), denoted by f_A, is called a fuzzy soft set over X , where f is a mapping given by $f : A \to I^X$ defined by $f_A(e) = \mu^e_{f_A}$ where $\mu^e_{f_A} = \overline{0}$ if $e \notin A$ and $\mu^e_{f_A} \neq \overline{0}$ if $e \in A$ where $\overline{0}(e) = 0\ \forall\ x \in X$. The family of all these fuzzy soft sets over X denoted by $FSS(X)_A$.

Proposition 2.1 ([5]). *Every fuzzy set may be considered a soft set.*

Definition 2.4 ([40]). The complement of a fuzzy soft set (f, A), denoted by $(f, A)'$, is defined by $(f, A)' = (f', A)$, $f'_A : E \to I^X$ is a mapping given by $\mu^e_{f'_A} = \overline{1} - \mu^e_{f_A}\ \forall\, e \in A$ where $\overline{1}(e) = 1\ \forall\ x \in X$. Clearly $(f'_A)' = f_A$.

Definition 2.5 ([32]). A fuzzy soft set f_A over X is said to be a NULL fuzzy soft set, denoted by $\tilde{0}_A$, if for all $e \in A$, $f_A(e) = \overline{0}$.

Definition 2.6 ([32]). A fuzzy soft set f_A over X is said to be an absolute fuzzy soft set, denoted by $\tilde{1}_A$, if for all $e \in A$, $f_A(e) = \overline{1}$. Clearly we have $(\tilde{1}_A)' = \tilde{0}_A$ and $(\tilde{0}_A)' = \tilde{1}_A$.

Definition 2.7 ([40]). Let $f_A,\ g_B \in FSS(X)_E$. Then, f_A is fuzzy soft subset of g_B, denoted by $f_A \sqsubseteq g_B$, if $A \subseteq B$ and $\mu^e_{f_A} \subseteq \mu^e_{g_B}\ \forall\ e \in A$, i.e. $\mu^e_{f_A}(x) \leq \mu^e_{g_B}(x)\ \forall\ x \in X\ and\ \forall\ e \in A$.

Definition 2.8 ([40]). The union of two fuzzy soft sets f_A and g_B over the common universe X is also a fuzzy soft set h_C, where $C = A \cup B$ and for all $e \in C$,
$h_C(e) = \mu^e_{h_c} = \mu^e_{f_A} \vee \mu^e_{g_B}\ \forall e \in C$. Here, we write $h_C = f_A \sqcup g_B$.

Definition 2.9 ([40]). The intersection of two fuzzy soft sets f_A and g_B over the common universe X is also a fuzzy soft set h_C, where $C = A \cap B$ and for all $e \in C$,
$h_C(e) = \mu^e_{h_c} = \mu^e_{f_A} \wedge \mu^e_{g_B}\ \forall e \in C$. Here, we write $h_C = f_A \sqcap g_B$.

Theorem 2.1 ([4]). *Let $\{(f,A)_j : j \in J\} \subseteq FSS(X)_E$. Then the following statements hold :*

(1) $[\sqcup_{j\in J}(f,A)_j]' = \sqcap_{j\in J}(f,A)'_j$.
(2) $[\sqcap_{j\in J}(f,A)_j]' = \sqcup_{j\in J}(f,A)'_j$.

Definition 2.10 ([40]). Let $\mathfrak{T}$ be a collection of fuzzy soft sets over a universe X with a fixed set of parameters E, then $\mathfrak{T}$ is called a fuzzy soft topology on X if

(1) $\tilde{1}_E, \tilde{0}_E \in \mathfrak{T}$, where $\tilde{0}_E(e) = \overline{0}$ and $\tilde{1}_E(e) = \overline{1}$, $\forall e \in E$.
(2) The union of any members of $\mathfrak{T}$, belongs to $\mathfrak{T}$.
(3) The intersection of any two members of $\mathfrak{T}$, belongs to $\mathfrak{T}$.

The triplet $(X, \mathfrak{T}, E)$ is called a fuzzy soft topological space over X. Also, each member of $\mathfrak{T}$ is called a fuzzy open soft in $(X, \mathfrak{T}, E)$. We denote the set of all fuzzy open soft sets by $FOS(X, \mathfrak{T}, E)$, or $FOS(X)$.

Definition 2.11 ([40]). Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space. A fuzzy soft set f_A over X is said to be fuzzy closed soft set in X, if its relative complement f'_A is fuzzy open soft set. We denote the set of all fuzzy closed soft sets by $FCS(X, \mathfrak{T}, E)$ or $FCS(X)$.

Definition 2.12 ([37]). Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. The fuzzy soft closure of f_A, denoted by $Fcl(f_A)$ is the intersection of all fuzzy closed soft super sets of f_A. i.e.,
$Fcl(f_A) = \sqcap\{h_D : h_D\ is\ fuzzy\ closed\ soft\ set\ and\ f_A \sqsubseteq h_D\}$.
The fuzzy soft interior of g_B, denoted by $Fint(g_B)$ is the fuzzy soft union of all fuzzy open soft subsets of g_B.i.e.,
$Fint(g_B) = \sqcup\{h_D : h_D\ is\ fuzzy\ open\ soft\ set\ and\ h_D \sqsubseteq g_B\}$.

Definition 2.13 ([29]). The fuzzy soft set $f_A \in FSS(X)_E$ is called fuzzy soft point if there exist $x \in X$ and $e \in E$ such that $\mu^e_{f_A}(x) = \alpha\ (0 < \alpha \leq 1)$ and $\mu^e_{f_A}(y) = \overline{0}$ for each $y \in X - \{x\}$, and this fuzzy soft point is denoted by x^e_α or f_e.

Definition 2.14 ([29]). The fuzzy soft point x^e_α is said to be belonging to the fuzzy soft set (g, A), denoted by $x^e_\alpha \tilde{\in} (g, A)$, if for the element $e \in A$, $\alpha \leq \mu^e_{g_A}(x)$.

Theorem 2.2 ([29]). *Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and f_e be a fuzzy soft point. Then, the following properties hold:*

(1) *If $f_e\tilde{\in}g_A$, then $f_e\tilde{\notin}g'_A$.*
(2) $f_e\tilde{\in}g_A \nRightarrow f'_e\tilde{\in}g'_A$.
(3) *Every non-null fuzzy soft set f_A can be expressed as the union of all the fuzzy soft points belonging to f_A.*

Definition 2.15 ([29]). A fuzzy soft set g_B in a fuzzy soft topological space $(X,\mathfrak{T},E)$ is called a fuzzy soft neighborhood of the fuzzy soft point x^e_α if there exists a fuzzy open soft set h_C such that $x^e_\alpha\tilde{\in}h_C \sqsubseteq g_B$. A fuzzy soft set g_B in a fuzzy soft topological space $(X,\mathfrak{T},E)$ is called a fuzzy soft neighborhood of the soft set f_A if there exists a fuzzy open soft set h_C such that $f_A \sqsubseteq h_C \sqsubseteq g_B$. The fuzzy soft neighborhood system of the fuzzy soft point x^e_α, denoted by $N_{\mathfrak{T}}(x^e_\alpha)$, is the family of all its fuzzy soft neighborhoods.

Definition 2.16 ([29]). Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and $Y \subseteq X$. Let h^Y_E be a fuzzy soft set over (Y,E) such that $h^Y_E : E \to I^Y$ such that $h^Y_E(e) = \mu^e_{h^Y_E}$,
$\mu^e_{h^Y_E}(x) = \begin{cases} 1, & x \in Y, \\ 0, & x \notin Y. \end{cases}$

Let $\mathfrak{T}_Y = \{h^Y_E \sqcap g_B : g_B \in \mathfrak{T}\}$, then the fuzzy soft topology $\mathfrak{T}_Y$ on $(Y,E)$is called fuzzy soft subspace topology for (Y,E) and $(Y,\mathfrak{T}_Y,E)$ is called fuzzy soft subspace of $(X,\mathfrak{T},E)$. If $h^Y_E \in \mathfrak{T}$ (resp. $h^Y_E \in \mathfrak{T}'$), then $(Y,\mathfrak{T}_Y,E)$ is called fuzzy open (resp. closed) soft subspace of $(X,\mathfrak{T},E)$.

Definition 2.17 ([37]). Let $FSS(X)_E$ and $FSS(Y)_K$ be families of fuzzy soft sets over X and Y, respectively. Let $u : X \to Y$ and $p : E \to K$ be mappings. Then, the map f_{pu} is called a fuzzy soft mapping from X to Y and denoted by $f_{pu} : FSS(X)_E \to FSS(Y)_K$ such that,

(1) If $f_A \in FSS(X)_E$. Then, the image of f_A under the fuzzy soft mapping f_{pu} is the fuzzy soft set over Y defined by $f_{pu}(f_A)$, where $\forall\, k \in p(E)$, $\forall\, y \in Y$,
$$f_{pu}(f_A)(k)(y) = \begin{cases} \bigvee_{u(x)=y} \ [\vee_{p(e)=k}(f_A(e))](x) & if\ x \in u^{-1}(y), \\ 0 & otherwise. \end{cases}$$
(2) If $g_B \in FSS(Y)_K$, then the pre-image of g_B under the fuzzy soft mapping f_{pu} is the fuzzy soft set over X defined by $f^{-1}_{pu}(g_B)$, where $\forall\, e \in p^{-1}(K)$, $\forall\, x \in X$,
$$f^{-1}_{pu}(g_B)(e)(x) = \begin{cases} g_B(p(e))(u(x)) & for\ p(e) \in B, \\ 0 & otherwise. \end{cases}$$

The fuzzy soft mapping f_{pu} is called surjective (resp. injective) if p and u are surjective (resp. injective), also it is said to be constant if p and u are constant.

Definition 2.18 ([37]). Let $(X,\mathfrak{T}_1,E)$ and $(Y,\mathfrak{T}_2,K)$ be two fuzzy soft topological spaces and $f_{pu} : FSS(X)_E \to FSS(Y)_K$ be a fuzzy soft mapping. Then, f_{pu} is called

(1) Fuzzy continuous soft if $f^{-1}_{pu}(g_B) \in \mathfrak{T}_1\ \forall\ (g_B) \in \mathfrak{T}_2$.
(2) Fuzzy open soft if $f_{pu}(g_A) \in \mathfrak{T}_2 \forall\ (g_A) \in \mathfrak{T}_1$.

Theorem 2.3 ([2]). *Let $FSS(X)_E$ and $FSS(Y)_K$ be two families of fuzzy soft sets. For the fuzzy soft function $f_{pu} : FSS(X)_E \to FSS(Y)_K$, the following statements hold,*

(1) $f_{pu}^{-1}((g,B)') = (f_{pu}^{-1}(g,B))' \forall\ (g,B) \in FSS(Y)_K$.
(2) $f_{pu}(f_{pu}^{-1}((g,B))) \sqsubseteq (g,B) \forall\ (g,B) \in FSS(Y)_K$. *If f_{pu} is surjective, then the equality holds.*
(3) $(f,A) \sqsubseteq f_{pu}^{-1}(f_{pu}((f,A))) \forall\ (f,A) \in FSS(X)_E$. *If f_{pu} is injective, then the equality holds.*
(4) $f_{pu}(\tilde{0}_E) = \tilde{0}_K$, $f_{pu}(\tilde{1}_E) \sqsubseteq \tilde{1}_K$. *If f_{pu} is surjective, then the equality holds.*
(5) $f_{pu}^{-1}(\tilde{1}_K) = \tilde{1}_E$ *and* $f_{pu}^{-1}(\tilde{0}_K) = \tilde{0}_E$.
(6) *If* $(f,A) \sqsubseteq (g,A)$, *then* $f_{pu}(f,A) \sqsubseteq f_{pu}(g,A)$.
(7) *If* $(f,B) \sqsubseteq (g,B)$, *then* $f_{pu}^{-1}(f,B) \sqsubseteq f_{pu}^{-1}(g,B)\ \forall\ (f,B),(g,B) \in FSS(Y)_K$.
(8) $f_{pu}^{-1}(\sqcup_{j\in J}(f,B)_j) = \sqcup_{j\in J} f_{pu}^{-1}(f,B)_j$ *and* $f_{pu}^{-1}(\sqcap_{j\in J}(f,B)_j) = \sqcap_{j\in J} f_{pu}^{-1}(f,B)_j$, $\forall\ (f,B)_j \in FSS(Y)_K$.
(9) $f_{pu}(\sqcup_{j\in J}(f,A)_j) = \sqcup_{j\in J} f_{pu}(f,A)_j$ *and* $f_{pu}(\sqcap_{j\in J}(f,A)_j) \sqsubseteq \sqcap_{j\in J} f_{pu}(f,A)_j$ $\forall\ (f,A)_j \in FSS(X)_E$. *If f_{pu} is injective, then the equality holds.*

Definition 2.19 ([17]). Let (X,τ,E) be a soft topological space and $F_A \in SS(X)_E$. If $F_A \tilde{\subseteq} cl(int(cl(F_A)))$, then F_A is called β-open soft set. We denote the set of all β-open soft sets by $\beta OS(X,\tau,E)$, or $\beta OS(X)$ and the set of all β-closed soft sets by $\beta CS(X,\tau,E)$, or $\beta CS(X)$.

Theorem 2.4 ([26]). *Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then*

(1) $f_A \in FSOS(X)$ *if and only if* $Fcl(f_A) = Fcl(Fint(f_A))$.
(2) *If* $g_B \in \mathfrak{T}$, *then* $g_B \sqcap Fcl(f_A) \sqsubseteq Fcl(g_B \sqcap g_B)$.

Definition 2.20 ([26]). Two fuzzy soft points $f_e = x_\alpha^e$ and $g_e = y_\beta^e$ are said to be distinct if and only if $x \neq y$.

3. Fuzzy β-open (closed) soft sets

Various generalizations of closed and open soft sets in soft topological spaces were studied by Kandil et al. [17], but for fuzzy soft topological spaces such generalization have not been studied so far. In this section, we move one step forward to introduce fuzzy β-open soft sets, fuzzy β-closed soft sets and study various properties and notions related to these structures.

Definition 3.1. Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. If $f_A \sqsubseteq Fcl(Fint(Fcl(f_A)))$, then f_A is called fuzzy β-open soft set. We denote the set of all fuzzy β-open soft sets by $F\beta OS(X,\mathfrak{T},E)$, or $F\beta OS(X)$ and the set of all fuzzy β-closed soft sets by $F\beta CS(X,\mathfrak{T},E)$, or $F\beta CS(X)$.

Theorem 3.1. *Let $(X,\mathfrak{T},E)$ be a fuzzy soft topological space and $f_A \in F\beta OS(X)$. Then*

(1) *Arbitrary fuzzy soft union of fuzzy β-open soft sets is fuzzy β-open soft.*
(2) *Arbitrary fuzzy soft intersection of fuzzy β-closed soft sets is fuzzy β-closed soft.*

Proof. (1) Let $\{(f,A)_j : j \in J\} \subseteq F\beta OS(X)$. Then, $\forall\, j \in J$, $(f,A)_j \sqsubseteq Fcl(Fint(Fcl(f,A)_j))$. It follows that

$$\begin{aligned}\sqcup_j (f,A)_j &\sqsubseteq \sqcup_j (Fcl(Fint(Fcl(f,A)_j)))\\ &= Fcl(\sqcup_j Fint(f,A)_j) \sqsubseteq Fcl(Fint(\sqcup_j Fcl(f,A)_j)\\ &= Fcl(Fint(Fcl(\sqcup_j (f,A)_j))).\end{aligned}$$

Thus $\sqcup_j (f,A)_j \in F\beta OS(X)\ \forall\, j \in J$.

(2) By a similar way.

□

Theorem 3.2. *Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then, $f_A \in F\beta OS(X)$ if and only if $Fcl(f_A) = Fcl(Fint(Fcl(f_A)))$.*

Proof. It is obvious. □

Definition 3.2. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space, $f_A \in FSS(X)_E$ and $f_e \tilde{\in} FSS(X)_E$. Then

(1) f_e is called fuzzy β-interior soft point of f_A if $\exists\, g_B \in F\beta OS(X)$ such that $f_e \tilde{\in} g_B \sqsubseteq f_A$. The set of all fuzzy β-interior soft points of f_A is called the fuzzy β-soft interior of f_A and is denoted by $F\beta int(f_A)$ consequently, $F\beta int(f_A) = \sqcup\{g_B : g_B \sqsubseteq f_A,\ g_B \in F\beta OS(X)\}$.

(2) f_e is called fuzzy β-closure soft point of f_A if $f_A \sqcap h_C \neq \tilde{0}_E\ \forall\, h_D \in F\beta OS(X)$. The set of all fuzzy β-closure soft points of f_A is called fuzzy β-soft closure of f_A and denoted by $F\beta cl(f_A)$. Consequently, $F\beta cl(f_A) = \sqcap\{h_D : h_D \in F\beta CS(X),\ f_A \sqsubseteq h_D\}$.

Theorem 3.3. *Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A, g_B \in FSS(X)_E$. Then, the following properties are satisfied for the fuzzy β-interior operator, denoted by $F\beta int$.*

(1) *$F\beta int(\tilde{1}_E) = \tilde{1}_E$ and $F\beta int(\tilde{0}_E)) = \tilde{0}_E$.*

(2) *$F\beta int(f_A) \sqsubseteq (f_A)$.*

(3) *$F\beta int(f_A)$ is the largest fuzzy β-open soft set contained in f_A.*

(4) *If $f_A \sqsubseteq g_B$, then $F\beta int(f_A) \sqsubseteq F\beta int(g_B)$.*

(5) *$F\beta int(F\beta int(f_A)) = F\beta int(f_A)$.*

(6) *$F\beta int(f_A) \sqcup F\beta int(g_B) \sqsubseteq F\beta int[(f_A) \sqcup (g_B)]$.*

(7) *$F\beta int[(f_A) \sqcap (g_B)] \sqsubseteq F\beta int(f_A) \sqcap F\beta int(g_B)$.*

Proof. Obvious. □

Theorem 3.4. *Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A, g_B \in FSS(X)_E$. Then, the following properties are satisfied for the fuzzy β-closure operator, denoted by $F\beta cl$.*

(1) *$F\beta cl(\tilde{1}_E) = \tilde{1}_E$ and $F\beta cl(\tilde{0}_E) = \tilde{0}_E$.*

(2) *$(f_A) \sqsubseteq F\beta cl(f_A)$.*

(3) *$F\beta cl(f_A)$ is the smallest fuzzy β-closed soft set contains f_A.*

(4) *If $f_A \sqsubseteq g_B$, then $F\beta cl(f_A) \sqsubseteq F\beta cl(g_B)$.*

(5) *$F\beta cl(F\beta cl(f_A)) = F\beta cl(f_A)$.*

(6) *$F\beta cl(f_A) \sqcup F\beta cl(g_B) \sqsubseteq F\beta cl[(f_A) \sqcup (g_B)]$.*

(7) $F\beta cl[(f_A) \sqcap (g_B)] \sqsubseteq F\beta cl(f_A) \sqcap F\beta cl(g_B)$.

Proof. It is obvious. □

Remark 3.1. Note that the family $\mathfrak{T}_\beta$ of all fuzzy β-open soft sets on $(X, \mathfrak{T}, E)$ forms a fuzzy supra soft topology, which is a collection of fuzzy soft sets contains $\tilde{1}_E, \tilde{0}_E$ and closed under arbitrary fuzzy soft union.

Lemma 3.1. *Every fuzzy open (resp. closed) soft set in a fuzzy soft topological space $(X, \mathfrak{T}, E)$ is fuzzy β-open (resp. closed) soft.*

Proof. Let $f_A \in FOS(X)$. Then $Fint(f_A) = f_A$. Since $f_A \sqsubseteq Fcl(f_A)$,
$f_A \sqsubseteq Fcl(Fint(f_A)) \sqsubseteq Fcl(Fint(Fcl(f_A)))$. Thus $f_A \in F\beta OS(X)$. □

Remark 3.2. The converse of Lemma 3.1 is not true in general as shown in the following example.

Example 3.1. Let $X = \{a, b, c\}$, $E = \{e_1, e_2, e_3\}$ and $A, B, C, D \subseteq E$ where,
$A = \{e_1, e_2\}$, $B = \{e_2, e_3\}$, $C = \{e_1, e_3\}$ and $D = \{e_2\}$.
Let $\mathfrak{T} = \{\tilde{1}_E, \tilde{0}_E, f_{1A}, f_{2B}, f_{3D}, f_{4E}, f_{5B}, f_{6D}\}$ where $f_{1A}, f_{2B}, f_{3D}, f_{4E}, f_{5B}, f_{6D}$ are fuzzy soft sets over X defined as follows :

$\mu^{e_1}_{f_{1A}} = \{a_{0.5}, b_{0.75}, c_{0.4}\}$, $\mu^{e_2}_{f_{1A}} = \{a_{0.3}, b_{0.8}, c_{0.7}\}$,
$\mu^{e_2}_{f_{2B}} = \{a_{0.4}, b_{0.6}, c_{0.3}\}$, $\mu^{e_3}_{f_{2B}} = \{a_{0.2}, b_{0.4}, c_{0.45}\}$,
$\mu^{e_2}_{f_{3D}} = \{a_{0.3}, b_{0.6}, c_{0.3}\}$,
$\mu^{e_1}_{f_{4E}} = \{a_{0.5}, b_{0.75}, c_{0.4}\}$, $\mu^{e_2}_{f_{4E}} = \{a_{0.4}, b_{0.8}, c_{0.7}\}$, $\mu^{e_3}_{f_{4E}} = \{a_{0.2}, b_{0.4}, c_{0.45}\}$,
$\mu^{e_2}_{f_{5B}} = \{a_{0.4}, b_{0.8}, c_{0.7}\}$, $\mu^{e_3}_{f_{5B}} = \{a_{0.2}, b_{0.4}, c_{0.45}\}$,
$\mu^{e_2}_{f_{6D}} = \{a_{0.3}, b_{0.8}, c_{0.7}\}$.

Then, $\mathfrak{T}$ defines a fuzzy soft topology on X. Thus the fuzzy soft set k_E where,
$\mu^{e_1}_{k_E} = \{a_{0.4}, b_{0.3}, c_{0.2}\}$, $\mu^{e_2}_{k_E} = \{a_{0.6}, b_{0.9}, c_{0.7}\}$, $\mu^{e_3}_{k_E} = \{a_{0.2}, b_{0.3}, c_{0.1}\}$.
is fuzzy β-open soft set of $(X, \mathfrak{T}, E)$, but it is not fuzzy open soft.

Theorem 3.5. *Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)$. Then*

(1) $F\beta int(f'_A) = \tilde{1} - [F\beta cl(f_A)]$.
(2) $F\beta cl(f'_A) = \tilde{1} - [F\beta int(f_A)]$.

Proof. (1) Since $F\beta cl(f_A) = \sqcap\{h_D : h_D \in F\beta CS(X),\ f_A \sqsubseteq h_D\}$,
$\tilde{1} - F\beta cl(f_A) = \sqcup\{h'_D : h'_D \in F\beta OS(X),\ h'_D \sqsubseteq f'_A\} = F\beta int(f'_A)$.
(2) By a similar way.

□

Theorem 3.6. *Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space, $f_A \in FOS(X)$ and $g_B \in F\beta OS(X)$. Then $f_A \sqcap g_B \in F\beta OS(X)$.*

Proof. Let $f_A \in FOS(X)$ and $g_B \in F\beta OS(X)$. Then, from Theorem 2.4 (2),

$$\begin{aligned} f_A \sqcap g_B &\sqsubseteq Fint(f_A) \sqcap Fcl(Fint(Fcl(g_B))) \\ &\sqsubseteq Fcl[Fint(f_A) \sqcap Fint(Fcl(g_B))] \\ &= Fcl(Fint[Fint(f_A) \sqcap Fcl(g_B)]) \\ &\sqsubseteq Fcl(Fint(Fcl[(f_A) \sqcap (g_B)]))). \end{aligned}$$

Thus $f_A \sqcap g_B \in F\beta OS(X)$. □

Theorem 3.7. *Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then $f_A \in F\beta CS(X)$ if and only if $Fint(Fcl(Fint(f_A))) \sqsubseteq f_A$.*

Proof. Let $f_A \in F\beta CS(X)$. Then f'_A is a fuzzy β-open soft set. This means that $f'_A \sqsubseteq Fcl(Fint(Fcl(\tilde{1}_E - f_A))) = \tilde{1}_E - (Fint(Fcl(Fint(f_A))))$. Thus $Fint(Fcl(Fint(f_A))) \sqsubseteq f_A$.
Conversely, let $Fint(Fcl(Fint(f_A)) \sqsubseteq f_A$. Then $\tilde{1}_E - f_A \sqsubseteq Fcl(Fint(Fcl(\tilde{1}_E - f_A))$. Thus $\tilde{1}_E - f_A$ is a fuzzy β-open soft set. So f_A is fuzzy β-closed soft set. □

Corollary 3.1. *Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then $f_A \in F\beta CS(X)$ if and only if $f_A = f_A \sqcup Fint(Fcl(Fint(f_A))$.*

4. Fuzzy β-continuous soft functions

In [2], Karal et al. defined the notion of a mapping on classes of fuzzy soft sets, which is a fundamental important in fuzzy soft set theory, to improve this work and they studied properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets. Kandil et al. [25] introduced some types of soft function in soft topological spaces. Here, we introduce the notions of fuzzy β-soft function in fuzzy soft topological spaces and study its basic properties.

Definition 4.1. Let $(X, \mathfrak{T}_1, E)$, $(Y, \mathfrak{T}_2, K)$ be fuzzy soft topological spaces and $f_{pu} : FSS(X)_E \to FSS(Y)_K$ be a fuzzy soft function. Then the function f_{pu} is called :

(1) Fuzzy β-continuous soft if $f_{pu}^{-1}(g_B) \in F\beta OS(X) \forall\, g_B \in \mathfrak{T}_2$.
(2) Fuzzy β-open soft if $f_{pu}(g_A) \in F\beta OS(Y) \forall\, g_A \in \mathfrak{T}_1$.
(3) Fuzzy β-closed soft if $f_{pu}(f_A) \in F\beta CS(Y) \forall\, f_A \in \mathfrak{T}'_1$.
(4) Fuzzy β-irresolute soft if $f_{pu}^{-1}(g_B) \in F\beta OS(X) \forall\, g_B \in F\beta OS(Y)$.
(5) Fuzzy β-irresolute open soft if $f_{pu}(g_A) \in F\beta OS(Y) \forall\, g_A \in F\beta OS(X)$.
(6) Fuzzy β-irresolute closed soft if $f_{pu}(f_A) \in F\beta CS(Y) \forall\, f_A \in F\beta CS(Y)$.

Example 4.1. Let $X = Y = \{a, b, c\}$, $E = \{e_1, e_2, e_3\}$ and $A \subseteq E$ where, $A = \{e_1, e_2\}$. Let $f_{pu} : (X, \mathfrak{T}_1, E) \to (Y, \mathfrak{T}_2, K)$ be the constant soft mapping where $\mathfrak{T}_1$ is the indiscrete fuzzy soft topology and $\mathfrak{T}_2$ is the discrete fuzzy soft topology such that $u(x) = a \ for\ each\ x \in X$ and $p(e) = e_1 \ \forall\ e \in E$. Let f_A be fuzzy soft set over Y defined as follows :

$\mu_{f_A}^{e_1} = \{a_{0.3}, b_{0.5}, c_{0.6}\}$, $\mu_{f_A}^{e_2} = \{a_{0.6}, b_{0.2}, c_{0.5}\}$.

Then, $f_A \in \mathfrak{T}_2$. Now, we find $f_{pu}^{-1}(f_A)$ as follows:

$f_{pu}^{-1}(f_A)(e_1)(a) = f_A(p(e_1))(u(a)) = f_A(e_1)(a) = 0.3$,
$f_{pu}^{-1}(f_A)(e_1)(b) = f_A(p(e_1))(u(b)) = f_A(e_1)(a) = 0.3$,
$f_{pu}^{-1}(f_A)(e_1)(c) = f_A(p(e_1))(u(c)) = f_A(e_1)(a) = 0.3$,
$f_{pu}^{-1}(f_A)(e_2)(a) = f_A(p(e_2))(u(a)) = f_A(e_1)(a) = 0.3$,
$f_{pu}^{-1}(f_A)(e_2)(b) = f_A(p(e_2))(u(b)) = f_A(e_1)(a) = 0.3$,
$f_{pu}^{-1}(f_A)(e_2)(c) = f_A(p(e_2))(u(c)) = f_A(e_1)(a) = 0.3$,
$f_{pu}^{-1}(f_A)(e_3)(a) = f_A(p(e_3))(u(a)) = f_A(e_1)(a) = 0.3$,
$f_{pu}^{-1}(f_A)(e_3)(b) = f_A(p(e_3))(u(b)) = f_A(e_1)(a) = 0.3$,
$f_{pu}^{-1}(f_A)(e_3)(c) = f_A(p(e_3))(u(c)) = f_A(e_1)(a) = 0.3$.

Thus $f_{pu}^{-1}(f_A) \notin F\beta OS(X)$. So f_{pu} is not fuzzy β-continuous soft function.

Theorem 4.1. *Every fuzzy continuous soft function is fuzzy β-continuous soft.*

Proof. It is clear from Lemma 3.1. □

Theorem 4.2. *Let $(X, \mathfrak{T}_1, E)$, $(Y, \mathfrak{T}_2, K)$ be fuzzy soft topological spaces and f_{pu} be a soft function such that $f_{pu} : FSS(X)_E \to FSS(Y)_K$. Then the followings are equivalent :*

(1) f_{pu} *is a fuzzy β-continuous soft function.*
(2) $f_{pu}^{-1}(h_B) \in F\beta CS(X)\ \forall\ h_B \in F\beta S(Y)$.
(3) $f_{pu}(F\beta cl(g_A) \sqsubseteq Fcl_{\mathfrak{T}_2}(f_{pu}(g_A))\ \forall\ g_A \in FSS(X)_E$.
(4) $F\beta cl(f_{pu}^{-1}(h_B)) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(h_B))\ \forall\ h_B \in FSS(Y)_K$.
(5) $f_{pu}^{-1}(Fint_{\mathfrak{T}_2}(h_B)) \sqsubseteq F\beta int(f_{pu}^{-1}(h_B))\ \forall\ h_B \in FSS(Y)_K$.

Proof. (1) $\Rightarrow$ (2)**:** Let h_B be a fuzzy closed soft set over Y. Then $h'_B \in FOS(Y)$ and $f_{pu}^{-1}(h'_B) \in F\beta OS(X)$ by Definition 4.1. Since $f_{pu}^{-1}(h'_B) = (f_{pu}^{-1}(h_B))'$ from Theorem 2.3, $f_{pu}^{-1}(h_B) \in F\beta CS(X)$.

(2) $\Rightarrow$ (3)**:** Let $g_A \in FSS(X)_E$. From (2) and Theorem 2.3, since $g_A \sqsubseteq f_{pu}^{-1}(f_{pu}(g_A)) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(f_{pu}(g_A))) \in F\beta CS(X)$, $g_A \sqsubseteq F\beta cl(g_A) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(f_{pu}(g_A)))$. Thus, from Theorem 2.3,

$$f_{pu}(F\beta cl(g_A)) \sqsubseteq f_{pu}(f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(f_{pu}(g_A)))) \sqsubseteq Fcl_{\mathfrak{T}_2}(f_{pu}(g_A))).$$

So $f_{pu}(F\beta cl(g_A)) \sqsubseteq Fcl_{\mathfrak{T}_2}(f_{pu}(g_A))$.

(3) $\Rightarrow$ (4)**:** Let $h_B \in FSS(Y)_K$ and $g_A = f_{pu}^{-1}(h_B)$. Then, from (3), $f_{pu}(F\beta cl f_{pu}^{-1}(h_B)) \sqsubseteq Fcl_{\mathfrak{T}_2}(f_{pu}(f_{pu}^{-1}(h_B)))$. From Theorem 2.3, it follows that

$$\begin{aligned} F\beta cl(f_{pu}^{-1}(h_B)) &\sqsubseteq f_{pu}^{-1}(f_{pu}(F\beta cl(f_{pu}^{-1}(h_B)))) \\ &\sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(f_{pu}(f_{pu}^{-1}(h_B)))) \\ &\sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(h_B)). \end{aligned}$$

Thus $F\beta cl(f_{pu}^{-1}(h_B)) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(h_B))$.

(4) $\Rightarrow$ (2)**:** Let h_B be a fuzzy closed soft set over Y. Then, from (4), $F\beta cl(f_{pu}^{-1}(h_B)) \sqsubseteq f_{pu}^{-1}(Fcl_{\mathfrak{T}_2}(h_B))\ \forall\ h_B \in FSS(Y)_K$. But clearly $f_{pu}^{-1}(h_B) \sqsubseteq F\beta cl(f_{pu}^{-1}(h_B))$. This means that $f_{pu}^{-1}(h_B) = F\beta cl(f_{pu}^{-1}(h_B))$. So $f_{pu}^{-1}(h_B) \in F\beta CS(X)$.

(1) $\Rightarrow$ (5)**:** Let $h_B \in FSS(Y)_K$. Then $f_{pu}^{-1}(Fint_{\mathfrak{T}_2}(h_B)) \in F\beta OS(X)$ from (1). Thus $f_{pu}^{-1}(Fint_{\mathfrak{T}_2}(h_B)) = F\beta int(f_{pu}^{-1} Fint_{\mathfrak{T}_2}(h_B)) \sqsubseteq F\beta int(f_{pu}^{-1}(h_B))$. So $f_{pu}^{-1}(Fint_{\mathfrak{T}_2}(h_B)) \sqsubseteq F\beta int(f_{pu}^{-1}(h_B))$.

(5) $\Rightarrow$ (1)**:** Let h_B be a fuzzy open soft set over Y. Then, from (5),

$$Fint_{\mathfrak{T}_2}(h_B) = h_B$$

and

$$f_{pu}^{-1}(Fint_{\mathfrak{T}_2}(h_B)) = f_{pu}^{-1}((h_B)) \sqsubseteq F\beta int(f_{pu}^{-1}(h_B)).$$ But, we have $$F\beta int(f_{pu}^{-1}(h_B)) \sqsubseteq f_{pu}^{-1}(h_B).$$

This means that $F\beta int(f_{pu}^{-1}(h_B)) = f_{pu}^{-1}(h_B) \in F\beta OS(X)$. Thus f_{pu} is a fuzzy β-continuous soft function. □

Theorem 4.3. *Let $(X, \mathfrak{T}_1, E)$ and $(Y, \mathfrak{T}_2, K)$ be fuzzy soft topological spaces and f_{pu} be a soft function such that $f_{pu} : FSS(X)_E \to FSS(Y)_K$. Then the following are equivalent :*

(1) *f_{pu} is a fuzzy β-open soft function.*
(2) *$f_{pu}(Fint_{\mathfrak{T}_1}(g_A)) \sqsubseteq F\beta int(f_{pu}(g_A)) \; \forall \; g_A \in FSS(X)_E$.*

Proof. (1) $\Rightarrow$ (2)**:** Let $g_A \in FSS(X)_E$. Since $Fint_{\mathfrak{T}_1}(g_A) \in \mathfrak{T}_1$, $f_{pu}(Fint_{\mathfrak{T}_1}(g_A)) \in F\beta OS(Y) \; \forall \; g_A \in \mathfrak{T}_1$ by (1). It follow that

$$f_{pu}(Fint_{\mathfrak{T}_1}(g_A)) = F\beta int(f_{pu}Fint_{\mathfrak{T}_1}(g_A)) \sqsubseteq F\beta int(f_{pu}(g_A)).$$

Thus $f_{pu}(Fint_{\mathfrak{T}_1}(g_A)) \sqsubseteq F\beta int(f_{pu}(g_A)) \; \forall \; g_A \in FSS(X)_E$.

(2) $\Rightarrow$ (1)**:** Let $g_A \in \mathfrak{T}_1$. By hypothesis,

$$f_{pu}(Fint_{\mathfrak{T}_1}(g_A)) = f_{pu}(g_A) \sqsubseteq F\beta int(f_{pu}(g_A)) \in F\beta OS(Y).$$

But $F\beta int(f_{pu}(g_A)) \sqsubseteq f_{pu}(g_A)$. Thus $F\beta int(f_{pu}(g_A)) = f_{pu}(g_A)$ $\in F\beta OS(Y) \; \forall \; g_A \in \mathfrak{T}_1$. So f_{pu} is a fuzzy β-open soft function.

□

Theorem 4.4. *Let $f_{pu} : FSS(X)_E \to FSS(Y)_K$ be a fuzzy β-open soft function. If there exist $k_D \in FSS(Y)_K$ and $l_C \in \mathfrak{T}_1'$ such that $f_{pu}^{-1}(k_D) \sqsubseteq l_C$, then there exists $h_B \in F\beta CS(Y)$ such that $k_D \sqsubseteq h_B$ and $f_{pu}^{-1}(h_B) \sqsubseteq l_C$.*

Proof. Let $k_D \in FSS(Y)_K$ and $l_C \in \mathfrak{T}_1'$ such that $f_{pu}^{-1}(k_D) \sqsubseteq l_C$. Then $f_{pu}(l_C') \sqsubseteq k_D'$ from Theorem 2.3 where $l_C' \in \mathfrak{T}_1$. Since f_{pu} is fuzzy β-open soft function, we have $f_{pu}(l_C') \in F\beta OS(Y)$. Take $h_B = [f_{pu}(l_C')]'$. Thus $h_B \in F\beta CS(Y)$ such that $k_D \sqsubseteq h_B$ and $f_{pu}^{-1}(h_B) = f_{pu}^{-1}([f_{pu}(l_C')]') \sqsubseteq f_{pu}^{-1}(k_D')' = f_{pu}^{-1}(k_D) \sqsubseteq l_C$. This completes the proof. □

Theorem 4.5. *Let $(X, \mathfrak{T}_1, E)$ and $(Y, \mathfrak{T}_2, K)$ be fuzzy soft topological spaces and f_{pu} be a soft function such that $f_{pu} : FSS(X)_E \to FSS(Y)_K$. Then the following are equivalent :*

(1) *f_{pu} is a fuzzy β-closed soft function.*
(2) *$F\beta cl(f_{pu}(h_A)) \sqsubseteq f_{pu}(Fcl_{\mathfrak{T}_1}(h_A)) \; \forall \; h_A \in FSS(X)_E$.*

Proof. It follows immediately from Theorem 4.3. □

5. Fuzzy soft β-separation axioms

Soft separation axioms for soft topological spaces were studied by Shabir and Naz [41]. Kandil et al. [25] introduced and studied the notions of soft β-separation axioms in soft topological spaces. Here, we introduce the notions of fuzzy soft β-separation axioms in fuzzy soft topological spaces and study some of its basic properties.

Definition 5.1. A fuzzy soft topological space $(X, \mathfrak{T}, E)$ is said to be a fuzzy soft β-T_o-space if for every pair of distinct fuzzy soft points f_e, g_e there exists a fuzzy β-open soft set containing one of the points but not the other.

Examples 5.1. (1) Let $X = \{a, b\}$, $E = \{e_1, e_2, e_3\}$ and $\mathfrak{T}$ be the discrete fuzzy soft topology on X. Then $(X, \mathfrak{T}, E)$ is fuzzy soft β-T_o-space.
(2) Let $X = \{a, b, c\}$, $E = \{e_1, e_2\}$ and $\mathfrak{T}$ be the indiscrete fuzzy soft topology on X. Then $\mathfrak{T}$ is not fuzzy soft β-T_o-space.

Theorem 5.1. *A soft subspace* $(Y, \mathfrak{T}_Y, E)$ *of a fuzzy soft* β*-*T_o*-space* $(X, \mathfrak{T}, E)$ *is fuzzy soft* β*-*T_o*.*

Proof. Let h_e, g_e be two distinct fuzzy soft points in (Y, E). Then these fuzzy soft points are also in (X, E). Thus there exists a fuzzy β-open soft set f_A in $\mathfrak{T}$ containing one of the fuzzy soft points but not the other. So $h^Y_E \sqcap f_A$ is a fuzzy β-open soft set in $(Y, \mathfrak{T}_Y, E)$ containing one of the fuzzy soft points but not the other from Definition 2.16. Hence $(Y, \mathfrak{T}_Y, E)$ is fuzzy soft β-T_o. □

Definition 5.2. A fuzzy soft topological space $(X, \mathfrak{T}, E)$ is said to be a fuzzy soft β-T_1-space if for every pair of distinct fuzzy soft points f_e, g_e there exist fuzzy β-open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$, $g_e \tilde{\notin} f_A$; and $f_e \tilde{\notin} g_B$, $g_e \tilde{\in} g_B$.

Example 5.1. Let $X = \{a, b\}$, $E = \{e_1, e_2, e_3\}$ and $\mathfrak{T}$ be the discrete fuzzy soft topology on X. Then $(X, \mathfrak{T}, E)$ is fuzzy soft β-T_1-space.

Theorem 5.2. *A fuzzy soft subspace* $(Y, \mathfrak{T}_Y, E)$ *of a fuzzy soft* β*-*T_1*-space* $(X, \mathfrak{T}, E)$ *is fuzzy soft* β*-*T_1*.*

Proof. It is similar to the proof of Theorem 5.1. □

Theorem 5.3. *If every fuzzy soft point of a fuzzy soft topological space* $(X, \mathfrak{T}, E)$ *is fuzzy* β*-closed soft, then* $(X, \mathfrak{T}, E)$ *is fuzzy soft* β*-*T_1*.*

Proof. Suppose that f_e and g_e be two distinct fuzzy soft points of (X, E). By hypothesis, f_e and g_e are fuzzy β-closed soft sets. Thusf'_e and g'_e are distinct fuzzy β-open soft sets where $f_e \tilde{\in} g'_e$, $g_e \tilde{\notin} g'_e$ and $f_e \tilde{\notin} f'_e$, $g_e \tilde{\in} f'_e$. So $(X, \mathfrak{T}, E)$ is fuzzy soft β-T_1. □

Definition 5.3. A fuzzy soft topological space $(X, \mathfrak{T}, E)$ is said to be a fuzzy soft β-T_2-space if for every pair of distinct fuzzy soft points f_e, g_e there exist disjoint fuzzy β-open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$ and $g_e \tilde{\in} g_B$.

Example 5.2. Let $X = \{a, b, c\}$, $E = \{e_1, e_2, e_3\}$ and $\mathfrak{T}$ be the discrete fuzzy soft topology on X. Then $(X, \mathfrak{T}, E)$ is fuzzy soft β-T_2-space.

Proposition 5.1. *For a fuzzy soft topological space* $(X, \mathfrak{T}, E)$ *we have :*
fuzzy soft β*-*T_2*-space* $\Rightarrow$ *fuzzy soft* β*-*T_1*-space* $\Rightarrow$ *fuzzy soft* β*-*T_o*-space.*

Proof.

(1) Let $(X, \mathfrak{T}, E)$ be a fuzzy soft β-T_2-space and let f_e, g_e be two distinct fuzzy soft points. Then there exist disjoint fuzzy β-open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$ and $g_e \tilde{\in} g_B$. Since $f_A \sqcap g_B = \tilde{0}_E$, $f_e \tilde{\notin} g_B$ and $g_e \tilde{\notin} f_A$. Thus there exist fuzzy β-open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$, $g_e \tilde{\notin} f_A$ and $f_e \tilde{\notin} g_B$, $g_e \tilde{\in} g_B$. So $(X, \mathfrak{T}, E)$ is fuzzy soft β-T_1-space.

(2) Let $(X, \mathfrak{T}, E)$ be a fuzzy soft β-T_1-space and let f_e, g_e be two distinct fuzzy soft points. Then there exist fuzzy β-open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$, $g_e \tilde{\notin} f_A$ and $f_e \tilde{\notin} g_B$, $g_e \tilde{\in} g_B$. Thus we obtain a fuzzy β-open soft set containing one of the fuzzy soft point but not the other. So $(X, \mathfrak{T}, E)$ is fuzzy soft β-T_o-space.

□

Theorem 5.4. *Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space . If $(X, \mathfrak{T}, E)$ is fuzzy soft β-T_2-space, for every pair of distinct fuzzy soft points f_e, g_e there exists a fuzzy β-closed soft set k_A such that containing one of the fuzzy soft points $g_e \tilde{\in} k_A$ but not the other $f_e \tilde{\notin} k_A$ and $g_e \tilde{\notin} F\beta cl(k_A)$.*

Proof. Let f_e, g_e be two distinct fuzzy soft points. By assumption, there exists disjoint fuzzy β-open soft sets b_A and h_B such that $f_e \tilde{\in} b_A$, $g_e \tilde{\in} h_B$. Thus $g_e \tilde{\in} b'_A$ and $f_e \tilde{\notin} b'_A$ from Theorem 2.2. So b'_A is a fuzzy β-closed soft set containing g_e but not f_e and $f_e \tilde{\notin} F\beta cl(b'_A) = b'_A$. □

Theorem 5.5. *A fuzzy soft subspace $(Y, \mathfrak{T}_Y, E)$ of fuzzy soft β-T_2-space $(X, \mathfrak{T}, E)$ is fuzzy soft β-T_2.*

Proof. Let j_e, k_e be two distinct fuzzy soft points in (Y, E). Then these fuzzy soft points are also in (X, E). Thus there exist disjoint fuzzy β-open soft sets f_A and g_B in $\mathfrak{T}$ such that $j_e \in f_A$ and $k_e \in g_B$. So $h^Y_E \sqcap f_A$ and $h^Y_E \sqcap g_B$ are disjoint fuzzy β-open soft sets in $\mathfrak{T}_Y$ such that $j_e \in h^Y_E \sqcap f_A$ and $k_e \in h^Y_E \sqcap g_B$. Hence $(Y, \mathfrak{T}_Y, E)$ is fuzzy soft β-T_2. □

Theorem 5.6. *If every fuzzy soft point of a fuzzy soft topological space $(X, \mathfrak{T}, E)$ is fuzzy β-closed soft, then $(X, \mathfrak{T}, E)$ is fuzzy soft β-T_2.*

Proof. It similar to the proof of Theorem 5.3. □

Definition 5.4. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space, h_C be a fuzzy β-closed soft set and g_e be a fuzzy soft point such that $g_e \tilde{\notin} h_C$. If there exist disjoint fuzzy β-open soft sets f_S and f_W such that $g_e \tilde{\in} f_S$ and $g_B \sqsubseteq f_W$. Then, $(X, \mathfrak{T}, E)$ is called fuzzy soft β-regular space. A fuzzy soft β-regular T_1-space is called a fuzzy soft β-T_3-space.

Proposition 5.2. *Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space, h_C be a fuzzy β-closed soft set and g_e be a fuzzy soft point such that $g_e \tilde{\notin} h_C$. If $(X, \mathfrak{T}, E)$ is fuzzy soft β-regular space, then there exists a fuzzy β-open soft set f_A such that $g_e \tilde{\in} f_A$ and $f_A \sqcap h_C = \tilde{0}_E$.*

Proof. Obvious from Definition 5.4. □

Theorem 5.7. *Let $(X, \mathfrak{T}, E)$ be a fuzzy soft β-regular space and be a fuzzy β-open soft set g_B such that $f_e \tilde{\in} g_B$. Then, there exists a fuzzy β-open soft set f_S such that $f_e \tilde{\in} f_S$ and $F\beta cl(f_S) \sqsubseteq g_B$.*

Proof. Let g_B be a fuzzy β-open soft set containing a fuzzy soft point f_e in a fuzzy soft β-regular space $(X, \mathfrak{T}, E)$. Then g'_B is a fuzzy β-closed soft such that $f_e \tilde{\notin} g'_B$ from Theorem 2.2. By hypothesis, there exist disjoint fuzzy β-open soft sets f_S and f_W such that $f_e \tilde{\in} f_S$ and $g'_B \sqsubseteq f_W$. Thus $f'_W \sqsubseteq g_B$ and $f_S \sqsubseteq f'_W$. So $F\beta cl(f_S) \sqsubseteq f'_W \sqsubseteq g_B$. Hence we have a fuzzy β-open soft set f_S containing f_e such that $F\beta cl(f_S) \sqsubseteq g_B$. □

Theorem 5.8. *Every fuzzy soft β-T_3-space, in which every fuzzy soft point is fuzzy β-closed soft, is fuzzy soft β-T_2-space.*

Proof. Let f_e, g_e be two distinct fuzzy soft points of a fuzzy soft β-T_3-space $(X, \mathfrak{T}, E)$. By hypothesis, g_e is fuzzy β-closed soft set and $f_e \tilde{\notin} g_e$. From the fuzzy soft β-regularity, there exist disjoint fuzzy β-open soft sets k_A and h_B such that $f_e \tilde{\in} k_A$ and $g_e \sqsubseteq h_B$. Thus $f_e \tilde{\in} k_A$ and $g_e \tilde{\in} h_B$. So $(X, \mathfrak{T}, E)$ is fuzzy soft β-T_2-space. □

Theorem 5.9. *A fuzzy soft subspace $(Y, \mathfrak{T}_Y, E)$ of a fuzzy soft β-T_3-space $(X, \mathfrak{T}, E)$ is fuzzy soft β-T_3.*

Proof. By Theorem 5.2, $(Y, \mathfrak{T}_Y, E)$ is fuzzy soft β-T_1-space. Now, we want to prove that $(Y, \mathfrak{T}_Y, E)$ is fuzzy soft β-regular space. Let k_E be a fuzzy β-closed soft set in (Y, E) and let g_e be a fuzzy soft point in (Y, E) such that $g_e \tilde{\notin} k_E$. Then $k_E = h_E^Y \sqcap g_B$ for some fuzzy β-closed soft set g_B in (X, E). Thus $g_e \tilde{\notin} h_E^Y \sqcap g_B$. But $g_e \tilde{\in} h_E^Y$. So $g_e \tilde{\notin} g_B$. Since $(X, \mathfrak{T}, E)$ is fuzzy soft β-T_3, there exist disjoint fuzzy β-open soft sets f_S and f_W in $\mathfrak{T}$ such that $g_e \tilde{\in} f_S$ and $g_B \sqsubseteq f_W$. It follows that $h_E^Y \sqcap f_S$ and $h_E^Y \sqcap f_W$ are disjoint fuzzy β-open soft sets in $\mathfrak{T}_Y$ such that $g_e \tilde{\in} h_E^Y \sqcap f_S$ and $k_E \sqsubseteq h_E^Y \sqcap f_W$. Hence $(Y, \mathfrak{T}_Y, E)$ is fuzzy soft β-T_3. □

Definition 5.5. Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space and h_C, g_B be disjoint fuzzy β-closed soft sets. If there exist disjoint fuzzy β-open soft sets f_S and f_W such that $h_C \sqsubseteq f_S$, $g_B \sqsubseteq f_W$. Then, $(X, \mathfrak{T}, E)$ is called fuzzy soft β-normal space. A fuzzy soft β-normal T_1-space is called a fuzzy soft β-T_4-space.

Theorem 5.10. *Let $(X, \mathfrak{T}, E)$ be a fuzzy soft topological space. Then the following are equivalent :*

(1) *$(X, \mathfrak{T}, E)$ is a fuzzy soft β-normal space.*
(2) *For every fuzzy β-closed soft set h_C and fuzzy β-open soft set g_B where $h_C \sqsubseteq g_B$, there exists a fuzzy β-open soft set f_S such that $h_C \sqsubseteq f_S$, $F\beta cl(f_S) \sqsubseteq g_B$.*

Proof. **(1) ⇒ (2) :** Let h_C be a β-closed soft set and let g_B be a fuzzy β-open soft set such that $h_C \sqsubseteq g_B$. Then h_C, g'_B are disjoint fuzzy β-closed soft sets. It follows, by (1), there exist disjoint fuzzy β-open soft sets f_S and f_W such that $h_C \sqsubseteq f_S$, $g'_B \sqsubseteq f_W$. Now, $f_S \sqsubseteq f'_W$. Thus $F\beta cl(f_S) \sqsubseteq F\beta cl f'_W = f'_W$, where g_B is fuzzy β-open soft set. Also, $f'_W \sqsubseteq g_B$. So $F\beta cl(f'_S) \sqsubseteq f'_W \sqsubseteq g_B$. Hence $h_C \sqsubseteq f_S$, $F\beta cl(f_S) \sqsubseteq g_B$.

(2) ⇒ (1) : Let g_A and g_B be disjoint fuzzy β-closed soft sets. Then $g_A \sqsubseteq g'_B$. By hypothesis, there exists a fuzzy β-open soft set f_S such that $g_A \sqsubseteq f_S$, $F\beta cl(f_S) \sqsubseteq g'_B$. Thus $g_B \sqsubseteq [F\beta cl(f_S)]'$, $g_A \sqsubseteq f_S$ and $[F\beta cl(f_S)]' \sqcap f_S = \tilde{0}_E$, where f_S and $[F\beta cl(f_S)]'$ are fuzzy β-open soft sets. So $(X, \mathfrak{T}, E)$ is fuzzy soft β-normal space.

□

Theorem 5.11. *A fuzzy β-closed fuzzy soft subspace $(Y, \mathfrak{T}_Y, E)$ of a fuzzy soft β-normal space $(X, \mathfrak{T}, E)$ is fuzzy soft β-normal.*

Proof. Let g_A and g_B be disjoint fuzzy β-closed soft sets in $\mathfrak{T}_Y$. Then $g_A = h_E^Y \sqcap f_C$ and $g_B = h_E^Y \sqcap f_D$ for some fuzzy β-closed soft sets f_C, f_D in (X, E). Thus f_C, f_D are disjoint fuzzy β-closed soft sets in $\mathfrak{T}$. Since $(X, \mathfrak{T}, E)$ is fuzzy soft β-normal, there exist disjoint fuzzy β-open soft sets f_S and f_W in $\mathfrak{T}$ such that $f_C \sqsubseteq f_S$, $f_D \sqsubseteq f_W$. It follows that $h_E^Y \sqcap f_S$ and $h_E^Y \sqcap f_W$ are disjoint fuzzy β-open soft sets in $\mathfrak{T}_Y$ such that $g_A = h_E^Y \sqcap f_C \sqsubseteq h_E^Y \sqcap f_S$ and $g_B = h_E^Y \sqcap f_D \sqsubseteq h_E^Y \sqcap f_W$. So $(Y, \mathfrak{T}_Y, E)$ is fuzzy soft β-normal. □

Theorem 5.12. *Let $(X, \mathfrak{T}_1, E)$ and $(Y, \mathfrak{T}_2, K)$ be fuzzy soft topological spaces and $f_{pu} : SS(X)_E \to SS(Y)_K$ be a fuzzy soft function which is bijective, fuzzy β-irresolute soft and fuzzy β-irresolute open soft. If $(X, \mathfrak{T}_1, E)$ is a fuzzy soft β-normal space, then $(Y, \mathfrak{T}_2, K)$ is also a fuzzy soft β-normal space.*

Proof. Let f_A, g_B be disjoint fuzzy β-closed soft sets in Y. Since f_{pu} is fuzzy β-irresolute soft, $f_{pu}^{-1}(f_A)$ and $f_{pu}^{-1}(g_B)$ are fuzzy β-closed soft set in X such that $f_{pu}^{-1}(f_A) \sqcap f_{pu}^{-1}(g_B) = f_{pu}^{-1}[f_A \sqcap g_B] = f_{pu}^{-1}[\tilde{0}_K] = \tilde{0}_E$ from Theorem 2.3. By hypothesis, there exist disjoint fuzzy β-open soft sets k_C and h_D in X such that $f_{pu}^{-1}(f_A) \sqsubseteq k_C$ and $f_{pu}^{-1}(g_B) \sqsubseteq h_D$. From Theorem 2.3, it follows that

$$f_A = f_{pu}[f_{pu}^{-1}(f_A)] \sqsubseteq f_{pu}(k_C) \ , g_B = f_{pu}[f_{pu}^{-1}(g_B)] \sqsubseteq f_{pu}(h_D)$$

and

$$f_{pu}(k_C) \sqcap f_{pu}(h_D) = f_{pu}[k_C \sqcap h_D] = f_{pu}[\tilde{0}_E] = \tilde{0}_K.$$

Since f_{pu} is fuzzy β-irresolute open soft function, $f_{pu}(k_C), f_{pu}(h_D)$ are fuzzy β-open soft sets in Y. So$(Y, \mathfrak{T}_2, K)$ is a fuzzy soft β-normal space. □

6. Conclusion

Recently, many scientists have studied the soft set theory, which is initiated by Molodtsov [34] and easily applied to many problems having uncertainties from social life. In the present work, we have continued to study the properties of fuzzy soft topological spaces. We introduce the some new concepts in fuzzy soft topological spaces such as fuzzy β-open soft sets, fuzzy β-closed soft sets, fuzzy β-soft interior, fuzzy β-soft closure and fuzzy β-separation axioms and have established several interesting properties. Since the authors introduced topological structures on fuzzy soft sets [8, 15, 42], so the β-topological properties, which introduced by Kandil et al.[25], is generalized here to the fuzzy soft sets which we hope to be useful in the fuzzy systems. To extend this work, we will study the connectedness and other interesting properties based on fuzzy β-open soft sets. Because there exists compact connections between soft sets and information systems [38, 44], we expect that, the results deducted from the studies on fuzzy soft topological space may improve these kinds of connections. We hope that the findings in this paper will help researcher enhance and promote the further study on fuzzy soft topology to carry out a general framework for their applications in practical life.

7. Acknowledgements

The authors express their sincere thanks to the reviewers for their valuable suggestions. The authors are also thankful to the editors-in-chief and managing editors for their important comments which helped to improve the presentation of the paper.

References

[1] A. M. Abd El-latif and Serkan Karatas, Supra b-open soft sets and supra b-soft continuity on soft topological spaces, Journal of Mathematics and Computer Applications Research 5 (1) (2015) 1–18.

[2] B. Ahmad and A. Kharal, Mappings on fuzzy soft classes, Adv. Fuzzy Syst. 2009, Art. ID 407890, 1–6.

[3] B. Ahmad and A. Kharal, Mappings on soft classes, New Math. Nat. Comput. 7 (3) (2011) 471–481.

[4] B. Ahmad and A. Kharal, On fuzzy soft sets, Adv. Fuzzy Syst. 2009, Art. ID 586507, 1–6.

[5] H. Aktas and N. Cagman, Soft sets and soft groups, Inform. Sci. 1 (2007) 2726–2735.

[6] M. I. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, On some new operations in soft set theory, Computers and Mathematics with Applications 57 (2009) 1547–1553.

[7] S. Atmaca and I.Zorlutuna, On fuzzy soft topological spaces, Ann. Fuzzy Math. Inform. 5 (2) (2013) 377–386.

[8] Bakir Tanay and M. Burcl Kandemir, Topological structure of fuzzy soft sets, Computer and mathematics with applications (61) (2011) 2952–2957.

[9] N. Cagman, F.Citak and S. Enginoglu, Fuzzy parameterized fuzzy soft set theory and its applications, Turkish Journal of Fuzzy Systems 1 (1) (2010) 21–35.

[10] N. Cagman and S. Enginoglu, Soft set theory and uni-Fint decision making, European Journal of Operational Research 207 (2010) 848–855.

[11] C. L. Change, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.

[12] S. A. El-Sheikh and A. M. Abd El-latif, Characterization of b-open soft sets in soft topological spaces, Journal of New Theory 2 (2015) 8–18.

[13] S. A. El-Sheikh, Rodyna A. Hosny and A. M. Abd El-latif, Characterizations of b-soft separation axioms in soft topological spaces, Inf. Sci. Lett. 4 (3) (2015) 125–133.

[14] S. A. El-Sheikh and A. M. Abd El-latif, Decompositions of some types of supra soft sets and soft continuity, International Journal of Mathematics Trends and Technology 9 (1) (2014) 37–56.

[15] Jianyu Xiao, Minming Tong, Qi Fan and Su Xiao, Generalization of Belief and Plausibility Functions to Fuzzy Sets, Applied Mathematics Information Sciences 6 (2012) 697–703.

[16] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Fuzzy soft semi connected properties in fuzzy soft topological spaces, Math. Sci. Lett. 4 (2015) 171–179.

[17] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, γ-operation and decompositions of some forms of soft continuity in soft topological spaces, Ann. Fuzzy Math. Inform. 7 (2) (2014) 181–196.

[18] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, γ-operation and decompositions of some forms of soft continuity of soft topological spaces via soft ideal, Ann. Fuzzy Math. Inform. 9 (3) (2015) 385–402.

[19] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Notes on γ-soft operator and some counter examples on (supra) soft continuity, Ann. Fuzzy Math. Inform. 10 (2) (2015) 203–213.

[20] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft connectedness via soft ideals, Journal of New Results in Science 4 (2014) 90-1-08.

[21] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft ideal theory, Soft local function and generated soft topological spaces, Appl. Math. Inf. Sci. 8 (4) (2014) 1595–1603.

[22] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft regularity and normality based on semi open soft sets and soft ideals, Appl. Math. Inf. Sci. Lett. 3 (2015) 47–55.

[23] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft semi compactness via soft ideals , Appl. Math. Inf. Sci. 8 (5) (2014) 2297–2306.

[24] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft semi (quasi) Hausdorff spaces via soft ideals, South Asian J. Math. 4 (6) (2014) 265–284.

[25] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft semi separation axioms and irresolute soft functions, Ann. Fuzzy Math. Inform. 8 (2) (2014) 305–318.
[26] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Some fuzzy soft topological properties based on fuzzy semi open soft sets, South Asian J. Math. 4 (4) (2014) 154–169.
[27] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Supra generalized closed soft sets with respect to an soft ideal in supra soft topological spaces, Appl. Math. Inf. Sci. 8 (4) (2014) 1731–1740.
[28] D. V. Kovkov, V. M. Kolbanov and D. A. Molodtsov, Soft sets theory-based optimization, Journal of Computer and Systems Sciences Finternational 46 (6) (2007) 872–880.
[29] J. Mahanta and P. K. Das, Results on fuzzy soft topological spaces, arXiv:1203.0634v1,2012.
[30] P. K. Maji, R. Biswas and A. R. Roy, Fuzzy soft sets, J. Fuzzy Math. 9 (3) (2001) 589–602.
[31] P. K. Maji, R. Biswas and A. R. Roy, intuitionistic fuzzy soft sets, J. Fuzzy Math. 9 (3) (2001) 677–691.
[32] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Computers and Mathematics with Applications 45 (2003) 555–562.
[33] P. Majumdar and S. K. Samanta, Generalised fuzzy soft sets, Computers and Mathematics with Applications 59 (2010) 1425–1432.
[34] D. A. Molodtsov, Soft set theory-firs tresults, Computers and Mathematics with Applications 37 (1999) 19–31.
[35] D. Molodtsov, V. Y. Leonov and D. V. Kovkov, Soft sets technique and its application, Nechetkie Sistemy i Myagkie Vychisleniya 1 (1) (2006) 8–39.
[36] A. Mukherjee and S. B. Chakraborty, On Fintuitionistic fuzzy soft relations, Bulletin of Kerala Mathematics Association 5 (1) (2008) 35–42.
[37] B. Pazar Varol and H. Aygun, Fuzzy soft topology, Hacettepe Journal of Mathematics and Statistics 41 (3) (2012) 407–419.
[38] D. Pei and D. Miao, From soft sets to information systems, in: X. Hu, Q. Liu, A. Skowron, T. Y. Lin, R. R. Yager, B. Zhang (Eds.), Proceedings of Granular Computing, in: IEEE 2 (2005) 617–621.
[39] S. Roy and T. K. Samanta, An introduction to open and closed sets on fuzzy topological spaces, Ann. Fuzzy Math. Inform. 6 (2) (2012) 425–431.
[40] S. Roy and T. K. Samanta, A note on fuzzy soft topological spaces, Ann. Fuzzy Math. Inform. 3 (2) (2012) 305–311.
[41] M. Shabir and M. Naz, On soft topological spaces, Comput. Math. Appl. 61 (2011) 1786–1799.
[42] B. Tanay and M. B. Kandemir, Topological structure of fuzzy soft sets, Computer and Math. with appl. 61 (2011) 412–418.
[43] Won Keun Min, A note on soft topological spaces, Computers and Mathematics with Applications 62 (2011) 3524–3528.
[44] Z. Xiao, L. Chen, B. Zhong and S. Ye, Recognition for soft information based on the theory of soft sets, in: J. Chen (Ed.), Proceedings of ICSSSM-05, IEEE 2 (2005) 1104–1106.
[45] Yong Chan Kim and Jung Mi Ko, Fuzzy G-closure operators, commun Korean Math. Soc. 18 (2) (2008) 325–340.
[46] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[47] Y. Zou and Z. Xiao, Data analysis approaches of soft sets under incomplete information, Knowledge-Based Systems 21 (2008) 941–945.

A. M. Abd El-latif (alaa_8560@yahoo.com)
Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt