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Solution of second order linear differential equation in fuzzy environment

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Abstract. In this paper the solution of a second order differential equation of type $\frac{d^2x(t)}{dt^2} = kx(t)$, $k < or, > 0$ is described in fuzzy environments. It is discussed for three different cases: Initial condition is fuzzy number, coefficient is fuzzy number and initial condition and coefficient are both fuzzy numbers. Here all the fuzzy numbers are taken as Generalized Trapezoidal Fuzzy Numbers (GTrFNs). Further numerical examples are illustrated.

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1. Introduction

1.1. **Fuzzy derivative and fuzzy differential equation.** The topic fuzzy differential equation(FDE) has been rapidly developing in recent years. The use of fuzzy differential equations is an innate way to model dynamic systems under possibilistic uncertainty [29]. The notion of the fuzzy derivative was first induced by Chang and Zadeh [11]. It was followed up by Dubois and Prade [12]. Other process has been discussed by Puri and Ralescu [23] and Goetschel and Voxman [14]. The concept of differential equations in a fuzzy environment was first formulated by Kaleva [20]. In fuzzy differential equation the concepts of all derivative is deliberated as either Hukuhara or generalized derivatives. The Hukuhara differentiability has a deficiency (see [3, 13]). The solution turns fuzzier as time goes by. Bede manifested that a large class of BVPs has no solution if the Hukuhara derivative is used [4]. To overcome this difficulty, the concept of a generalized derivative was developed ([3, 8]) and fuzzy differential equations were discussed using this concept (see [5, 9, 10, 26]). Khastan and Nieto found solutions for a large enough class of boundary value problems using the generalized derivative [21].Bede in [6] discussed the generalized differentiability for fuzzy valued functions. The disadvantage of strongly generalized differentiability of a function in comparison H-differentiability is that, a fuzzy differential equation has no unique solution [3]. Stefanini and Bede (see [26, 27]) by the concept of generalization of the Hukuhara difference for compact convex set and introduced generalized Hukuhara differentiability for fuzzy valued function and they demonstrated that, this concept of differentiability have relationships with weakly generalized differentiability and strongly generalized differentiability. Recently Gasilov et. al. [15] solve the fuzzy initial value problem by a new technique where Barros et. al. [7] solve fuzzy differential equation via fuzzification of the derivative operator.

1.2. **Second order fuzzy differential equation.** Second order fuzzy Differential Equation (FDE) is also most important among all FDE. Second order FDE has many applications. There are many different methods for solving this FDE. In [22] the authors consider two point fuzzy boundary value problems (BVP). Wang and Gue [28] consider second order fuzzy differential equation and solve the problem by adomian method. Gasilov et al. in [16] described a new solution procedure by linear transformation on second order BVP. Repeatedly in [15] , Gasilov et al. take second order initial value problem. The solution strategy is same as the previous one. Using fuzzy Laplace transformation (FLT),Toluti and Ahmadi [24] solving fuzzy two order differential equation. Ahmad et al. [2] apply FLT in fuzzy two point BVP. There exists some paper where numerical solutions are derived. In [17] approximate solution of second order fuzzy BVP was found. Variational iteration methods are applying in [1]. Jamshidi and Avazpour [18] find solution using shooting method. Rabiei et al. [25] taking improved runge-kutta nystrom method for solving second order FDE. Ismail et al. [19] Nth Order Two Point Fuzzy Boundary Value Problems by Optimal Homotopy Asymptotic Method.

There are many approaches in solve the second order FDE. These are (i) The one approach is Hukuhara or generalized derivative. There is some difficulty in using Hukuhara derivative approach. To overcome the difficulty generalized derivative was developed. (ii) The second approaches are extension principle. In this method first we solve the associated crisp differential equation and then fuzzify the solutions. (iii) The third approaches is the fuzzy problem transformed into a crisp problem.(iv) The fourth one is the method base on linear transform. Split up the problem into two parts, corresponding crisp problem and the fuzzy problems. Both the solutions are unique. (v) The another approaches is numerical solution of this FDE.

1.3. **Motivation.** Many researcher works on second order fuzzy differential equation. All the authors take only the fuzzy initial value i.e., the initial condition is fuzzy number. But if the only coefficients and coefficients and initial condition both of the second order differential equation is fuzzy number then how its behavior is. For these we take possible three cases: Initial condition is fuzzy, coefficients are fuzzy and both initial condition and coefficients are fuzzy number. In other hand previously all authors take fuzzy number with maximum gradation one. But it is not necessary that the maximum gradation is one. To develop in generalized sense we take generalized fuzzy number.

1.4. **Novelties.** In spite of above mentioned developments on second order fuzzy differential equation, the following lacunas still exists in the formulation and solution of second order fuzzy differential equation, which are summarized below:

(i) Though there are some articles of second order fuzzy differential equation problem was solved but till now none has solve FDE in fuzzy environment i.e., Initial conditions are fuzzy number, coefficients are fuzzy number, initial conditions and coefficient are both fuzzy numbers.

(ii) Here also second order differential equation is solved with generalized trapezoidal fuzzy number.

1.5. **Structure of the paper.** The paper is organized as follows. In Section 2, we recall some fundamental results on fuzzy number and fuzzy calculus. In Section 3, second order FDE is discussed in generalized trapezoidal fuzzy environment. Numerical examples are also given in Section 4. Application is given in Section 5. Finally we conclude the paper and tell some future research scope in Section 6.

2. Preliminaries

Definition 2.1. (**Fuzzy Set**) A fuzzy set $\tilde{A}$ is defined by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \epsilon A, \mu_{\tilde{A}}(x) \epsilon [0,1]\}$. In the pair $(x, \mu_{\tilde{A}}(x))$ the first element belongs to the classical set A, the second element $\mu_{\tilde{A}}(x)$, belongs to the interval $[0,1]$, called membership function.

Definition 2.2. (**α-cut of a fuzzy set**) The α-level set (or, interval of confidence at level α or α-cut) of the fuzzy set $\tilde{A}$ of X that have membership values in A greater than or equal to α i.e., $\tilde{A} = \{x : \mu_{\tilde{A}}(x) \geq \alpha, x \epsilon X, \alpha \epsilon [0,1]\}$.

Definition 2.3. (**Fuzzy number**) A fuzzy number is an extension of a regular number in the sense that it does not refer to one single value but rather to a connected set of possible values, where each possible value has its own weight between 0 and 1.This weight is called membership function. Thus a fuzzy number is a convex and normal fuzzy set.

Definition 2.4. (**Trapezoidal fuzzy number**) A TrIFN $\tilde{A}^i$ is a subset of IFN in R with following membership function and non membership function as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x < a_2 \\ 1 & \text{if } a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3} & \text{if } a_3 < x \leq a_4 \\ 0 & \text{otherwise} \end{cases}$$

where $a_1 \leq a_2 \leq a_3 \leq a_4$ and the TrFN is denoted by

$\tilde{A}_{TrFN} = (a_1, a_2, a_3, a_4)$

Definition 2.5. (**Generalized trapezoidal fuzzy number**) A GTrIFN $\tilde{A}^i$ is a subset of IFN in R with following membership function and non membership function as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \omega\frac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x < a_2 \\ \omega & \text{if } a_2 \leq x \leq a_3 \\ \omega\frac{a_4-x}{a_3-a_3} & \text{if } a_3 < x \leq a_4 \\ 0 & \text{otherwise} \end{cases}$$

where $a_1 \leq a_2 \leq a_3 \leq a_4$ and GTrFN is denoted by

$\tilde{A}_{GTrFN} = (a_1, a_2, a_3, a_4; \omega)$

Now we have proposed a chart of generalized fuzzy number

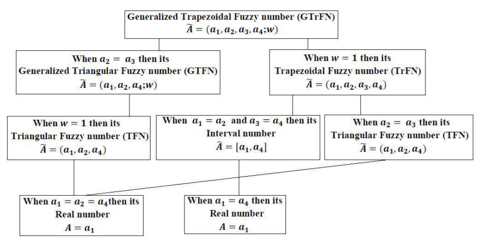

FIGURE 1. Chart of TrFN and GTrFN

Definition 2.6 ([20]). (**Generalized Hukuhara difference**) The generalized Hukuhara difference of two fuzzy number $u, v \in \Re_F$ is defines as follows

$$u \ominus_{gH} v = w \text{ is equivalent to } \begin{cases} (i) u = v \oplus w \\ (ii) v = u \oplus (-1)w \end{cases}$$

Consider $[w]_\alpha = [w_1(\alpha), w_2(\alpha)]$, then $w_1(\alpha) = min\{u_1(\alpha) - v_1(\alpha), u_2(\alpha) - v_2(\alpha)\}$ and $w_2(\alpha) = max\{u_1(\alpha) - v_1(\alpha), u_2(\alpha) - v_2(\alpha)\}$

Here the parametric representation of a fuzzy valued function $f : [a, b] \to \Re_F$ is expressed by $[f(t)]_\alpha = [f_1(t, \alpha), f_2(t, \alpha)]$, $t \in [a, b], \alpha \in [0, 1]$.

Definition 2.7 ([20]). (**Generalized Hukuhara derivative for first order**) The generalized Hukuhara derivative of a fuzzy valued function $f : (a, b) \to \Re_F$ at t_0 is defined as $f^{'}(t_0) = lim_{h \to 0} \frac{f(t_0+h) \ominus_{gH} f(t_0)}{h}$

If $f^{'}(t_0) \in \Re_F$ satisfying (2.1) exists, we say that f is generalized Hukuhara differentiable at t_0.

Also we say that $f(t)$ is (i)-gH differentiable at t_0 if

$$[f^{'}(t_0)]_\alpha = [f_1^{'}(t_0,\alpha), f_2^{'}(t_0,\alpha)]$$

and $f(t)$ is (ii)-gH differentiable at t_0 if

$$[f^{'}(t_0)]_\alpha = [f_2^{'}(t_0,\alpha), f_1^{'}(t_0,\alpha)]$$

Definition 2.8 ([23]). (**Generalized Hukuhara derivative for second order**) The second order generalized Hukuhara derivative of a fuzzy valued function $f : (a, b) \to \Re_F$ at t_0 is defined as

$$f^{''}(t_0) = lim_{h\to 0} \frac{f^{'}(t_0+h)\ominus_{gH} f^{'}(t_0)}{h}$$

If $f^{''} \in \Re_F$, we say that $f^{'}(t_0)$ is generalized Hukuhara at t_0.
Also we say that $f^{'}(t_0)$ is (i)-gH differentiable at t_0 if

$$f^{''}(t_0,\alpha) = \begin{cases} (f_1^{'}(t_0,\alpha), f_2^{'}(t_0,\alpha)) & \text{if f be (i)-gH differentiable on (a,b)} \\ (f_2^{'}(t_0,\alpha), f_1^{'}(t_0,\alpha)) & \text{if f be (ii)-gH differentiable on (a,b)} \end{cases}$$

for all $\alpha \in [0, 1]$, and that $f^{'}(t_0)$ is (ii)-gH differentiable at t_0 if

$$f^{''}(t_0,\alpha) = \begin{cases} (f_2^{'}(t_0,\alpha), f_1^{'}(t_0,\alpha)) & \text{if f be (i)-gH differentiable on (a,b)} \\ (f_1^{'}(t_0,\alpha), f_2^{'}(t_0,\alpha)) & \text{if f be (ii)-gH differentiable on (a,b)} \end{cases}$$

for all $\alpha \in [0, 1]$.

Definition 2.9. (**Second order fuzzy ordinary differential equation (FODE)**) Consider the second order linear homogeneous ordinary differential equation $\frac{d^2x(x)}{dt^2} = kx(t)$,$k > 0$ with initial conditions $x(t_0) = a$ and $\frac{dx(t_0)}{dt} = b$.

The above ODE is called FODE if any one of the following three cases holds:

(i) Only a and b is generalized fuzzy number (Type-I).

(ii) Only k is a generalized fuzzy number (Type-II).

(iii) Both k, a and b are generalized fuzzy numbers (Type-III).

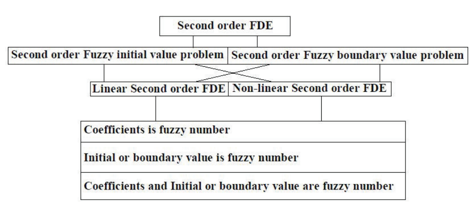

Figure 2. Second order fuzzy differential equation

Definition 2.10. (**Strong and week solution**) If the solution of fuzzy differential equation is of the form $[x_1(t,\alpha), x_2(t,\alpha)]$, the solution is called strong solution when

$\frac{dx_1(t,\alpha)}{d\alpha} > 0$, $\frac{dx_2(t,\alpha)}{d\alpha} < 0 \ \forall \alpha \in [0,\omega], x_1(t,\omega) \leq x_2(t,\omega)$.
Otherwise it is weak solution.

3. Solution procedure of 2^{nd} order linear homogeneous FODE

The solution procedures of 2^{nd} order linear homogeneous FODE of Type-I, Type-II and Type-III are described taking the coefficients positive and negative respectively. Here fuzzy numbers are taken as GTrFNs.

3.1. **Solution procedure of 2^{nd} order linear homogeneous FODE when coefficient is positive.** For this section two different cases arise

3.1.1 Solution Procedure of 2^{nd} Order Linear Homogeneous FODE of Type-I i.e., Initial value is Fuzzy number

Consider the initial value problem

$$\frac{d^2x(t)}{dt^2} = kx(t), k > 0 \tag{3.1}$$

With fuzzy initial condition $x(t_0) = \widetilde{a} = (a_1, a_2, a_3, a_4; \omega)$ and $\frac{dx(t_0)}{dt} = \widetilde{b} = (b_1, b_2, b_3, b_4; \omega)$

Here we solve the given problem for (i)-gH and (ii)-gH differentiability concepts respectively.

Here four cases arise

Case 3.1.1.1: when $x(t)$ and $\frac{dx(t)}{dt}$ is (i)-gH differentiable
Case 3.1.1.2: when $x(t)$ is (i)-gH differentiable and $\frac{dx(t)}{dt}$ is (ii)-gH differentiable
Case 3.1.1.3: when $x(t)$ is (ii)-gH differentiable and $\frac{dx(t)}{dt}$ is (i)-gH differentiable
Case 3.1.1.4: when $x(t)$ and $\frac{dx(t)}{dt}$ is (ii)-gH differentiable

Using the concept of Generalized Hukuhara differentiability the **Case 3.1.1.1** and **Case 3.1.1.4** are same where the **Case 3.1.1.2** and **Case 3.1.1.3** are same.

Solution of Case 3.1.1.1 and Case 3.1.1.4

In these case we have from (3.1) two differential equation

$$\frac{d^2x_1(t,\alpha)}{dt^2} = kx_1(t,\alpha) \tag{3.2}$$

and

$$\frac{d^2x_2(t,\alpha)}{dt^3} = kx_2(t,\alpha) \tag{3.3}$$

With initial condition $x_1(t_0,\alpha) = a_1 + \frac{\alpha l_{\tilde{a}}}{\omega}$, $x_2(t_0,\alpha) = a_4 - \frac{\alpha r_{\tilde{a}}}{\omega}$, $\frac{dx_1(t_0,\alpha)}{dt} = b_1 + \frac{\alpha l_{\tilde{b}}}{\omega}$ and $\frac{dx_2(t_0,\alpha)}{dt} = b_4 - \frac{\alpha r_{\tilde{b}}}{\omega}$

Where, $l_{\tilde{a}} = a_2 - a_1$,$r_{\tilde{a}} = a_4 - a_3$, $l_{\tilde{b}} = b_2 - b_1$ and $r_{\tilde{b}} = b_4 - b_3$.

The general solution of (3.2) is given by

$$x_1(t,\alpha) = c_1 e^{\sqrt{k}t} + c_2 e^{-\sqrt{k}t} \tag{3.4}$$

Using initial condition we have

$$a_1 + \frac{\alpha l_{\tilde{a}}}{\omega} = c_1 e^{\sqrt{k}t_0} + c_2 e^{-\sqrt{k}t_0} \tag{3.5}$$

and

$$\frac{1}{\sqrt{k}}(b_1 + \frac{\alpha l_{\tilde{b}}}{\omega}) = c_1 e^{\sqrt{k}t_0} - c_2 e^{-\sqrt{k}t_0} \tag{3.6}$$

Solving we get

$$c_1 = \frac{1}{2}\{(a_1 + \frac{\alpha l_{\tilde{a}}}{\omega}) + \frac{1}{\sqrt{k}}(b_1 + \frac{\alpha l_{\tilde{b}}}{\omega})\}e^{-\sqrt{k}t_0} \tag{3.7}$$

and

$$c_2 = \frac{1}{2}\{(a_1 + \frac{\alpha l_{\tilde{a}}}{\omega}) - \frac{1}{\sqrt{k}}(b_1 - \frac{\alpha l_{\tilde{b}}}{\omega})\}e^{\sqrt{k}t_0} \tag{3.8}$$

Hence from (3.4) we have

$x_1(t,\alpha) = \frac{1}{2}\{(a_1 + \frac{\alpha l_{\tilde{a}}}{\omega}) + \frac{1}{\sqrt{k}}(b_1 + \frac{\alpha l_{\tilde{b}}}{\omega})\}e^{-\sqrt{k}(t-t_0)} + \frac{1}{2}\{(a_1 + \frac{\alpha l_{\tilde{a}}}{\omega}) - \frac{1}{\sqrt{k}}(b_1 - \frac{\alpha l_{\tilde{b}}}{\omega})\}e^{-\sqrt{k}(t-t_0)}$

Similarly from (3.3) we have

$x_2(t,\alpha) = \frac{1}{2}\{(a_4 - \frac{\alpha r_{\tilde{a}}}{\omega}) + \frac{1}{\sqrt{k}}(b_4 - \frac{\alpha r_{\tilde{b}}}{\omega})\}e^{-\sqrt{k}(t-t_0)} - \frac{1}{2}\{(a_4 - \frac{\alpha r_{\tilde{a}}}{\omega}) - \frac{1}{\sqrt{k}}(b_4 - \frac{\alpha r_{\tilde{b}}}{\omega})\}e^{-\sqrt{k}(t-t_0)}$

Solution of Case 3.1.1.2 and Case 3.1.1.3

In these case we have from (3.1) two differential equation

$$\frac{d^2x_2(t,\alpha)}{dt^2} = kx_1(t,\alpha) \tag{3.9}$$

$$\frac{d^2x_1(t,\alpha)}{dt^3} = kx_2(t,\alpha) \tag{3.10}$$

With initial condition $x_1(t_0,\alpha) = a_1 + \frac{\alpha l_{\tilde{a}}}{\omega}$, $x_2(t_0,\alpha) = a_4 - \frac{\alpha r_{\tilde{a}}}{\omega}$, $\frac{dx_1(t_0,\alpha)}{dt} = b_1 + \frac{\alpha l_{\tilde{b}}}{\omega}$ and $\frac{dx_2(t_0,\alpha)}{dt} = b_4 - \frac{\alpha r_{\tilde{b}}}{\omega}$

Where, $l_{\tilde{a}} = a_2 - a_1$,$r_{\tilde{a}} = a_4 - a_3$, $l_{\tilde{b}} = b_2 - b_1$ and $r_{\tilde{b}} = b_4 - b_3$.

The general solution are

$$x_1(t,\alpha) = d_1e^{\sqrt{k}t} + d_2e^{-\sqrt{k}t} + d_3\sin\sqrt{k}t + d_4\cos\sqrt{k}t \tag{3.11}$$

and

$$x_2(t,\alpha) = d_1e^{\sqrt{k}t} + d_2e^{-\sqrt{k}t} - d_3\sin\sqrt{k}t - d_4\cos\sqrt{k}t \tag{3.12}$$

Using initial condition we have

$a_1 + \frac{\alpha l_{\tilde{a}}}{\omega} = d_1e^{\sqrt{k}t_0} + d_2e^{-\sqrt{k}t_0} + d_3\sin\sqrt{k}t_0 + d_4\cos\sqrt{k}t_0$

$\frac{1}{\sqrt{k}}(b_1 + \frac{\alpha l_{\tilde{b}}}{\omega}) = d_1e^{\sqrt{k}t_0} - d_2e^{-\sqrt{k}t_0} + d_3\sin\sqrt{k}t_0 - d_4\cos\sqrt{k}t_0$

$a_4 - \frac{\alpha r_{\tilde{a}}}{\omega} = d_1e^{\sqrt{k}t_0} + d_2e^{-\sqrt{k}t_0} - d_3\sin\sqrt{k}t_0 - d_4\cos\sqrt{k}t_0$

$\frac{1}{\sqrt{k}}(b_4 - \frac{\alpha r_{\tilde{b}}}{\omega}) = d_1e^{\sqrt{k}t_0} - d_2e^{-\sqrt{k}t_0} - d_3\sin\sqrt{k}t_0 + d_4\cos\sqrt{k}t_0$

Solving this above equation we get

$d_1 = \frac{1}{4}\{a_1 + a_4 + \frac{\alpha(l_{\tilde{a}} - r_{\tilde{a}})}{\omega} + \frac{1}{\sqrt{k}}(b_1 + b_4 + \frac{\alpha(l_{\tilde{b}} - r_{\tilde{b}})}{\omega})\}e^{-\sqrt{k}t_0}$

$d_2 = \frac{1}{4}\{a_1 + a_4 + \frac{\alpha(l_{\tilde{a}} - r_{\tilde{a}})}{\omega} - \frac{1}{\sqrt{k}}(b_1 + b_4 + \frac{\alpha(l_{\tilde{b}} - r_{\tilde{b}})}{\omega})\}e^{\sqrt{k}t_0}$

$d_3 = \frac{1}{4}\frac{1}{\sin\sqrt{k}t_0}\{a_1 - a_4 + \frac{\alpha(l_{\tilde{a}} + r_{\tilde{a}})}{\omega} + \frac{1}{\sqrt{k}}(b_1 - b_4 + \frac{\alpha(l_{\tilde{b}} + r_{\tilde{b}})}{\omega})\}$

$d_4 = \frac{1}{4}\frac{1}{\sin\sqrt{k}t_0}\{a_1 - a_4 + \frac{\alpha(l_{\tilde{a}} + r_{\tilde{a}})}{\omega} - \frac{1}{\sqrt{k}}(b_1 - b_4 + \frac{\alpha(l_{\tilde{b}} + r_{\tilde{b}})}{\omega})\}$

Hence the solution is (3.11) and (3.12) when the constants are the above value.

3.1.2 Solution Procedure of 2nd Order Linear Homogeneous FODE of Type-II

Consider the initial value problem

$$\frac{d^2x(t)}{dt^2} = \widetilde{k}x(t) \tag{3.13}$$

With $\widetilde{k} = (k_1, k_2, k_3, k_4; \lambda) > 0$ and
With fuzzy initial condition $x(t_0) = a$ and $\frac{dx(t_0)}{dt} = b$

Here we solve the given problem for (i)-gH and (ii)-gH differentiability concepts respectively.

Here four cases arise

Case 3.1.2.1: when $x(t)$ and $\frac{dx(t)}{dt}$ is (i)-gH differentiable
Case 3.1.2.2: when $x(t)$ is (i)-gH differentiable and $\frac{dx(t)}{dt}$ is (ii)-gH differentiable
Case 3.1.2.3: when $x(t)$ is (ii)-gH differentiable and $\frac{dx(t)}{dt}$ is (i)-gH differentiable
Case 3.1.2.4: when $x(t)$ and $\frac{dx(t)}{dt}$ is (ii)-gH differentiable

Using the concept of Generalized Hukuhara differentiability the **Case 3.1.2.1** and **Case 3.1.2.4** are same where the **Case 3.1.2.2** and **Case 3.1.2.3** are same.

Solution of Case 3.1.2.1 and Case 3.1.2.4

In this case we have from (3.13) two differential equations

$$\frac{d^2x_1(t,\alpha)}{dt^2} = k_1(\alpha)x_1(t,\alpha) \tag{3.14}$$

$$\frac{d^2x_2(t,\alpha)}{dt^2} = k_2(\alpha)x_2(t,\alpha) \tag{3.15}$$

With initial conditions $x_1(t_0,\alpha) = a$, $x_2(t_0,\alpha) = a$, $\frac{dx_1(t_0,\alpha)}{dt} = b$ and $\frac{dx_2(t_0,\alpha)}{dt} = b$ and coefficients coefficients $k_1(\alpha) = k_1 + \frac{\alpha l_{\tilde{k}}}{\lambda}$ and $k_2(\alpha) = k_4 - \frac{\alpha r_{\tilde{k}}}{\lambda}$

Using the initial condition the solution is given by

$x_1(t,\alpha) = \frac{1}{2}\{a + \frac{1}{\sqrt{k_1+\frac{\alpha l_{\tilde{k}}}{\lambda}}}b\}e^{\sqrt{k_1+\frac{\alpha l_{\tilde{k}}}{\lambda}}(t-t_0)} + \frac{1}{2}\{a - \frac{1}{\sqrt{k_1+\frac{\alpha l_{\tilde{k}}}{\lambda}}}b\}e^{-\sqrt{k_1+\frac{\alpha l_{\tilde{k}}}{\lambda}}(t-t_0)}$

and

$x_2(t,\alpha) = \frac{1}{2}\{a + \frac{1}{\sqrt{k_4 - \frac{\alpha r_{\widetilde{k}}}{\lambda}}} b\} e^{\sqrt{k_4 - \frac{\alpha r_{\widetilde{k}}}{\lambda}}(t-t_0)} + \frac{1}{2}\{a - \frac{1}{\sqrt{k_4 - \frac{\alpha r_{\widetilde{k}}}{\lambda}}} b\} e^{-\sqrt{k_4 - \frac{\alpha r_{\widetilde{k}}}{\lambda}}(t-t_0)}$

Solution of Case 3.1.2.2 and Case 3.1.2.3

In this case we have from (3.13) two differential equations

$$\frac{d^2x_2(t,\alpha)}{dt^2} = k_1(\alpha)x_1(t,\alpha) \tag{3.16}$$

$$\frac{d^2x_1(t,\alpha)}{dt^2} = k_2(\alpha)x_2(t,\alpha) \tag{3.17}$$

With initial conditions $x_1(t_0,\alpha) = a$, $x_2(t_0,\alpha) = a$, $\frac{dx_1(t_0,\alpha)}{dt} = b$ and $\frac{dx_2(t_0,\alpha)}{dt} = b$ and coefficients $k_1(\alpha) = k_1 + \frac{\alpha l_{\widetilde{k}}}{\lambda}$ and $k_2(\alpha) = k_4 - \frac{\alpha r_{\widetilde{k}}}{\lambda}$

The solution is written as

$x_1(t,\alpha) = d_1 e^{\sqrt{p(\alpha)}t} + d_2 e^{-\sqrt{p(\alpha)}t} + d_3 \sin\sqrt{p(\alpha)}t + d_4 \cos\sqrt{p(\alpha)}t$

$x_2(t,\alpha) = \sqrt{\frac{k_2(\alpha)}{k_1(\alpha)}}[d_1 e^{\sqrt{p(\alpha)}t} + d_2 e^{-\sqrt{p(\alpha)}t} - d_3 \sin\sqrt{p(\alpha)}t - d_4 \cos\sqrt{p(\alpha)}t]$

Where, $p(\alpha) = \sqrt{k_1(\alpha)k_2(\alpha)}$
$d_1 = \frac{1}{4}(a + \frac{b}{\sqrt{p(\alpha)}})(1 + \sqrt{\frac{k_1(\alpha)}{k_2(\alpha)}})e^{-\sqrt{p(\alpha)}t_0}$
$d_2 = \frac{1}{4}(a - \frac{b}{\sqrt{p(\alpha)}})(1 + \sqrt{\frac{k_1(\alpha)}{k_2(\alpha)}})e^{\sqrt{p(\alpha)}t_0}$
$d_3 = \frac{1}{2}(a \sin\sqrt{p(\alpha)}t_0 + \frac{b\cos\sqrt{p(\alpha)}t_0}{\sqrt{p(\alpha)}})(1 - \sqrt{\frac{k_1(\alpha)}{k_2(\alpha)}})$
$d_4 = \frac{1}{2}(a \cos\sqrt{p(\alpha)}t_0 - \frac{b\sin\sqrt{p(\alpha)}t_0}{\sqrt{p(\alpha)}})(1 - \sqrt{\frac{k_1(\alpha)}{k_2(\alpha)}})$

3.1.1.3 Solution Procedure of 2_{nd} Order Linear Homogeneous FODE of Type-III

Consider the initial value problem

$$\frac{d^2x(t)}{dt^2} = \widetilde{k}x(t) \tag{3.18}$$

and $\widetilde{k} = (k_1, k_2, k_3, k_4; \lambda) > 0$

With fuzzy initial condition $x(t_0) = \widetilde{a} = (a_1, a_2, a_3, a_4; \omega)$ and $\frac{dx(t_0)}{dt} = \widetilde{b} = (b_1, b_2, b_3, b_4; \omega)$

Here we solve the given problem for (i)-gH and (ii)-gH differentiability concepts respectively.

Here four cases arise

Case 3.1.3.1: when $x(t)$ and $\frac{dx(t)}{dt}$ is (i)-gH differentiable
Case 3.1.3.2: when $x(t)$ is (i)-gH differentiable and $\frac{dx(t)}{dt}$ is (ii)-gH differentiable
Case 3.1.3.3: when $x(t)$ is (ii)-gH differentiable and $\frac{dx(t)}{dt}$ is (i)-gH differentiable
Case 3.1.3.4: when $x(t)$ and $\frac{dx(t)}{dt}$ is (ii)-gH differentiable

Using the concept of Generalized Hukuhara differentiability the **Case 3.1.3.1** and **Case 3.1.3.4** are same where the **Case 3.1.3.2** and **Case 3.1.3.3** are same.

<u>Solution of Case 3.1.3.1 and Case 3.1.3.4</u>

In this case we have from (3.18) two differential equations

$$\frac{d^2x_1(t,\alpha)}{dt^2} = k_1(\alpha)x_1(t,\alpha) \tag{3.19}$$

$$\frac{d^2x_2(t,\alpha)}{dt^2} = k_2(\alpha)x_2(t,\alpha) \tag{3.20}$$

With initial conditions $x_1(t_0,\alpha) = a_1 + \frac{\alpha l_{\tilde{a}}}{\omega}$, $x_2(t_0,\alpha) = a_4 - \frac{\alpha r_{\tilde{a}}}{\omega}$, $\frac{dx_1(t_0,\alpha)}{dt} = b_1 + \frac{\alpha l_{\tilde{b}}}{\omega}$ and $\frac{dx_2(t_0,\alpha)}{dt} = b_4 - \frac{\alpha r_{\tilde{a}}}{\omega}$

With coefficients $k_1(\alpha) = k_1 + \frac{\alpha l_{\tilde{k}}}{\lambda}$ and $k_2(\alpha) = k_4 - \frac{\alpha r_{\tilde{k}}}{\lambda}$

Using the initial condition the solution is given by

$x_1(t,\alpha) = \frac{1}{2}\{(a_1 + \frac{\alpha l_{\tilde{a}}}{\eta}) + \frac{1}{\sqrt{k_1+\frac{\alpha l_{\tilde{k}}}{\eta}}}(b_1 + \frac{\alpha l_{\tilde{b}}}{\eta})\}e^{\sqrt{k_1+\frac{\alpha l_{\tilde{k}}}{\eta}}(t-t_0)} + \frac{1}{2}\{(a_1 + \frac{\alpha l_{\tilde{a}}}{\eta}) - \frac{1}{\sqrt{k_1+\frac{\alpha l_{\tilde{k}}}{\eta}}}(b_1 + \frac{\alpha l_{\tilde{b}}}{\eta})\}e^{-\sqrt{k_1+\frac{\alpha l_{\tilde{k}}}{\eta}}(t-t_0)}$

and

$x_2(t,\alpha) = \frac{1}{2}\{(a_4 - \frac{\alpha r_{\tilde{a}}}{\eta}) + \frac{1}{\sqrt{k_4-\frac{\alpha r_{\tilde{k}}}{\eta}}}(b_4 - \frac{\alpha r_{\tilde{a}}}{\eta})\}e^{\sqrt{k_4-\frac{\alpha r_{\tilde{k}}}{\eta}}(t-t_0)} + \frac{1}{2}\{(a_4 - \frac{\alpha r_{\tilde{a}}}{\eta}) - \frac{1}{\sqrt{k_4-\frac{\alpha r_{\tilde{k}}}{\eta}}}(b_4 - \frac{\alpha r_{\tilde{a}}}{\eta})\}e^{-\sqrt{k_4-\frac{\alpha r_{\tilde{k}}}{\eta}}(t-t_0)}$

Where, $\eta = min\{\omega, \lambda\}$

Solution of Case 3.1.3.2 and Case 3.1.3.3

In this case we have from (3.18) two differential equations

$$\frac{d^2x_2(t,\alpha)}{dt^2} = k_1(\alpha)x_1(t,\alpha) \tag{3.21}$$

$$\frac{d^2x_1(t,\alpha)}{dt^2} = k_2(\alpha)x_2(t,\alpha) \tag{3.22}$$

With initial conditions $x_1(t_0,\alpha) = a_1 + \frac{\alpha l_{\tilde{a}}}{\omega}$, $x_2(t_0,\alpha) = a_4 - \frac{\alpha r_{\tilde{a}}}{\omega}$, $\frac{dx_1(t_0,\alpha)}{dt} = b_1 + \frac{\alpha l_{\tilde{b}}}{\omega}$ and $\frac{dx_2(t_0,\alpha)}{dt} = b_4 - \frac{\alpha r_{\tilde{a}}}{\omega}$

With coefficients $k_1(\alpha) = k_1 + \frac{\alpha l_{\tilde{k}}}{\lambda}$ and $k_2(\alpha) = k_4 - \frac{\alpha r_{\tilde{k}}}{\lambda}$

The solution is written as

$x_1(t,\alpha) = d_1 e^{\sqrt{p(\alpha)}t} + d_2 e^{-\sqrt{p(\alpha)}t} + d_3 \sin\sqrt{p(\alpha)}t + d_4 \cos\sqrt{p(\alpha)}t$

$x_2(t,\alpha) = \sqrt{\frac{k_1(\alpha)}{k_2(\alpha)}}[d_1 e^{\sqrt{p(\alpha)}t} + d_2 e^{-\sqrt{p(\alpha)}t} - d_3 \sin\sqrt{p(\alpha)}t - d_4 \cos\sqrt{p(\alpha)}t]$

Where, $p(\alpha) = \sqrt{k_1(\alpha)k_2(\alpha)}$

$d_1 = \frac{1}{4}[\{(a_1 + \frac{\alpha l_{\tilde{a}}}{\eta}) + \sqrt{\frac{k_2(\alpha)}{k_1(\alpha)}}(a_4 - \frac{\alpha r_{\tilde{a}}}{\eta})\} + \frac{1}{\sqrt{p(\alpha)}}\{(b_1 + \frac{\alpha l_{\tilde{b}}}{\eta}) + \sqrt{\frac{k_2(\alpha)}{k_1(\alpha)}}(b_4 - \frac{\alpha r_{\tilde{a}}}{\eta})\}]e^{-\sqrt{p(\alpha)}t_0}$

$d_2 = \frac{1}{4}[\{(a_1 + \frac{\alpha l_{\tilde{a}}}{\eta}) + \sqrt{\frac{k_2(\alpha)}{k_1(\alpha)}}(a_4 - \frac{\alpha r_{\tilde{a}}}{\eta})\} - \frac{1}{\sqrt{p(\alpha)}}\{(b_1 + \frac{\alpha l_{\tilde{b}}}{\eta}) + \sqrt{\frac{k_2(\alpha)}{k_1(\alpha)}}(b_4 - \frac{\alpha r_{\tilde{a}}}{\eta})\}]e^{-\sqrt{p(\alpha)}t_0}$

$d_3 = \frac{1}{4}[\{(a_1 + \frac{\alpha l_{\tilde{a}}}{\eta}) - \sqrt{\frac{k_2(\alpha)}{k_1(\alpha)}}(a_4 - \frac{\alpha r_{\tilde{a}}}{\eta})\}\sin\sqrt{p(\alpha)}t_0 + \frac{1}{\sqrt{p(\alpha)}}\{(b_1 + \frac{\alpha l_{\tilde{b}}}{\eta}) - \sqrt{\frac{k_2(\alpha)}{k_1(\alpha)}}(b_4 - \frac{\alpha r_{\tilde{a}}}{\eta})\}\cos\sqrt{p(\alpha)}t_0]$

$d_4 = \frac{1}{4}[\{(a_1 + \frac{\alpha l_{\tilde{a}}}{\eta}) - \sqrt{\frac{k_2(\alpha)}{k_1(\alpha)}}(a_4 - \frac{\alpha r_{\tilde{a}}}{\eta})\}\cos\sqrt{p(\alpha)}t_0 - \frac{1}{\sqrt{p(\alpha)}}\{(b_1 + \frac{\alpha l_{\tilde{b}}}{\eta}) - \sqrt{\frac{k_2(\alpha)}{k_1(\alpha)}}(b_4 - \frac{\alpha r_{\tilde{a}}}{\eta})\}\sin\sqrt{p(\alpha)}t_0]$

Where, $\eta = min\{\omega, \lambda\}$

3.2. **Solution Procedure of 2_{nd} Order Linear Homogeneous FODE when Coefficient is Negative.** We split up this case into three subcases

Solution procedure of 2^{nd} order linear homogeneous FODE of Type-I

Consider the initial value problem

$$\frac{d^2x(t)}{dt^2} = -mx(t), k > 0 \tag{3.23}$$

With fuzzy initial condition $x(t_0) = \widetilde{a} = (a_1, a_2, a_3, a_4; \omega)$ and $\frac{dx(t_0)}{dt} = \widetilde{b} = (b_1, b_2, b_3, b_4; \omega)$

Here we solve the given problem for (i)-gH and (ii)-gH differentiability concepts respectively.

Here four cases arise

Case 3.2.1.1: when $x(t)$ and $\frac{dx(t)}{dt}$ is (i)-gH differentiable
Case 3.2.1.2: when $x(t)$ is (i)-gH differentiable and $\frac{dx(t)}{dt}$ is (ii)-gH differentiable
Case 3.2.1.3: when $x(t)$ is (ii)-gH differentiable and $\frac{dx(t)}{dt}$ is (i)-gH differentiable
Case 3.2.1.4: when $x(t)$ and $\frac{dx(t)}{dt}$ is (ii)-gH differentiable

Using the concept of Generalized Hukuhara differentiability the **Case 3.2.1.1** and **Case 3.2.1.4** are same where the **Case 3.2.1.2** and **Case 3.2.1.3** are same.

<u>Solution of Case 3.2.1.1 and Case 3.2.1.4</u>

In these case we have from (3.23) two differential equation

$$\frac{d^2x_1(t,\alpha)}{dt^2} = -mx_2(t,\alpha) \tag{3.24}$$

$$\frac{d^2x_2(t,\alpha)}{dt^3} = -mx_1(t,\alpha) \tag{3.25}$$

With initial condition $x_1(t_0,\alpha) = a_1 + \frac{\alpha l_{\widetilde{a}}}{\omega}$, $x_2(t_0,\alpha) = a_4 - \frac{\alpha r_{\widetilde{a}}}{\omega}$, $\frac{dx_1(t_0,\alpha)}{dt} = b_1 + \frac{\alpha l_{\widetilde{b}}}{\omega}$ and $\frac{dx_2(t_0,\alpha)}{dt} = b_4 - \frac{\alpha r_{\widetilde{b}}}{\omega}$

Where, $l_{\widetilde{a}} = a_2 - a_1$,$r_{\widetilde{a}} = a_4 - a_3$, $l_{\widetilde{b}} = b_2 - b_1$ and $r_{\widetilde{b}} = b_4 - b_3$.

The general solution are

$$x_1(t,\alpha) = d_1 e^{\sqrt{m}t} + d_2 e^{-\sqrt{m}t} + d_3 \sin\sqrt{m}t + d_4 \cos\sqrt{m}t \tag{3.26}$$

and

$$x_2(t,\alpha) = -d_1 e^{\sqrt{m}t} - d_2 e^{-\sqrt{m}t} + d_3 \sin\sqrt{m}t + d_4 \cos\sqrt{m}t \tag{3.27}$$

Using initial condition we have

$a_1 + \frac{\alpha l_{\tilde{a}}}{\omega} = d_1 e^{\sqrt{m}t_0} + d_2 e^{-\sqrt{m}t_0} + d_3 \sin\sqrt{m}t_0 + d_4 \cos\sqrt{m}t_0$

$\frac{1}{\sqrt{m}}(b_1 + \frac{\alpha l_{\tilde{b}}}{\omega}) = d_1 e^{\sqrt{m}t_0} - d_2 e^{-\sqrt{m}t_0} + d_3 \cos\sqrt{m}t_0 - d_4 \sin\sqrt{m}t_0$

$a_4 - \frac{\alpha r_{\tilde{a}}}{\omega} = -d_1 e^{\sqrt{m}t_0} - d_2 e^{-\sqrt{m}t_0} + d_3 \sin\sqrt{m}t_0 - d_4 \cos\sqrt{m}t_0$

$\frac{1}{\sqrt{m}}(b_4 - \frac{\alpha r_{\tilde{b}}}{\omega}) = -d_1 e^{\sqrt{m}t_0} + d_2 e^{-\sqrt{m}t_0} + d_3 \cos\sqrt{k}t_0 - d_4 \sin\sqrt{m}t_0$

Solving this above equation we get

$d_1 = \frac{1}{4}\{a_1 - a_4 + \frac{\alpha(l_{\tilde{a}}+r_{\tilde{a}})}{\omega} + \frac{1}{\sqrt{m}}(b_1 - b_4 + \frac{\alpha(l_{\tilde{b}}+r_{\tilde{b}})}{\omega})\}e^{-\sqrt{m}t_0}$

$d_2 = \frac{1}{4}\{a_1 - a_4 + \frac{\alpha(l_{\tilde{a}}+r_{\tilde{a}})}{\omega} - \frac{1}{\sqrt{m}}(b_1 - b_4 + \frac{\alpha(l_{\tilde{b}}+r_{\tilde{b}})}{\omega})\}e^{\sqrt{m}t_0}$

$d_3 = \frac{1}{2}\{a_1 + a_4 + \frac{\alpha(l_{\tilde{a}}-r_{\tilde{a}})}{\omega} \sin\sqrt{m}t_0 + \frac{1}{\sqrt{m}}(b_1 + b_4 + \frac{\alpha(l_{\tilde{b}}-r_{\tilde{b}})}{\omega}) \cos\sqrt{m}t_0\}$

$d_4 = \frac{1}{2}\{a_1 + a_4 + \frac{\alpha(l_{\tilde{a}}-r_{\tilde{a}})}{\omega} \cos\sqrt{m}t_0 - \frac{1}{\sqrt{m}}(b_1 + b_4 + \frac{\alpha(l_{\tilde{b}}-r_{\tilde{b}})}{\omega}) \sin\sqrt{m}t_0\}$

Hence the solution is (3.26) and (3.27) when the constants are the above value.

Solution of Case 3.2.1.2 and Case 3.2.1.3

In these case we have from (3.23) two differential equation

$$\frac{d^2 x_2(t,\alpha)}{dt^2} = -m x_2(t,\alpha) \tag{3.28}$$

$$\frac{d^2 x_1(t,\alpha)}{dt^3} = -m x_1(t,\alpha) \tag{3.29}$$

With initial condition $x_1(t_0,\alpha) = a_1 + \frac{\alpha l_{\tilde{a}}}{\omega}$, $x_2(t_0,\alpha) = a_4 - \frac{\alpha r_{\tilde{a}}}{\omega}$, $\frac{dx_1(t_0,\alpha)}{dt} = b_1 + \frac{\alpha l_{\tilde{b}}}{\omega}$ and $\frac{dx_2(t_0,\alpha)}{dt} = b_4 - \frac{\alpha r_{\tilde{b}}}{\omega}$

Where, $l_{\tilde{a}} = a_2 - a_1$,$r_{\tilde{a}} = a_4 - a_3$, $l_{\tilde{b}} = b_2 - b_1$ and $r_{\tilde{b}} = b_4 - b_3$.

Using the above initial conditions we get the solution as

$x_1(t,\alpha) = (a_1 + \frac{\alpha l_{\tilde{a}}}{\omega}) \cos\sqrt{m}(t - t_0) + \frac{1}{\sqrt{m}}(b_1 + \frac{\alpha l_{\tilde{b}}}{\omega}) \sin\sqrt{m}(t - t_0)$

and
$x_2(t,\alpha) = (a_4 - \frac{\alpha r_{\tilde{a}}}{\omega}) \cos\sqrt{m}(t - t_0) + \frac{1}{\sqrt{m}}(b_4 - \frac{\alpha r_{\tilde{b}}}{\omega}) \sin\sqrt{m}(t - t_0)$

3.2.2 Solution Procedure of 2^{nd} Order Linear Homogeneous FODE of Type-II

Consider the initial value problem

$$\frac{d^2x(t)}{dt^2} = -\widetilde{m}x(t), \widetilde{m} = (m_1, m_2, m_3, m_4; \lambda) > 0 \tag{3.30}$$

With fuzzy initial condition $x(t_0) = a$ and $\frac{dx(t_0)}{dt} = b$

Here we solve the given problem for (i)-gH and (ii)-gH differentiability concepts respectively.

Here four cases arise

Case 3.2.2.1: when $x(t)$ and $\frac{dx(t)}{dt}$ is (i)-gH differentiable
Case 3.2.2.2: when $x(t)$ is (i)-gH differentiable and $\frac{dx(t)}{dt}$ is (ii)-gH differentiable
Case 3.2.2.3: when $x(t)$ is (ii)-gH differentiable and $\frac{dx(t)}{dt}$ is (i)-gH differentiable
Case 3.2.2.4: when $x(t)$ and $\frac{dx(t)}{dt}$ is (ii)-gH differentiable

Using the concept of Generalized Hukuhara differentiability the **Case 3.2.2.1** and **Case 3.2.2.4** are same where the **Case 3.2.2.2** and **Case 3.2.2.3** are same.

Solution of Case 3.2.2.1 and Case 3.2.2.4

In this case we have from (3.30) two differential equations

$$\frac{d^2x_1(t,\alpha)}{dt^2} = -m_2(\alpha)x_2(t,\alpha) \tag{3.31}$$

$$\frac{d^2x_2(t,\alpha)}{dt^2} = -m_1(\alpha)x_1(t,\alpha) \tag{3.32}$$

With initial conditions $x_1(t_0,\alpha) = a$, $x_2(t_0,\alpha) = a$, $\frac{dx_1(t_0,\alpha)}{dt} = b$ and $\frac{dx_2(t_0,\alpha)}{dt} = b$
With coefficients $m_1(\alpha) = m_1 + \frac{\alpha l_{\widetilde{m}}}{\lambda}$ and $m_2(\alpha) = m_4 - \frac{\alpha r_{\widetilde{m}}}{\lambda}$

The solution is written as

$$x_1(t,\alpha) = d_1 e^{\sqrt{q(\alpha)}t} + d_2 e^{-\sqrt{q(\alpha)}t} + d_3 \sin\sqrt{q(\alpha)}t + d_4 \cos\sqrt{q(\alpha)}t \tag{3.33}$$

(3.34) $$\sqrt{\frac{m_1(\alpha)}{m_2(\alpha)}}x_2(t,\alpha) = -d_1 e^{\sqrt{q(\alpha)}t} - d_2 e^{-\sqrt{q(\alpha)}t} + d_3 \sin\sqrt{q(\alpha)}t + d_4 \cos\sqrt{q(\alpha)}t$$

(3.35)

Where, $q(\alpha) = \sqrt{m_1(\alpha)m_2(\alpha)}$

$d_1 = \frac{1}{4}(a + \frac{b}{\sqrt{q(\alpha)}})(1 - \sqrt{\frac{m_1(\alpha)}{m_2(\alpha)}})e^{-\sqrt{q(\alpha)}t_0}$

$d_2 = \frac{1}{4}(a - \frac{b}{\sqrt{q(\alpha)}})(1 - \sqrt{\frac{m_1(\alpha)}{m_2(\alpha)}})e^{\sqrt{q(\alpha)}t_0}$

$d_3 = \frac{1}{2}(a \sin\sqrt{q(\alpha)}t_0 + \frac{b\cos\sqrt{q(\alpha)}t_0}{\sqrt{q(\alpha)}})(1 + \sqrt{\frac{m_1(\alpha)}{m_2(\alpha)}})$

$d_4 = \frac{1}{2}(a \cos\sqrt{q(\alpha)}t_0 - \frac{b\sin\sqrt{q(\alpha)}t_0}{\sqrt{q(\alpha)}})(1 + \sqrt{\frac{m_1(\alpha)}{m_2(\alpha)}})$

Solution of Case 3.2.2.3 and Case 3.2.2.3

In this case we have from (3.1.2.1) two differential equations

(3.36) $$\frac{d^2x_2(t,\alpha)}{dt^2} = -m_2(\alpha)x_2(t,\alpha)$$

(3.37) $$\frac{d^2x_1(t,\alpha)}{dt^2} = -m_1(\alpha)x_1(t,\alpha)$$

With initial conditions $x_1(t_0,\alpha) = a$, $x_2(t_0,\alpha) = a$, $\frac{dx_1(t_0,\alpha)}{dt} = b$ and $\frac{dx_2(t_0,\alpha)}{dt} = b$
With coefficients $k_1(\alpha) = k_1 + \frac{\alpha l_{\tilde{k}}}{\lambda}$ and $k_2(\alpha) = k_4 - \frac{\alpha r_{\tilde{k}}}{\lambda}$

The general solution is given by

(3.38) $$x_1(t,\alpha) = c_1 \sin\sqrt{m_1(t,\alpha)}t + c_2\cos\sqrt{m_1(\alpha)}t$$

(3.39) $$x_2(t,\alpha) = c_3 \sin\sqrt{m_2(t,\alpha)}t + c_4\cos\sqrt{m_2(\alpha)}t$$

Where, $c_1 = a\sin\sqrt{m_1(\alpha)}t_0 + \frac{b\cos\sqrt{m_1(\alpha)}t_0}{\sqrt{m_1(\alpha)}}$

$c_2 = a\sin\sqrt{m_1(\alpha)}t_0 - \frac{b\cos\sqrt{m_1(\alpha)}t_0}{\sqrt{m_1(\alpha)}}$

$c_3 = a\sin\sqrt{m_2(\alpha)}t_0 + \frac{b\cos\sqrt{m_2(\alpha)}t_0}{\sqrt{m_2(\alpha)}}$

$c_4 = a\sin\sqrt{m_2(\alpha)}t_0 - \frac{b\cos\sqrt{m_2(\alpha)}t_0}{\sqrt{m_2(\alpha)}}$

3.2.3 Solution Procedure of 2_{nd} Order Linear Homogeneous FODE of Type-III

Consider the initial value problem

$$\frac{d^2x(t)}{dt^2} = -\widetilde{m}x(t), \widetilde{m} = (m_1, m_2, m_3, m_4; \lambda) > 0 \tag{3.40}$$

With fuzzy initial condition $x(t_0) = \widetilde{a} = (a_1, a_2, a_3, a_4; \omega)$ and $\frac{dx(t_0)}{dt} = \widetilde{b} = (b_1, b_2, b_3, b_4; \omega)$

Here we solve the given problem for (i)-gH and (ii)-gH differentiability concepts respectively.

Here four cases arise

Case 3.2.3.1: when $x(t)$ and $\frac{dx(t)}{dt}$ is (i)-gH differentiable
Case 3.2.3.2: when $x(t)$ is (i)-gH differentiable and $\frac{dx(t)}{dt}$ is (ii)-gH differentiable
Case 3.2.3.3: when $x(t)$ is (ii)-gH differentiable and $\frac{dx(t)}{dt}$ is (i)-gH differentiable
Case 3.2.3.4: when $x(t)$ and $\frac{dx(t)}{dt}$ is (ii)-gH differentiable

Using the concept of Generalized Hukuhara differentiability the **Case 3.2.3.1** and **Case 3.2.3.4** are same where the **Case 3.2.3.2** and **Case 3.2.3.3** are same.

Solution of Case 3.2.3.1 and Case 3.2.3.4

In this case we have from (3.40) two differential equations

$$\frac{d^2x_1(t,\alpha)}{dt^2} = -m_2(\alpha)x_2(t,\alpha) \tag{3.41}$$

$$\frac{d^2x_2(t,\alpha)}{dt^2} = -m_1(\alpha)x_1(t,\alpha) \tag{3.42}$$

With initial conditions $x_1(t_0,\alpha) = a_1 + \frac{\alpha l_{\widetilde{a}}}{\omega}$, $x_2(t_0,\alpha) = a_4 - \frac{\alpha r_{\widetilde{a}}}{\omega}$, $\frac{dx_1(t_0,\alpha)}{dt} = b_1 + \frac{\alpha l_{\widetilde{b}}}{\omega}$ and $\frac{dx_2(t_0,\alpha)}{dt} = b_4 - \frac{\alpha r_{\widetilde{a}}}{\omega}$

With coefficients $m_1(\alpha) = m_1 + \frac{\alpha l_{\widetilde{m}}}{\lambda}$ and $m_2(\alpha) = m_4 - \frac{\alpha r_{\widetilde{m}}}{\lambda}$

The solution is written as

$$x_1(t,\alpha) = e_1 e^{\sqrt{q(\alpha)}t} + e_2 e^{-\sqrt{e(\alpha)}t} + e_3 \sin\sqrt{q(\alpha)}t + e_4 \cos\sqrt{q(\alpha)}t \tag{3.43}$$

$$(3.44)\qquad x_2(t,\alpha)=\sqrt{\frac{m_1(\alpha)}{m_2(\alpha)}}[-e_1e^{\sqrt{q(\alpha)}t}-e_2e^{-\sqrt{q(\alpha)}t}+e_3\sin\sqrt{e(\alpha)}t+e_4\cos\sqrt{q(\alpha)}t]$$

Where, $q(\alpha)=\sqrt{m_1(\alpha)m_2(\alpha)}$

$e_1=\frac{1}{4}[\{(a_1+\frac{\alpha l_{\tilde{a}}}{\eta})+\sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}}(a_4-\frac{\alpha r_{\tilde{a}}}{\eta})\}+\frac{1}{\sqrt{q(\alpha)}}\{(b_1+\frac{\alpha l_{\tilde{b}}}{\eta})-\sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}}(b_4-\frac{\alpha r_{\tilde{a}}}{\eta})\}]e^{-\sqrt{q(\alpha)}t_0}$

$e_2=\frac{1}{4}[\{(a_1+\frac{\alpha l_{\tilde{a}}}{\eta})+\sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}}(a_4-\frac{\alpha r_{\tilde{a}}}{\eta})\}-\frac{1}{\sqrt{q(\alpha)}}\{(b_1+\frac{\alpha l_{\tilde{b}}}{\eta})-\sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}}(b_4-\frac{\alpha r_{\tilde{a}}}{\eta})\}]e^{-\sqrt{q(\alpha)}t_0}$

$e_3=\frac{1}{2}[\{(a_1+\frac{\alpha l_{\tilde{a}}}{\eta})+\sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}}(a_4-\frac{\alpha r_{\tilde{a}}}{\eta})\}\sin\sqrt{q(\alpha)}t_0+\frac{1}{\sqrt{q(\alpha)}}\{(b_1+\frac{\alpha l_{\tilde{b}}}{\eta})+\sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}}(b_4-\frac{\alpha r_{\tilde{a}}}{\eta})\}\cos\sqrt{q(\alpha)}t_0]$

$e_4=\frac{1}{2}[\{(a_1+\frac{\alpha l_{\tilde{a}}}{\eta})+\sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}}(a_4-\frac{\alpha r_{\tilde{a}}}{\eta})\}\cos\sqrt{q(\alpha)}t_0-\frac{1}{\sqrt{q(\alpha)}}\{(b_1+\frac{\alpha l_{\tilde{b}}}{\eta})+\sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}}(b_4-\frac{\alpha r_{\tilde{a}}}{\eta})\}\sin\sqrt{q(\alpha)}t_0]$

Where, $\eta=min\{\omega,\lambda\}$

Solution of Case 3.2.3.2 and Case 3.2.3.3

In this case we have from (3.40) two differential equations

$$(3.45)\qquad \frac{d^2x_2(t,\alpha)}{dt^2}=-m_2(\alpha)x_2(t,\alpha)$$

$$(3.46)\qquad \frac{d^2x_1(t,\alpha)}{dt^2}=-m_1(\alpha)x_1(t,\alpha)$$

With initial conditions $x_1(t_0,\alpha)=a_1+\frac{\alpha l_{\tilde{a}}}{\omega}$, $x_2(t_0,\alpha)=a_4-\frac{\alpha r_{\tilde{a}}}{\omega}$, $\frac{dx_1(t_0,\alpha)}{dt}=b_1+\frac{\alpha l_{\tilde{b}}}{\omega}$ and $\frac{dx_2(t_0,\alpha)}{dt}=b_4-\frac{\alpha r_{\tilde{a}}}{\omega}$

With coefficients $m_1(\alpha)=m_1+\frac{\alpha l_{\widetilde{m}}}{\lambda}$ and $m_2(\alpha)=m_4-\frac{\alpha r_{\widetilde{m}}}{\lambda}$

The solution is written as

$$(3.47)\qquad x_1(t,\alpha)=c_1\sin\sqrt{m_1(\alpha)}t+c_2\cos\sqrt{m_1(\alpha)}t$$

and

$$(3.48)\qquad x_2(t,\alpha)=c_3\sin\sqrt{m_2(\alpha)}t+c_4\cos\sqrt{m_2(\alpha)}t$$

Where, $c_1 = (a_1 + \frac{\alpha l_{\tilde{a}}}{\eta}) \sin \sqrt{m_1(t,\alpha)} t_0 + \frac{1}{\sqrt{m_1(\alpha)}} (b_1 + \frac{\alpha l_{\tilde{b}}}{\eta}) \cos \sqrt{m_1(\alpha)} t_0$

$c_2 = (a_1 + \frac{\alpha l_{\tilde{a}}}{\eta}) \cos \sqrt{m_1(t,\alpha)} t_0 + \frac{1}{\sqrt{m_1(\alpha)}} (b_1 - \frac{\alpha l_{\tilde{b}}}{\eta}) \sin \sqrt{m_1(\alpha)} t_0$

$c_3 = (a_4 - \frac{\alpha r_{\tilde{a}}}{\eta}) \sin \sqrt{m_2(t,\alpha)} t_0 + \frac{1}{\sqrt{m_2(\alpha)}} (b_4 - \frac{\alpha r_{\tilde{a}}}{\eta}) \cos \sqrt{m_2(\alpha)} t_0$

$c_3 = (a_4 - \frac{\alpha r_{\tilde{a}}}{\eta}) \cos \sqrt{m_2(t,\alpha)} t_0 - \frac{1}{\sqrt{m_2(\alpha)}} (b_4 - \frac{\alpha r_{\tilde{a}}}{\eta}) \sin \sqrt{m_2(\alpha)} t_0$

$m_1(\alpha) = m_1 + \frac{\alpha l_{\widetilde{m}}}{\eta}$ and $m_2(\alpha) = m_4 - \frac{\alpha r_{\widetilde{m}}}{\eta}$

Where $\eta = min\{\omega, \lambda\}$

4. Numerical example

Example 4.1: Solve $\frac{d^2x(t)}{dt^2} = x(t)$ with $x(0.5) = \widetilde{1}, x^{'}(0.5) = 0$, where $\widetilde{1} = (0.8, 1, 1.3, 1.6; 1)$

Solution: When $x(t)$ and $\frac{dx(t)}{dt}$ are both (i)-gH or, (ii)-gH differentiable the solution is

$x_1(t,\alpha) = \frac{1}{2}(0.8 + 0.2\alpha)(e^{t-0.5} + e^{-t+0.5})$

$x_2(t,\alpha) = \frac{1}{2}(1.6 - 0.3\alpha)(e^{t-0.5} - e^{-t+0.5})$

Table 1. Solution for $t = 2$

α	$x_1(t,\alpha;0.7)$	$x_2(t,\alpha;0.7)$
0	1.8819	3.4068
0.1	1.9290	3.3430
0.2	1.9760	3.2791
0.3	2.0231	3.2152
0.4	2.0701	3.1513
0.5	2.1172	3.0875
0.6	2.1642	3.0236
0.7	2.2113	2.9597
0.8	2.2583	2.8958
0.9	2.3054	2.8319
1	2.3524	2.7681

Remarks: From the above table and graph we conclude that $x_1(t,\alpha)$ is increasing and $x_2(t,\alpha)$ decreasing function. Hence we conclude that the solution is a strong solution.

Example 4.2: Solve $\frac{d^2x(t)}{dt^2} = -4x(t)$ with $x(0) = \widetilde{1}, x^{'}(0) = 0$, where $\widetilde{1} = (0.8, 1, 1.3, 1.5; 0.8)$

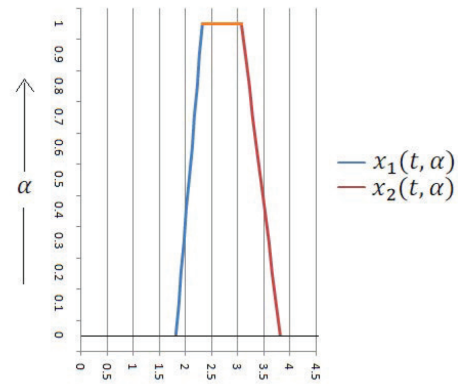

FIGURE 3. Graph of solutions for t=2

Solution: When $x(t)$ and $\frac{dx(t)}{dt}$ are both (i)-gH or, (ii)-gH differentiable the solution is

$$x_1(t,\alpha) = \frac{1}{4}(-0.7 + 0.5\alpha)(e^{2t} + e^{-2t}) + 1.15\cos 2t$$

$$x_2(t,\alpha) = -\frac{1}{4}(-0.7 + 0.5\alpha)(e^{2t} + e^{-2t}) + 1.15\cos 2t$$

TABLE 2. Solution for $t = 2$

α	$x_1(t,\alpha;0.7)$	$x_2(t,\alpha;0.7)$
0	0.6097	1.6899
0.1	0.6483	1.6513
0.2	0.6869	1.6127
0.3	0.7255	1.5742
0.4	0.7641	1.5356
0.5	0.8026	1.4970
0.6	0.8412	1.4584
0.7	0.8798	1.4199
0.8	0.9184	1.3813

Remarks: From the above table and graph we conclude that $x_1(t,\alpha)$ is increasing and $x_2(t,\alpha)$ decreasing function. Hence we conclude that the solution is a strong solution.

When $x(t)$ is (i)-gH and $\frac{dx(t)}{dt}$ is (ii)-gH or, $x(t)$ is (ii)-gH and $\frac{dx(t)}{dt}$ is (i)-gH differentiable the the solution is

$$x_1(t,\alpha) = (0.8 + 0.25\alpha)\cos 2t \text{ and } x_2(t,\alpha) = (1.3 - 0.25\alpha)\cos 2t$$

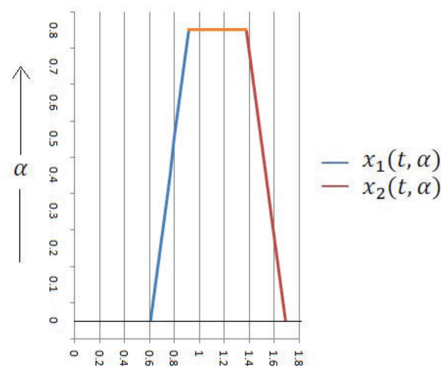

FIGURE 4. Graph of solutions for t=2

TABLE 3. Solution for $t = 2$

α	$x_1(t,\alpha;0.7)$	$x_2(t,\alpha;0.7)$
0	0.7999	1.2998
0.1	0.8249	1.2748
0.2	0.8499	1.2498
0.3	0.8749	1.2248
0.4	0.8999	1.1998
0.5	0.9249	1.1748
0.6	0.9499	1.1498
0.7	0.9749	1.1248
0.8	0.9998	1.0998

Remarks: From the above table and graph we conclude that $x_1(t,\alpha)$ is increasing and $x_2(t,\alpha)$ decreasing function. Hence we conclude that the solution is a strong solution.

5. APPLICATION

An 8-lb weight is placed upon the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position, thereby stretching the spring $\frac{1}{2}$ft. The weight is then pulled down about $\widetilde{\frac{1}{4}}$ft [i.e.,$(\frac{1}{8},\frac{1}{4},\frac{1}{2},\frac{3}{4};1)$ ft] bellow its equilibrium position and released at t=0 with an initial velocity 1 ft/sec [i.e.,$(\frac{1}{2},1,2,\frac{5}{2})$ ft/sec], directed downward. Neglecting the resistance of the medium and assuming that no external forces are present, solve the model.

Solution: By Hooks law $F = KS$, which gives $8 = K\frac{1}{2}$ and so $K = 16$ lb/ft. Also $m = \frac{m}{g} = \frac{8}{32}$

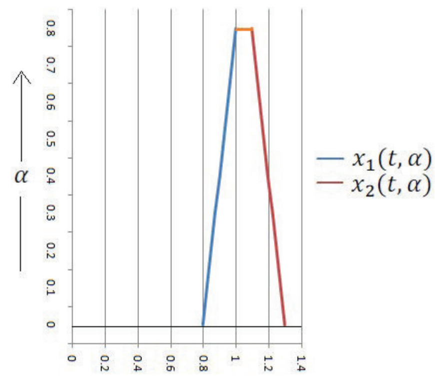

FIGURE 5. Graph of solutions for t=2

The differential equation is

$\frac{d^2x(t)}{dt^2} = -64x(t)$, $x(0) = (\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1)$ and $x^{'}(0) = (\frac{1}{2}, 1, 2, \frac{5}{2})$

The solution when $x(t)$ is (i)-gH differentiable and $\frac{dx(t)}{dt}$ is (ii)-gH differentiable or, $x(t)$ is (ii)-gH differentiable and $\frac{dx(t)}{dt}$ is (i)-gH differentiable then from (5.1) we have

$$\frac{d^2x_2(t,\alpha)}{dt^2} = -64x_2(t,\alpha) \tag{5.1}$$

and

$$\frac{d^2x_1(t,\alpha)}{dt^2} = -64x_1(t,\alpha) \tag{5.2}$$

With, $x_1(0,\alpha) = \frac{1+\alpha}{8}$ and $x_2(0,\alpha) = \frac{3-\alpha}{4}$, $\frac{dx_1(0,\alpha)}{dt} = \frac{1+\alpha}{2}$ and $\frac{dx_2(0,\alpha)}{dt} = \frac{5-\alpha}{2}$

Solving we get
$x_1(t,\alpha) = \frac{1+\alpha}{32}\cos 8t + \frac{3(1+\alpha)}{32}\sin 8t$

and
$x_2(t,\alpha) = \frac{7-3\alpha}{32}\cos 8t + \frac{17-5\alpha}{32}\sin 8t$

Remarks: From the above table and graph we conclude that $x_1(t,\alpha)$ is increasing and $x_2(t,\alpha)$ decreasing function. Hence we conclude that the solution is a strong solution.

Table 4. Solution for $t = 0.05$

α	$x_1(t, \alpha; 0.7)$	$x_2(t, \alpha; 0.7)$
0	0.0653	0.4084
0.1	0.0718	0.3936
0.2	0.0783	0.3789
0.3	0.0849	0.3642
0.4	0.0914	0.3495
0.5	0.0979	0.3348
0.6	0.1045	0.3200
0.7	0.1110	0.3053
0.8	0.1175	0.2906
0.9	0.1241	0.2759
1	0.1306	0.2612

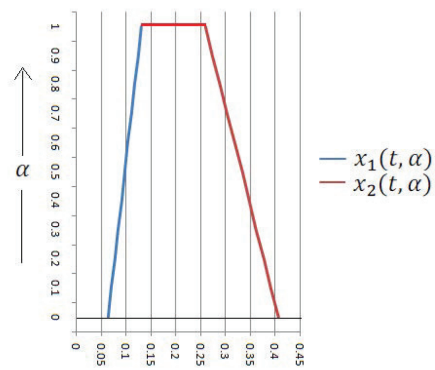

Figure 6. Graph of solutions for t=0.05

6. Conclusion

In this paper the solution of a second order differential equation in fuzzy environments are described. It is discussed for three different cases: Only initial condition, coefficient and both initial condition and coefficient are taken as fuzzy numbers. Here fuzzy numbers are taken as Generalized Trapezoidal Fuzzy Numbers (GTrFNs). The result are very useful in the field of fuzzy differential equation in theoretical and applied sense. In future research we solve n-th order differential equation with different type fuzzy environments and applied the results in different field like economics, engineering and sciences.

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References

[1] A. Armand and Z. Gouyandeh, Solving two-point fuzzy boundary value problem using variational iteration method, Communications on Advanced Computational Science with Applications. 2013 (2013) 1–10.

[2] L. Ahmad, M. Farooq, S. Ullah and S. Abdullah, Solving fuzzy two-point boundary value problem using fuzzy Laplace tansform (Archive).

[3] B. Bede and S. G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, Fuzzy Sets and Systems 151 (2005) 581-599.

[4] B. Bede, A note on two-point boundary value problems associated with non-linear fuzzy differential equations, Fuzzy Sets and Systems 157 (2006) 986–989.

[5] B. Bede, I. J. Rudas and A. L. B Encsik, First order linear fuzzy differential equations under generalized differentiability, Inform. Sci. 177 (2007) 1648–1662.

[6] B. Bede and L. Stefanini, Generalized differentiability of fuzzy-valued functions, Fuzzy Sets and Systems 230 (2013) 119– 141.

[7] L. C. Barros, L. T. Gomes and P. A Tonelli, Fuzzy differential equations: An approach via fuzzification of the derivative operator, Fuzzy Sets and Systems 230 (2013) 39–52.

[8] Y. Chalco-Cano and H. Roman-Flores,O n the new solution of fuzzy differential equations, Chaos Solitons Fractals 38 (2008) 112–119.

[9] Y. Chalco-Cano, M. A. Rojas-Medar and H. Roman-Flores, Sobre ecuaciones diferencial esdifusas, Bol. Soc. Esp. Mat. Apl. 41 (2007) 91–99.

[10] Y. Chalco-Cano ,H. Roman-Flores and M. A. Rojas-Medar, Fuzzy differential equations with generalized derivative, in:Proceedings of the 27th North American Fuzzy Information Processing Society International Conference,IEEE. 2008.

[11] S. L. Chang and L. A. Zadeh, On fuzzy mapping and control, IEEE Transaction on Systems Man Cybernetics 2 (1972) 30–34.

[12] D. Dubois and H. Prade, Towards fuzzy differential calculus: Part 3, Differentiation, Fuzzy Sets and Systems 8 (1982) 225–233.

[13] P. Diamond and P. Kloeden, Metric Spaces of Fuzzy Sets, World Scientific, Singapore 1994.

[14] R. Goetschel and W. Voxman, Elementary fuzzy calculus, Fuzzy Sets and Systems 18 (1986) 31–43.

[15] N. Gasilov, S. E. Amrahov, A. G. Fatullayev and A. Khastan, A new approach to fuzzy initial value problem 18 (2) (2014) 217–225.

[16] N. Gasilov, S. E. Amrahov and A. G. Fatullayev, Solution of linear differential equation with fuzzy boundary values, Fuzzy Sets and System 257 (2014) 169–183.

[17] X. Guo, D. Shang and X. Lu, Fuzzy approximate solutions of second-order fuzzy linear boundary value Problems, Boundary Value Problems 2013 (2013) 1–17.

[18] L. Jamshidi and L. Avazpour, Solution of the Fuzzy Boundary Value Differential Equations Under Generalized Differentiability By Shooting Method. 2012 (2012) 1–19.

[19] A. F. Jameel , Ahmad Izani Md Ismail,Approximate Solution of Nth Order Two Point Fuzzy Boundary Value Problems by Optimal Homotopy Asymptotic Method, International Journal of Modern Mathematical Sciences 6 (2) (2013) 107–120.

[20] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems 24 (1987) 301–317.

[21] A. Khastan and J. J. Nieto, A boundary value problem for second-order fuzzy differential equations, Nonlinear Analysis 72 (2010) 3583–3593.

[22] H. K. Liu, Comparison results of two-point fuzzy boundary value problems, World Academy of Science, Engineering and Technology 51 (2011) 463–469.

[23] M. L. Puri and D. A. Ralescu, Differentials of fuzzy functions, J. Math. Anal. Appl. 91 (1983) 552–558.

[24] S. J. Ramazannia Tolouti and M. Barkhordari Ahmadi, Fuzzy laplace transform on two order derivative and solving fuzzy two order differential equation, Int. J. Industrial Mathematics 2 (4) (2010) 279–293.

[25] F. Rabiei, F. Ismail, A. Ahmadian and S. Salahshour, Numerical Solution of Second-Order Fuzzy Differential Equation Using Improved Runge-Kutta Nystrom Method, Mathematical Problems in Engineering 2013 (2013) 1–10.

[26] L. Stefanini, A generalization of Hukuhara difference for interval and fuzzy arithmetic, in: D. Dubois, M.A. Lubiano, H. Prade, M. A. Gil, P. Grzegorzewski, O. Hryniewicz (Eds.), Soft Methods for Handling Variability and Imprecision, in: Series on Advances in Soft Computing 48 (2008).

[27] L. Stefanini and B. Bede, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, Nonlinear Analysis 71 (2009) 1311–1328.

[28] L. Wang and S. Guo, Adomian method for second-order fuzzy differential Equation, World Academy of Science, Engineering and Technology 76 (2011) 613–616.

[29] L. Zadeh, Toward a generalized theory of uncertainty (GTU) an outline, Inform. Sci. 172 (2005) 1–40.

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