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Fuzzy metric space and generating space of quasi-metric family

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Abstract. The skeleton of this work consists of a relation between fuzzy metric space and generating space of quasi-metric family (GSQMF). Here we have attempted to establish two decomposition theorems. In the first theorem, we deduce GSQMF from a fuzzy metric space. In the second theorem, from a GSQMF, fuzzy metric space is derived. Lastly we try to show that under certain conditions these two fuzzy metrics are similar.

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1. Introduction

L.A. Zadeh [15] first introduced an idea of fuzzy set in 1965. After that, in 1975, Kramosil and Michalek [7] have presented a concept of fuzzy metric space which is very similar to that of generalized Menger space [4]. In 1984, Kaleva and Seikkala [5] introduced a concept of fuzzy metric space which generalizes the notion of a metric space by setting the distance between two points by a nonnegative fuzzy number proving some fixed point theorems. Many authors [1, 2, 8, 9, 10, 13] have developed fuzzy metric space theory in different ways. Chang et al.[3] gave a definition of generating space of quasi-metric family which is a most generalized structure unifying those of fuzzy metric space in the sense of Kaleva & Seikkala [5] and Menger probabilistic metric spaces [14]. They [3, 12] also established several fixed point theorems and minimization theorems in complete generating space of quasi-metric family. In this paper, we have attempted to establish two decomposition theorems. In the first theorem, we deduce GSQMF from a fuzzy metric space and in the second theorem, from a GSQMF we deduce fuzzy metric space.

The organization of the paper is as follows:

A brief introduction of the work is given in section 1. Section 2 comprises some preliminary results. GSQMF from fuzzy metric space is deduced in section 3. A fuzzy metric space is derived from a GSQMF in the spectrum of the section 4. In section 5, it has been proved that under certain conditions two fuzzy metrics are identical. A brief conclusion of this manuscript is given in section 6. Throughout this paper straightforward proofs are omitted.

2. Preliminaries

In this section some preliminary results are given which will be used in this paper.

Definition 2.1 ([6])**.** A binary operation $* : [0,\ 1] \times [0,\ 1] \to\ [0,\ 1]$ is called t-norm if the following axioms are satisfied for all $a,\ b,\ d \in [0,\ 1]$:
(T1) $a * 1 = a$ (boundary condition),
(T2) $b \leq\ d$ implies $a * b \leq\ a * d$ (monotonicity),
(T3) $a * b = b * a$ (commutativity),
(T4) $a * (b * d) = (a * b) * d$ (associativity).

Definition 2.2 ([6])**.** $*$ is said to be continuous if for any sequences $\{a_n\},\ \{b_n\}$ in $[0,\ 1]$ with $\lim\limits_{n\to\infty}\ a_n = a$ and $\lim\limits_{n\to\infty}\ b_n = b$ implies $\lim\limits_{n\to\infty}\ (a_n * b_n) = (a * b)$.

Definition 2.3 ([9])**.** The 3-tuple $(X,\ M,\ *)$ is said to be a fuzzy metric space if X is a nonempty arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions :
(M1) $M(x,\ y,\ 0) = 0$,
(M2) $M(x,\ y,\ t) = 1 \quad \forall\ \ t > 0 \quad$ iff $\quad x = y$,
(M3) $M(x,\ y,\ t) = M(y,\ x,\ t) \quad \forall x,\ y \in X,\ \forall\, t \in\ [0, \infty)$,
(M4) $M(x,\ y,\ t) * M(y,\ z,\ s) \leq M(x,\ z,\ t + s)$,
(M5) $\lim\limits_{t\to\infty} M(x,\ y,\ t) = 1$.

Definition 2.4 ([11])**.** Let X be a nonempty set and $\{d_\alpha :\ \ \alpha \in\ \ (0,\ 1)\}$ be a family of mappings from $X \times\ X$ into $[0,\ \infty)$. Then $(X,\ d_\alpha :\ \alpha \in\ (0,\ 1))$ is called a generating space of quasi-metric family if it satisfies the following conditions :
(QM1) $d_\alpha(x,\ y) = 0 \quad \forall \alpha \in (0,\ 1)$ iff $x = y$,
(QM2) $d_\alpha(x,\ y) = d_\alpha(y,\ x) \quad \forall x,\ y \in\ X$ and $\forall \alpha \in (0,\ 1)$,
(QM3) for any $\alpha \in (0,\ 1)$ there exists a $\beta \in (0,\ \alpha]$ such that
$d_\alpha(x,\ y) \leq\ d_\beta(x,\ z)\ + d_\beta(z,\ y) \quad$ for all $x,\ y,\ z \in X$,
(QM4) for any $x,\ y \in\ X,\ \ d_\alpha(x,\ y)$ is non-increasing in α.

Definition 2.5 ([11])**.** Let $(X,\ d_\alpha\ :\ \ \alpha\ \in\ \ (0,1))$ be a generating space of quasi-metric family, then it is called a generating space of sub-strong quasi-metric family, strong quasi-metric family and semi-metric family respectively, if (QM3) is strengthened to (QM3u), (QM3t) and (QM3e), where
(QN3u) for any $\alpha \in (0,\ 1)$ there exists $\beta \in (0,\ \alpha]$ such that

$$d_\alpha(x_m,\ x_{m+p}) \leq \sum_{i=0}^{p-1} d_\beta(x_{m+i},\ x_{m+i+1})$$

for any $p \in Z^+$ and $x_{m+i} \in X (i = 1, 2, ..., p - 1)$,
(QM3t) for any $\alpha \in (0,\ 1)$ there exists a $\beta \in (0, \alpha]$ such that

$$d_\alpha(x,\ z) \leq\ d_\alpha(x,\ y) + d_\beta(y,\ z)$$

for $x,\ y,\ z \in X$,

(QM3e) for any $\alpha \in (0,\ 1)$, it holds that

$$d_\alpha(x,\ z) \leq d_\alpha(x,\ y) + d_\alpha(y,\ z)$$

for $x,\ y,\ z \in X$.

Definition 2.6 ([11]). Let $(X,\ d_\alpha :\ \alpha \in\ (0,\ 1))$ be a generating space of semi-metric family, where $(d_\alpha :\ \alpha \in\ (0,\ 1))$ satisfies the following additional condition : If $x \neq\ y$ in X, then $d_\alpha(x,\ y)\ > 0\ \forall \alpha \in (0,\ 1)$.
Then $(X,\ d_\alpha\ :\ \ \alpha \in\ \ (0,\ 1))$ is called a generating space of metric family and $(d_\alpha :\ \alpha \in\ (0,\ 1))$ is called a metric family on X.

3. Decomposition theorem and examples

In this section, we deduce GSQMF from fuzzy metric space.

Theorem 3.1. *Let $(X,\ M,\ *)$ be a fuzzy metric space.*
For $\alpha \in (0,\ 1)$ we define

$$d_\alpha(x,\ y) = \bigwedge\{t > 0\ :\ M(x,\ y,\ t) \geq\ (1-\alpha)\}.$$

Then $\{d_\alpha : \alpha \in (0,\ 1)\}$ is a quasi-metric family on X and $(X,\ d_\alpha :\ \alpha \in\ (0,\ 1))$ is a generating space of quasi-metric family.

Proof. (QM1) Let $x = y$. Then $M(x,\ y,\ t) = 1,\ \forall t > 0$. Thus $d_\alpha(x,\ y) = 0\ \ \forall \alpha \in (0,\ 1)$. Conversely if $d_\alpha(x,\ y) = 0\ \ \forall \alpha \in (0,\ 1)$, then $M(x,\ y,\ t) \geq\ (1-\alpha)\ \ \forall t > 0\ \forall \alpha \in (0,\ 1)$. Thus $M(x,\ y,\ t) = 1\ \forall t > 0$. So $x = y$.

(QM2) Since M satisfies (M3), (QM2) holds from definition.

(QM3) Since $*$ is continuous, for any $\alpha \in (0,\ 1)$ there exists a $\beta \in (0,\ \alpha]$ such that $(1-\beta) * (1-\beta) = (1-\alpha)$.
Now

$$\begin{aligned} & d_\beta(x,\ y)\ +\ d_\beta(y,\ z) \\ & = \bigwedge\{t > 0\ :\ M(x,\ y,\ t) \geq\ (1-\beta)\} + \bigwedge\{s > 0\ :\ M(y,\ z,\ s) \geq\ (1-\beta)\} \\ & \geq \bigwedge\{t + s > 0\ :\ M(x,\ y,\ t) \geq\ (1-\beta),\ M(y,\ z,\ s) \geq\ (1-\beta)\}. \end{aligned}$$

On the other hand, $M(x,\ y,\ t) \geq\ (1-\beta),\ M(y,\ z,\ s) \geq\ (1-\beta)$. Thus

$$M(x,\ z,\ t+s) \geq\ M(x,\ y,\ t) * M(y,\ z,\ s) \geq\ (1-\beta) * (1-\beta) = (1-\alpha).$$

So $d_\beta(x,\ y)\ +\ d_\beta(y,\ z)\ \geq \bigwedge\{t + s > 0\ :\ M(x+z,\ t+s) \geq (1-\alpha)\} = d_\alpha(x,\ z)$.

(QM4) Clearly $d_\alpha(x,\ y)$ is non-increasing for $\alpha \in (0,\ 1)$ from definition. Hence $(X,\ d_\alpha :\ \alpha \in\ (0,\ 1))$ is a generating space of quasi-metric family. □

Note 3.2. *If $*$ satisfies the condition given by :*
(T5) $\forall\ \alpha,\ \beta \in (0,\ 1),\ \alpha * \beta\ > 0$,
then, from Theorem 3.1, it is clear that

$$d_{(1-(1-\alpha)*(1-\beta))}(x,\ z) \leq\ d_\alpha(x,\ y)\ +\ d_\beta(y,\ z)\ \ \forall\ x,\ y,\ z \in\ X,\ \forall\ \alpha,\ \beta \in (0,\ 1).$$

Note 3.3. *In Theorem 3.1, if we take the continuous t-norm $*$ defined by $a*b = min\{a,\ b\}\ \ \forall\ a,\ b \in\ [0,\ 1]$ then $(X,\ d_\alpha:\ \alpha \in\ (0,\ 1))$ is a generating space of semi-metric family.*

Theorem 3.4. *Let $(X,\ M,\ min)$ be a fuzzy metric space.*
For $\alpha \in (0,\ 1)$ we define

$$d_\alpha(x,\ y) = \bigwedge\{t > 0\ :\ M(x,\ y,\ t) \geq\ (1-\alpha)\}.$$

Then $(X,\ d_\alpha:\ \alpha \in\ (0,\ 1))$ is a generating space of metric family iff $M(x,\ y,\ t)$ is continuous at $t = 0$ for all $x,\ y(\neq\ x) \in\ X$.

Proof. From Theorem 3.1, $(X,\ d_\alpha:\ \alpha \in\ (0,\ 1))$ is a generating space of semi-metric family. For complete the proof, we need to show that
$d_\alpha(x,\ y)\ > 0\ \forall\alpha \in (0,\ 1)$ and for all $x,\ y(\neq\ x)$ in X.

If possible let $\exists\ x,\ y(\neq\ x)\ \in\ X$ such that $d_{\alpha_0}(x,\ y) =\ 0$ for some $\alpha_0 \in (0,\ 1)$. Then $M(x,\ y,\ t) \geq\ (1-\alpha_0)\ \forall\ t > 0$. But $M(x,\ y,\ 0) = 0$, which contradicts the fact that $M(x,\ y,\ t)$ is continuous at $t = 0$ for all $x,\ y(\neq\ x) \in\ X$.

Conversely suppose M does not satisfy the condition $(M6)$. Then $\exists\ x,\ y(\neq\ x) \in\ X$ for which $N(x,\ y,\ .)$ is not continuous at $t = 0$.
i.e. $\exists\ \alpha_0 \in\ (0,\ 1)$ such that $M(x,\ y,\ t) \geq\ (1-\alpha_0)\ \forall\ t > 0$. Thus $d_{\alpha_0}(x,\ y) = 0$.
Hence the is complete. □

Example 3.5. Let $X = R^2$. For $x = (x_1,\ x_2),\ y = (y_1,\ y_2) \in\ X$ and $t \in\ [0,\ \infty)$ define

$$M(x,\ y,\ t) = \begin{cases} \frac{t^2}{(t+|x_1-y_1|)(t+|x_2-y_2|)} & for\ \ t > 0 \\ 0\ \ \ for\ \ t = 0 \end{cases}$$

Then $(X,\ M,\ *)$ is a fuzzy metric space for the continuous t-norm $*$ defined by $a*b = a.b\ \ \forall\ a,\ b \in\ [0,\ 1]$.

Proof. (M1) $M(x,\ y,\ 0) = 0$ (from definition).

(M2) $\forall\ t > 0,\ \ M(x,\ y,\ t) = 1$
$\Rightarrow\ \frac{t^2}{(t+|x_1-y_1|)(t+|x_2-y_2|)} = 1$
$\Rightarrow\ t^2 = t^2 + t(|x_1-y_1| + |x_2-y_2|) + |x_1-y_1||x_2-y_2|$
$\Rightarrow\ t(|x_1-y_1| + |x_2-y_2|) + |x_1-y_1||x_2-y_2| = 0\ \ \forall\ t > 0$
$\Rightarrow\ |x_1-y_1| = 0$ and $|x_2-y_2| = 0$.
i.e. $x_1 = y_1$ and $x_2 = y_2\ \ \Rightarrow\ x = y$.
Conversely if $x = y$ then $M(x,\ y,\ t) = 1\ \ \forall t >\ 0$ (from definition).

(M3) Follows from definition.

(M4)

$$\begin{aligned} M(x,\ z,\ t+s) &= \frac{(t+s)^2}{(t+s+|x_1-z_1|)(t+s+|x_2-z_2|)} \\ &\geq\ \frac{(t+s)^2}{((t+s)+|x_1-y_1|+|y_1-z_1|)((t+s)+|x_2-y_2|+|y_2-z_2|)} \end{aligned}$$

and

$$M(x,\ y,\ t)*M(y,\ z,\ s) = \frac{t^2s^2}{(t+|x_1-y_1|)(t+|x_2-y_2|)(s+|y_1-z_1|)(s+|y_2-z_2|)},$$

and it is not difficult to verify that

$$\begin{aligned}&(t+s)^2(t+|x_1-y_1|)(t+|x_2-y_2|)(s+|y_1-z_1|)(s+|y_2-z_2|)\\ &\geq\ t^2s^2((t+s)+|x_1-y_1|+|y_1-z_1|)((t+s)+|x_2-y_2|+|y_2-z_2|).\end{aligned}$$

So M(4) holds.

(M5) $\lim\limits_{t\to\infty} M(x,\ y,\ t)=1.$

Hence $(X,\ M,\ *)$ is a fuzzy metric space.

In the above example, if we take $*='min'$, then $(X,\ M,\ *)$ is not a fuzzy metric space as illustrated below :

Let $x=(0,\ 0),\ y=(0,\ 1),\ z=(1,\ 1)$ and $t=s=1$.

Then $M(x,\ y,\ t)=\frac{1}{2}$, $\quad M(y,\ z,\ s)=\frac{1}{2}$ and $M(x,\ z,\ t+s)=\frac{4}{9}$.

So M does not satisfies (M4) for $*='min'$.

In this example define

$$d_\alpha(x,\ y)=\ \bigwedge\{t>0:\ M(x,\ y,\ t)\geq\ (1-\alpha)\}.$$

Then $(X,\ d_\alpha:\ \ \alpha\in\ \ (0,\ 1))$ is a generating space of quasi-metric family. but not a generating space of semi-metric family although $M(x,\ y,\ .)$ is continuous at $t=0,\ \forall\ x,\ y(\neq\ x)\in\ X$.

Solution:

$$M(x,\ y,\ t)=\begin{cases}\frac{t^2}{(t+|x_1-y_1|)(t+|x_2-y_2|)} & for\ \ t>0\\ 0\quad for\ \ t=0\end{cases}$$

Now, $M(x,\ y,\ t)\geq\ (1-\alpha)$

$\Rightarrow\ \frac{t^2}{(t+|x_1-y_1|)(t+|x_2-y_2|)}\geq\ (1-\alpha)$

$\Rightarrow\ t^2\geq(1-\alpha)t^2+t(1-\alpha)(|x_1-y_1|+|x_2-y_2|)+(1-\alpha)|x_1-y_1||x_2-y_2|$

$\Rightarrow\ \alpha\ t^2-t(1-\alpha)(|x_1-y_1|+|x_2-y_2|)-(1-\alpha)|x_1-y_1||x_2-y_2|\geq\ 0$

$\Rightarrow\ \alpha\ (t-a)(t-b)\geq\ 0,$

where

$$a(\ =\frac{(1-\alpha)(|x_1-y_1|+|x_2-y_2|)+\sqrt{(1-\alpha)^2(|x_1-y_1|+|x_2-y_2|)^2+4\alpha(1-\alpha)|x_1-y_1||x_2-y_2|}}{2\alpha})$$

and

$$b(=\frac{(1-\alpha)(|x_1-y_1|+|x_2-y_2|)-\sqrt{(1-\alpha)^2(|x_1-y_1|+|x_2-y_2|)^2+4\alpha(1-\alpha)|x_1-y_1||x_2-y_2|}}{2\alpha})$$

are the roots of the equation

$$\alpha\ t^2-t(1-\alpha)(|x_1-y_1|+|x_2-y_2|)-(1-\alpha)|x_1-y_1||x_2-y_2|\ =0.$$

By Descartes's rule of sign, this equation has only one positive real root a.

So, $\forall\alpha\in(0,\ 1)$,

$$\begin{aligned}d_\alpha(x,\ y)=&\ \bigwedge\{t>0:\ M(x,\ y,\ t)\geq(1-\alpha)\}\\ &=\frac{1}{2\alpha}[(1-\alpha)(|x_1-y_1|+|x_2-y_2|)\\ &+\sqrt{(1-\alpha)^2(|x_1-y_1|+|x_2-y_2|)^2+4\alpha(1-\alpha)|x_1-y_1||x_2-y_2|}].\end{aligned}$$

By Theorem 3.1, $\{d_\alpha : \alpha \in (0,\ 1)\}$ is a quasi-metric family on X and $(X,\ d_\alpha :\ \alpha \in\ (0,\ 1))$ is a generating space of quasi-metric family.

Next we shall show $\{d_\alpha : \alpha \in (0,\ 1)\}$ is not a semi-metric family.
Take $x = (0,\ 0),\ y = (1,\ 0)$ and $z = (1,\ 1)$.
Then, $\forall \alpha \in (0,\ 1)$,

$$d_\alpha(x,\ y) = \tfrac{(1-\alpha)}{\alpha}\ ,\quad d_\alpha(y,\ z) = \tfrac{(1-\alpha)}{\alpha}$$

and

$$d_\alpha(x,\ z) = \tfrac{(1-\alpha)+\sqrt{(1-\alpha)(1+3\alpha)}}{\alpha} \text{ for } \alpha \in (0,\ 1).$$

Now

$$\begin{aligned} d_\alpha(x,\ y) + d_\alpha(y,\ z) &= \frac{2(1-\alpha)}{\alpha} \\ &= \frac{(1-\alpha)+(1-\alpha)}{\alpha} < \frac{(1-\alpha)+\sqrt{(1-\alpha)(1+3\alpha)}}{\alpha} \\ &= d_\alpha(x,\ z)\ ,\ \forall \alpha \in (0,\ 1). \end{aligned}$$

Thus d_α fails to satisfy the triangle inequality. So $(X,\ d_\alpha :\ \alpha \in\ (0,\ 1))$ is a generating space of quasi-metric family but not a generating space of semi-metric family. □

4. Construction of fuzzy metric space from GSQMF

In this section, we construct a fuzzy metric space from a GSQMF, under certain condition.

Theorem 4.1. *Let $(X,\ d_\alpha :\ \alpha \in\ (0,\ 1))$ is a generating space of quasi-metric family.*
We assume that
$d_{(1-(1-\alpha)*(1-\beta))}(x,\ z) \leq\ d_\alpha(x,\ y) + d_\beta(y,\ z)\ \forall\ x,\ y,\ z \in\ X,\ \forall\ \alpha,\ \beta \in (0,\ 1).$
with respect to some continuous t-norm $*$ satisfying (T5). Now we define a function $M' :\ X^2 \times\ [0,\ \infty) \to [0,\ 1]$ as

$$M'(x,\ y,\ t) = \begin{cases} \bigvee\{\alpha \in (0,\ 1) :\ d_{(1-\alpha)}(x,\ y) \leq\ t\} & for\quad t > 0 \\ 0 \quad for \quad t = 0 \end{cases}$$

Then $(X,\ M',\ *)$ is a fuzzy metric space with respect to the t-norm $*$.

Proof. (M1) It is immediate from definition.

(M2) Let $\forall\ t > 0,\quad M'(x,\ y,\ t) = 1$.
For any $t > 0$ and any $\epsilon \in\ (0,\ 1),\ \ \exists\ \alpha(t,\ \epsilon) > \epsilon$ such that $d_{(1-\alpha(t,\ \epsilon))}(x,\ y)\ \leq\ t$.
Since $t > 0$ is arbitrary, $d_{(1-\epsilon)}(x,\ y) = 0\ \ \forall\ \epsilon \in\ (0,\ 1)$.
$\Rightarrow\ \ x = y$(By (QM1)).
Conversely if $x = y$, then for $t > 0$
$d_{(1-\alpha)}(x,\ y) = 0\ \leq\ t\ \ \forall \alpha \in (0,\ 1)$.
So $M'(x,\ y,\ t) = 1\ \ \forall\ t > 0$.
Thus ($\forall t > 0,\quad M'(x,\ y,\ t) = 1$) iff $x = y$.

(M3) Follows from definition.

(M4) we have to show that $\forall s,\ t \in [0,\ \infty)$
$M'(x,\ z,\ s+t) \geq\ M'(x,\ y,\ s) *\ M'(y,\ z,\ t)\ \ \forall\ x,\ y,\ z \in\ X.$
If $M'(x,\ y,\ s) =\ 0$ or $M'(y,\ z,\ t) =\ 0$ then the relation is obvious.
Let $s > 0,\ t > 0,\ 0 <\ M'(x,\ y,\ s),\ 0 <\ M'(y,\ z,\ t)$.
Let $\alpha,\ \beta \in (0,\ 1)$ such that $0 < (1-\alpha) <\ M'(x,\ y,\ s),\ 0 < (1-\beta) <\ M'(y,\ z,\ t)$
Then $d_\alpha(x,\ y) \leq\ s$ and $d_\beta(y,\ z) \leq\ t$.
Since $d_{(1-(1-\alpha)*(1-\beta))}(x,\ z) \leq\ d_\alpha(x,\ y) + d_\beta(y,\ z) \leq\ t+s$
$\forall\ x,\ y,\ z \in\ X,\ \forall\ \alpha,\ \beta \in (0,\ 1)$.
Therefore $M'(x,\ z,\ s+t) \geq\ (1-\alpha)*(1-\beta)$.
Since $\alpha,\ \beta \in (0,\ 1)$ is arbitrary and $*$ is continuous,
$M'(x,\ z,\ s+t) \geq\ M'(x,\ y,\ s) *\ M'(y,\ z,\ t)$
(M5) Follows from definition.
Thus $(X,\ M',\ *)$ is a fuzzy metric for continuous t-norm satisfying (T5).

Now a natural question that may arise is- What is the relation between M and M'?
In the following Section we discuss this issue. □

5. Relation between M and M'

In this section we established a relation between equipotent fuzzy metric and their corresponding quasi-metric families. Finally we show that under certain condition two fuzzy metrics M and M' are identical.

Definition 5.1. Let X be any nonempty set and M a fuzzy metric on X. We define
$$M(x,\ y,\ t+)\ = M_+(x,\ y,\ t)\ = \lim_{s \downarrow t} M(x,\ y,\ s)$$
and
$$M(x,\ y,\ t-)\ = M_-(x,\ y,\ t) = \lim_{s \uparrow t} M(x,\ y,\ s).$$

Theorem 5.2. *Let X be any nonempty set and $M_1,\ \ M_2$ be two fuzzy metrics on X. Then $\forall x,\ y \in\ X,\ \forall t \in\ [0,\ \infty),\ M_1(x,\ y,\ t+) = M_2(x,\ y,\ t+)$ and $M_1(x,\ y,\ t-) = M_2(x,\ y,\ t-)$ iff $d^1_\alpha(x,\ y) = d^2_\alpha(x,\ y)$, $\forall\alpha \in (0,\ 1)$, where $\{d^1_\alpha :\ \alpha \in\ (0,\ 1)\}$ and $\{d^2_\alpha :\ \alpha \in\ (0,\ 1)\}$ denote the corresponding quasi-metric families of M_1 and M_2 respectively.*

Proof. First we suppose that $d^1_\alpha(x,\ y) = d^2_\alpha(x,\ y)\ \ \forall\alpha \in\ (0,\ 1)$.
If possible, suppose for some $t = t_0\ \in\ [0,\ \infty),\ M_1(x,\ y,\ t_0+) \neq M_2(x,\ y,\ t_0+)$.
Without loss of generality, we may assume that
$$M_1(x,\ y,\ t_0+) < M_2(x,\ y,\ t_0+). \tag{5.1}$$
Choose $\beta \in (0,\ 1)$ such that $M_1(x,\ y,\ t_0+) < (1-\beta) < M_2(x,\ y,\ t_0+)$. Note that
$$d^1_\alpha(x,\ y) = \bigwedge\{t > 0 :\ M_1(x,\ y,\ t) \geq (1-\alpha)\},\ \alpha \in (0,1) \tag{5.2}$$
and
$${}^2_\alpha(x,\ y) = \bigwedge\{t > 0 :\ M_2(x,\ y,\ t) \geq (1-\alpha)\},\ \alpha \in (0,\ 1). \tag{5.3}$$
Now $M_1(x,\ y,\ t_0) \leq\ M_1(x,\ y,\ t_0+) < (1-\beta) < M_2(x,\ y,\ t_0+)$ implies that $\exists\ \epsilon(\beta) > 0$ such that $M_1(x,\ y,\ t_0+\epsilon) < 1-\beta$. By using (5.1), (5.2) and (5.3) we

have $d^2_\beta(x,\ y) \leq t_0$, $d^1_\beta(x,\ y) \geq t_0+\epsilon$, which is a contradiction to the hypothesis. Therefore $M_1(x,\ y,\ t+) = M_2(x,\ y,\ t+)\ \forall t \in\ [0,\ \infty)$.
Similarly $M_1(x,\ y,\ t-) = M_2(x,\ y,\ t-)\ \forall t \in\ [0,\ \infty)$.

Conversely suppose that $M_1(x,\ y,\ t+) = M_2(x,\ y,\ t+)$, $M_1(x,\ y,\ t-) = M_2(x,\ y,\ t-)$ hold $\forall t \in\ [0,\ \infty)$.
We have to show that $d^1_\alpha(x,\ y) = d^2_\alpha(x,\ y)\ \forall\alpha \in (0,\ 1)$.
If possible suppose that $\exists\alpha_0 \in (0,\ 1)$ such that $d^1_{\alpha_0}(x,\ y) \neq d^2_{\alpha_0}(x,\ y)$.
Without loss of generality, we may suppose that

$$d^1_{\alpha_0}(x,\ y) > d^2_{\alpha_0}(x,\ y). \tag{5.4}$$

Choose k_1, k_2, k_3 such that $d^1_{\alpha_0}(x,\ y) > k_1 > k_2 > k_3 > d^2_{\alpha_0}(x,\ y)$. Then, by using (5.2), we have,

$$M_1(x,\ y,\ k_1) < (1-\alpha_0),\quad M_2(x,\ y,\ k_3) \geq (1-\alpha_0). \tag{5.5}$$

Now from (5.4) and (5.5), we get

$$(1-\alpha_0) > M_1(x,\ y,\ k_1) \geq M_1(x,\ y,\ k_2+)$$

and

$$M_2(x,\ y,\ k_2-) \geq M_2(x,\ y,\ k_3) \geq (1-\alpha_0).$$

Combining the above two results we have
$M_1(x,\ y,\ k_2+) < (1-\alpha_0) \leq M_2(x,\ y,\ k_2-) \leq M_2(x,\ y,\ k_2+)$
$\Rightarrow M_1(x,\ y,\ k_2+) < M_2(x,\ y,\ k_2+)$ a contradiction to the assumption.
Thus $d^1_\alpha(x,\ y) = d^2_\alpha(x,\ y)\ \forall\alpha \in (0,\ 1)\ \ \forall x,\ y \in\ X$.
This completes the proof. □

Example 5.3. Let $X = R^2$ and $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$ and $d(x, y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$.
Define $M_1,\ M_2 : X^2 \times [0, \infty) \to\ [0, 1]$ by

$$M_1(x,\ y,\ t) = \begin{cases} \frac{t}{t+d(x,y)} & \text{for}\ \ t > 0 \\ 0 & \text{for}\ \ t = 0. \end{cases}$$

and

$$M_2(x,\ y,\ t) = \begin{cases} \frac{t}{t+2d(x,y)} & \text{for}\ \ t > 0 \\ 0 & \text{for}\ \ t = 0. \end{cases}$$

Then $(X,\ M_1,\ *)$ and $(X,\ M_2,\ *)$ are two fuzzy metric spaces and $d^1_\alpha(x,\ y) = \frac{(1-\alpha)}{\alpha}d(x, y)$ and $d^2_\alpha(x,\ y) = \frac{2(1-\alpha)}{\alpha}d(x, y)\ \forall\alpha \in (0,\ 1)$.
Here if $x \neq\ y$, $M_1(x,\ y,\ t+) \neq\ M_2(x,\ y,\ t+)$ and
$M_1(x,\ y,\ t-) \neq\ M_2(x,\ y,\ t-)\ \ \forall t > 0$ and $d^1_\alpha(x,\ y) \neq\ d^2_\alpha(x,\ y)$, $\forall\alpha \in (0,\ 1)$.
Again if $x = y$, $M_1(x,\ y,\ t+) = M_2(x,\ y,\ t+)$ and
$M_1(x,\ y,\ t-) = M_2(x,\ y,\ t-)\ \ \forall t > 0$ and $d^1_\alpha(x,\ y) = d^2_\alpha(x,\ y)$, $\forall\alpha \in (0,\ 1)$.

Definition 5.4. Let X be any nonempty set and M_1, M_2 be two fuzzy metrics on X. M_1 and M_2 are said to be equipotent if $M_1(x,\ y,\ t-) = M_2(x,\ y,\ t-)$ and $M_1(x,\ y,\ t+) = M_2(x,\ y,\ t+)$, $\forall x,\ y \in\ X$, $\forall t \in\ [0,\ \infty)$.

Note 5.5. *It can be easily verified that the above relation is an equivalence relation.*

Theorem 5.6. *Let* $(X,\ M,\ *)$ *be a fuzzy metric space for a continuous t-norm* $*$ *satisfying (T5) and*
$d_\alpha(x,\ y) = \bigwedge\{t > 0 :\ M(x,\ y,\ t) \geq (1-\alpha)\},\ \alpha \in (0,\ 1).$
Let $M' :\ X^2 \times\ [0,\ \infty)\ \to [0,\ 1]$ *as*

$$M'(x,\ y,\ t) = \begin{cases} \bigvee\{\alpha \in (0,\ 1) :\ d_{(1-\alpha)}(x,\ y) \leq\ t\} & for\ \ t > 0 \\ 0 \quad for\ \ t = 0 \end{cases}$$

Then M' *is a fuzzy metric on* X *and* M *and* M' *are equipotent.*

Proof. By Theorem 4.1, it follows that M' is a fuzzy metric on X. Then we have

$$d_\alpha(x,\ y) = \bigwedge\{t > 0 : M(x,\ y,\ t) \geq\ (1-\alpha)\}, \alpha \in (0,\ 1). \tag{5.6}$$

Now we have to show that,
$M(x,\ y,\ t-) = M'(x,\ y,\ t-)$ and $M(x,\ y,\ t+) = M'(x,\ y,\ t+)$, $\forall x \in\ X$, $\forall t \in\ [0,\ \infty)$.
If possible, suppose that for some $t = t_0 \in\ [0,\ \infty)$ and some $x,\ y \in\ X$,
$M(x,\ y,\ t_0-) \neq M'(x,\ y,\ t_0-)$.
Without loss of generality, we may suppose that $M(x,\ y,\ t_0-) < M'(x,\ y,\ t_0-)$.
Choose β such that $M(x,\ y,\ t_0-) < (1-\beta) < M'(x,\ y,\ t_0-)$. Then $\exists\ \ \epsilon\ > 0$ such that

$$t_0 - \epsilon < t < t_0\ ,\ M(x,\ y,\ t) < (1-\beta) < M'(x,\ y,\ t).$$

Now for $t_0 - \epsilon < t < t_0$, $M(x,\ y,\ t) < (1-\beta)\ \Rightarrow d_\beta(x,\ y) \geq t_0$ by using 5.6.
$M'(x,\ y,\ t) > (1-\beta)\ \Rightarrow d_\beta(x,\ y) \leq t$, where $t \in (t_0 - \epsilon, t_0)$ (by using definition of M').
Thus we arrive at a contradiction. So $M(x,\ y,\ t_0-) = M'(x,\ y,\ t_0-)$. Similarly we can verify that $M(x,\ y,\ t+) = M'(x,\ y,\ t+)$. Hence M and M' are equipotent. □

Example 5.7. Let $X = l^\infty$ be the sequence space. Define

$$d'(x,\ y) =\ Sup\{|x_n - y_n|\}\ ,\quad d(x,\ y) =\ Sup\{\tfrac{|x_n - y_n|}{2}\},$$

where $x = (x_1,\ x_2,\ ....,\ x_n,\ ....)$ and $y = (y_1,\ y_2,\ ....,\ y_n,\ .....)$.
We now define $M : X^2 \times [0,\ \infty) \to [0,\ 1]$ by

$$M(x,\ y,\ t) = \begin{cases} 1 & \text{if}\ \ t > d'(x,\ y) \\ \frac{1}{2} & \text{if}\ \ d(x,\ y) < t \leq d'(x,\ y) \\ 0 & \text{if}\ \ t \leq d(x,\ y). \end{cases}$$

Then $(X,\ M, * = min)$ is a fuzzy metric space and for $\alpha \in\ (0,\ 1)$

$$d_\alpha(x,\ y) = \begin{cases} d'(x,\ y) & \text{if}\ \ 0 < \alpha < \frac{1}{2} \\ d(x,\ y) & \text{if}\ \ \frac{1}{2} \leq\ \alpha < 1 \end{cases}$$

and

$$M'(x,\ y,\ t) = \begin{cases} 1 & \text{if}\ \ t \geq\ d'(x,\ y) \\ \frac{1}{2} & \text{if}\ \ d(x,\ y) \leq\ t < d'(x,\ y) \\ 0 & \text{if}\ \ t < d(x,\ y). \end{cases}$$

Here $(X,\ M', * = min)$ is also a fuzzy metric space but M and M' are not equal though they are equipotent.

If we assume, for $x,\ y(\neq x),\ M(x,\ y,\ .)$ is a continuous function on $[0,\ \infty)$ then the relation between M and M' becomes the relation of identity. In fact, we have the following theorem:

Theorem 5.8. *Let $(X,\ M,\ *)$ be a fuzzy metric space for a continuous t-norm $*$ satisfying (T5). We assume that, $M(x,\ y,\ .)$ is a continuous function on $[0,\ \infty)$ for all $x,\ y(\neq x)\in\ X$.*
Let us define

$$d_\alpha(x,\ y)=\bigwedge\{t>0:\ M(x,\ y,\ t)\geq(1-\alpha)\},\ \alpha\in(0,\ 1)$$

and $M':\ X^2\times\ [0,\ \infty)\ \rightarrow[0,\ 1]$ be a function defined by

$$M'(x,\ y,\ t)=\begin{cases}\bigvee\{\alpha\in(0,\ 1):\ d_{(1-\alpha)}(x,\ y)\leq\ t\} & for\ \ t>0.\\ 0\quad for\ \ t=0\end{cases}$$

Then
(i) $\{d_\alpha:\alpha\in(0,\ 1)\}$ is a quasi-metric family on X.
(ii) M' is a fuzzy metric on X.
(iii) $M'=M$.

Proof. Proof of (i) and (ii) follows from Theorem 3.1 and Theorem 4.1 respectively.
To prove (iii), first we prove the following lemma. □

Lemma 5.9. *Let $(X,\ M,\ *)$ be a fuzzy metric space, $x_0,\ y_0(\neq x_0)\in X$ and $\{d_\alpha:\alpha\in(0,\ 1)\}$ be the corresponding quasi-metric family on X corresponding to the fuzzy metric M.*
(1) If $M(x_0,\ y_0,\ .)$ is upper semi continuous and if for $t_0>0,\ M(x_0,\ y_0,\ t_0)=(1-\alpha_0)\in(0,\ 1)$, then $M(x_0,\ y_0,\ d_{\alpha_0}(x_0,\ y_0))\ =(1-\alpha_0)$.
(2) If $M(x_0,\ y_0,\ .)$ is continuous, then for any $\alpha\in(0,\ 1),\ M(x_0,\ y_0,\ d_\alpha(x_0,\ y_0))\ =(1-\alpha)$.
(3) If $M(x_0,\ y_0,\ .)$ is continuous and strictly increasing for $t>0$,
then $M(x_0,\ y_0,\ t)=(1-\alpha)\Leftrightarrow d_\alpha(x_0,\ y_0)=t$.

Proof. (1) From definition,

$$d_{\alpha_0}(x_0,\ y_0)=\bigwedge\{s>0:\ M(x_0,\ y_0,\ s)\geq(1-\alpha_0)\}.\tag{5.7}$$

Since $M(x_0,\ y_0,\ t_0)\ =(1-\alpha_0)$, from (5.7), we get

$$d_{\alpha_0}(x_0,\ y_0)\ \leq t_0.\tag{5.8}$$

Since $M(x_0,\ y_0,\ .)$ is nondecreasing, from (5.8), we have

$$(1-\alpha_0)=M(x_0,\ y_0,\ t_0)\ \geq\ M(x_0,\ y_0,\ d_{\alpha_0}(x_0,\ y_0)).\tag{5.9}$$

Thus $M(x_0,\ y_0,\ d_{\alpha_0}(x_0,\ y_0))\ \leq(1-\alpha_0)$.
If possible suppose that $M(x_0,\ y_0,\ d_{\alpha_0}(x_0,\ y_0))\ <(1-\alpha_0)$. Then by the upper semi continuity of $M(x_0,\ y_0,\ .)$, $\exists t'\ >d_{\alpha_0}(x_0,\ y_0)$ such that $M(x_0,\ y_0,\ t')\ <(1-\alpha_0)$. Thus

$$d_{\alpha_0}(x_0,\ y_0)=\bigwedge\{s>0:\ M(x_0,\ y_0,\ s)\geq(1-\alpha_0)\}\ \geq t'>d_{\alpha_0}(x_0,\ y_0).$$

This is a contradiction. So, from (5.9), $M(x_0,\ y_0,\ d_{\alpha_0}(x_0,\ y_0))\ = (1-\alpha_0)$.

(2) Since $M(x_0,\ y_0,\ .)$ is continuous, by (M1) and (M5), for each $\alpha \in (0\ ,\ 1)$, $\exists t > 0$ such that $M(x_0,\ y_0,\ t)\ = (1-\alpha)$.

Then by (1), the proof follows.

(3) It follows from (1) and (2), by using the strict increasing property of $M(x_0,\ y_0,\ .)$.

Now we prove the Theorem 5.6(iii).

We consider the following cases.

Let $(x_0,\ y_0,\ t_0) \in X^2 \times\ [0,\ \infty)$ and $M(x_0,\ y_0,\ t_0) = (1-\alpha_0)$.

Case I: Let $t_0 \leq 0$.

Then, $M(x_0,\ y_0,\ t_0) = M'(x_0,\ y_0,\ t_0) = 0$.

Case II: $x_0 = y_0,\ t_0 > 0$.

Then $M(x_0,\ y_0,\ t_0) = M'(x_0,\ y_0,\ t_0) = 1$.

Case III: $x_0 \neq y_0$ and $t_0 \in\ [0,\ \infty)$ such that $M(x_0,\ y_0,\ t_0) = 0$.

For $\alpha \in (0,\ 1),\ d_\alpha(x_0,\ y_0) = \bigwedge\{t > 0 :\ M(x_0,\ y_0,\ t) \geq (1-\alpha)\}$.

By Lemma 5.7(2), we have $M(x_0,\ y_0,\ d_\alpha(x_0,\ y_0))\ = (1-\alpha)\ \ \forall \alpha \in (0,\ 1)$.

Since $M(x_0,\ y_0,\ t_0) = 0\ <\ (1-\alpha)$, it follows that $t_0 < d_\alpha(x_0,\ y_0)\ \forall\alpha\ \in (0,\ 1)$.

So $M'(x_0,\ y_0,\ t_0) = \bigvee\{\alpha \in (0,\ 1) :\ d_\alpha(x_0,\ y_0) \leq t_0\} = \bigvee \phi = 0$.

Therefore $M(x_0,\ y_0,\ t_0) = M'(x_0,\ y_0,\ t_0) = 0$.

Case IV: Suppose $x_0 \neq y_0$ and $t_0 > 0$ such that $0 < M(x_0,\ y_0,\ t_0) < 1$.

Let $M(x_0,\ y_0,\ t_0) = (1-\alpha_0)$. Then $0 < \alpha_0 < 1$. Now

$$M'(x,\ y,\ t) = \bigvee\{\alpha \in (0,\ 1) :\ d_{(1-\alpha)}(x,\ y) \leq t\} \tag{5.10}$$

and

$$d_\alpha(x,\ y) = \bigwedge\{t > 0 :\ M(x,\ y,\ t) \geq (1-\alpha)\}. \tag{5.11}$$

Since $M(x_0,\ y_0,\ t_0) = (1-\alpha_0)$, from (5.11) ,we have

$$d_{\alpha_0}(x_0,\ y_0) \leq t_0. \tag{5.12}$$

Using (5.12), from (5.10), we get

$$M'(x_0,\ y_0,\ t_0) \geq (1-\alpha_0). \tag{5.13}$$

Thus $M'(x_0,\ y_0,\ t_0) \geq M(x_0,\ y_0,\ t_0)$. For $\alpha \in (0,\ \alpha_0)$, let $d_\alpha(x_0,\ y_0) = t'$. By lemma 5.7(2), $M(x_0,\ y_0,\ t') = (1-\alpha)$. So $M(x_0,\ y_0,\ t') = (1-\alpha) > (1-\alpha_0) = M(x_0,\ y_0,\ t_0)$. Since $M(x_0,\ y_0,\ .)$ is monotonically increasing, $M(x_0,\ y_0,\ t') > M(x_0,\ y_0,\ t_0)$ implies that $t'\ > t_0$. Hence, for $\alpha \in (0,\ \alpha_0)$, $d_\alpha(x_0,\ y_0) = t' \not\leq t_0$. Therefore

$$M'(x_0,\ y_0,\ t_0) \leq (1-\alpha_0) = M(x_0,\ y_0,\ t_0. \tag{5.14}$$

By (5.13) and (5.14), we have $M'(x_0,\ y_0,\ t_0) = M(x_0,\ y_0,\ t_0)$.

Case V: Suppose $x_0 \neq y_0$ and $t_0 \in\ [0,\ \infty)$ such that $M(x_0,\ y_0,\ t_0) = 1$.

Note that for each $\alpha \in (0\ ,\ 1)$ and for any $\ x,\ y \in\ X$,

$$M'(x,\ y,\ t) = \begin{cases} \bigvee\{\alpha \in (0,\ 1) :\ d_{(1-\alpha)}(x,\ y) \leq\ t\}\ \ for\ \ t > 0 \\ 0\ \ \ for\ \ t = 0 \end{cases}$$

and

$$d_\alpha = \bigwedge\{t > 0 :\ M(x,\ y,\ t) \geq (1-\alpha)\}. \tag{5.15}$$

From (5.15), we have $d_{(1-\alpha)(x_0,\ y_0)} \ \leq t_0\ \forall \alpha \in (0,\ 1)$. Thus $M'(x_0,\ y_0,\ t_0) = 1$. So $M(x_0,\ y_0,\ t_0) = M'(x_0,\ y_0,\ t_0) = 1$. Hence $M(x,\ y,\ t) = M'(x,\ y,\ t) \quad \forall (x,\ y,\ t) \in X^2 \times\ [0,\ \infty)$. □

6. Conclusion

Though the decomposition theorems play a pivotal role in the development of fuzzy functional analysis, but it is observed that the validity of this theorems requires a stringent restriction on the underlying t-norm in the definition of fuzzy metric.To retain the generality of t-norm, the concept of Generating space of quasi-metric family comes in front. GSQMF has the common characteristic properties of both fuzzy metric space in the sense of Kaleva & Seikkala [5] and Menger probabilistic metric space [14]. So the GSQMF is more general concept than that of fuzzy metric but in a restricted situation these two spaces are similar.

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