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Topological structure of rough set models for binary relations

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Abstract. This paper investigates two classes of generalized rough set models based on two definitions, namely, the element and granule-based definitions. Four types of topological structures induced by the two rough set models are discussed. Furthermore, we explore the relationship between different topologies corresponding to the generalized rough set models. This study presents diverse research perspectives for the combination of rough set theory and topology theory.

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1. Introduction

Rough set theory was firstly proposed by Pawlak [10]. It is an extension of set theory for the analysis of a vague and inexact description of objects. This theory is widely applied to feature selection, rule extraction, decision supporting, data mining and knowledge discovery from large data sets [8, 11, 13, 16]. The foundation of its object classification is equivalence relation. The upper and lower approximation operators are two core notions in rough set theory. In real world databases, data sets usually take on variant forms. Pawlak rough approximations are based on equivalence relations, which is not suitable in several situations. So from the angle of applications, it is found to be more important to look into situations where the binary relation may not be an equivalence relation but only an arbitrary. To address this issue, many researchers have recently proposed several data processing methods using generalized rough set models [2, 6, 7, 8, 9].

Topology is an important mathematical tool for the study of information systems, which has independent theoretical framework, background and broad applications [3]. We can introduce topological methods to rough set theory and study the relationship between topological theory and rough set theory, which has a deep theoretical and practical significance beyond doubt. Some researchers carried out this exploration[4, 5, 13, 14, 15, 20]. Kortelainen [5] considered relationships among modified sets, topological spaces and rough sets based on a pre-order. Kondo [4] proved that every reflexive relation in a set can induce a topology and proposed a kind of compactness condition. Besides he got that a topology which satisfies the compactness condition can determine the lower and upper approximation operators induced by a similarity relation. In addition, connections between fuzzy rough set theory and fuzzy topology were also investigated [1, 15].

However, the relationship between general rough sets and topologies need to be further studied. Pawlak [10] indicated that $T = \{X \subseteq U | \underline{R}(X) = X = \overline{R}(X)\}$ is a clopen topology on U, where $\underline{R}(X)$ and $\overline{R}(X)$ are Pawlak rough lower and upper approximations of X. In fact, there are some equivalent representations for the above T.

$$\begin{aligned} T =&\{X \subseteq U | X \subseteq \underline{R}(X)\} \\ =&\{X \subseteq U | X \supseteq \overline{R}(X)\} \\ =&\{X \subseteq U | X = \underline{R}(X)\} \\ =&\{X \subseteq U | X = \overline{R}(X)\}. \end{aligned} \tag{1.1}$$

Generally, if we replace the equivalence relation with a general binary relation in the above four sets, they are no longer equivalent. In addition, for generalized rough sets induced by binary relations, Yao [19] gave an element-based definition, and Zhang et al. [21] gave a granule-based definition. So by the formula (1.1), we can propose eight types of sets based on the above two definitions. Obviously, they may not be equivalent. Naturally, we need to study the eight sets. Is it possible for each of the eight sets to be a topology induced by binary relations? What are the conditions for them to generate a topology respectively? What's the relationship among the eight topologies? No doubt, these are interesting problems which need to be further discussed. In this paper, we shall present some answers to these questions.

The remaining part of this paper is organized as follows. In Section 2, some basic concepts and results about generalized rough sets and topology are reviewed. In Section 3, we investigate four topological structures of generalized rough sets. Meanwhile, different relations inducing the same topology are explored. Conclusion is in Section 4.

2. Preliminaries

In this section, we shall briefly review basic concepts and results of the relation based rough sets and topology. For more details, we refer to [7, 17, 20, 21].

In this paper, we always assume that U is a finite universe, R is a binary relation on U.

$\forall x, y \in U$, if $(x, y) \in R$, then x is the predecessor of y and y is the successor of x.

The sets

$$R_s(x) = \{y \in U \mid (x, y) \in R\},\ R_p(x) = \{y \in U \mid (y, x) \in R\}$$

are called the successor and predecessor neighborhood of x respectively.

R is called serial if $\forall x \in U$, $\exists y \in U$ such that $y \in R_s(x)$; R is called inverse serial if $\forall x \in U$, $\exists y \in U$ such that $y \in R_p(x)$; R is called reflexive if $\forall x \in U$ such that $x \in R_s(x)$; R is called symmetric if $\forall x, y \in U$, $x \in R_s(y)$ implies $y \in R_s(x)$; R is called transitive if $\forall x, y, z \in U$, $y \in R_s(x)$ and $z \in R_s(y)$ imply $z \in R_s(x)$.

Definition 2.1 ([7, 18, 21]). Let U be a universe and R be a binary relation on U. The pair (U, R) is called a generalized approximation space. For any subset X of U, two types of generalized rough lower and the upper approximations of X are defined by the following:
Element-based definition:
$\underline{apr}'_R(X) = \{x \in U \mid R_s(x) \subseteq X\}, \quad \overline{apr}'_R(X) = \{x \in U \mid R_s(x) \cap X \neq \varnothing\};$
Granule-based definition:
$\underline{apr}''_R(X) = \bigcup\{R_s(x) \mid R_s(x) \subseteq X\}, \quad \overline{apr}''_R(X) = \bigcup\{R_s(x) \mid R_s(x) \cap X \neq \varnothing\}.$

When there is no confusion, we omit the lowercase R. For example, we denote $\underline{apr}'_R(X)$ by $\underline{apr}'(X)$.

Proposition 2.2 ([7, 20, 21]). *In a generalized approximation space (U, R), the lower approximation $\underline{apr}'(X)$ and the upper approximation $\overline{apr}'(X)$ of X satisfy the following properties: $\forall X, Y \subseteq U$,*
(1) $\overline{apr}'(\{x\}) = R_p(x)$ *for all* $x \in U$;
(2) $\underline{apr}'(U) = U$, $\overline{apr}'(\varnothing) = \varnothing$;
(3) $\underline{apr}'(X) = \sim \overline{apr}'(\sim X)$, $\overline{apr}'(X) = \sim \underline{apr}'(\sim X)$;
(4) $X \subseteq Y \Rightarrow \underline{apr}'(X) \subseteq \underline{apr}'(Y)$, $\overline{apr}'(X) \subseteq \overline{apr}'(Y)$;
(5) $\underline{apr}'(X \cap Y) = \underline{apr}'(X) \cap \underline{apr}'(Y)$, $\overline{apr}'(X \cup Y) = \overline{apr}'(X) \cup \overline{apr}'(Y)$;
(6) $\underline{apr}'(X \cup Y) \supseteq \underline{apr}'(X) \cup \underline{apr}'(Y)$, $\overline{apr}'(X \cap Y) \subseteq \overline{apr}'(X) \cap \overline{apr}'(Y)$;
where $\sim X$ *is the complement of* X *with respect to* U.

Proposition 2.3 ([21]). *In a generalized approximation space (U, R), the lower approximation $\underline{apr}''(X)$ and the upper approximation $\overline{apr}''(X)$ of X satisfy the following properties: $\forall X, Y \subseteq U$,*
(1) $\overline{apr}''(\varnothing) = \varnothing$;
(2) $\underline{apr}''(X) \subseteq X$;
(3) $\underline{apr}''(X) = \underline{apr}''\underline{apr}''(X)$, $\overline{apr}''(X) \subseteq \overline{apr}''\overline{apr}''(X)$;
(4) $X \subseteq Y \Rightarrow \underline{apr}''(X) \subseteq \underline{apr}''(Y)$, $\overline{apr}''(X) \subseteq \overline{apr}''(Y)$;
(5) $\underline{apr}''(X \cap Y) \subseteq \underline{apr}''(X) \cap \underline{apr}''(Y)$, $\overline{apr}''(X \cap Y) \subseteq \overline{apr}''(X) \cap \overline{apr}''(Y)$;
(6) $\underline{apr}''(X \cup Y) \supseteq \underline{apr}''(X) \cup \underline{apr}''(Y)$, $\overline{apr}''(X \cup Y) = \overline{apr}''(X) \cup \overline{apr}''(Y)$;
(7) If R is inverse serial, then $\underline{apr}''(U) = U$, $X \subseteq \overline{apr}''(X)$.

Proposition 2.4 ([21]). *Let R be a binary relation on U. R is inverse serial if and only if $R_s(U) = \bigcup_{x \in U} R_s(x) = U$.*

Proposition 2.5 ([21]). *Let R be a binary relation on U. The following conditions are equivalent: $\forall X \subseteq U$,*
(1) R is reflexive; (2) $\underline{apr}'(X) \subseteq X$; (3) $X \subseteq \overline{apr}'(X)$.

Proposition 2.6 ([21]). *Let R be a binary relation on U. The following conditions are equivalent: $\forall X \subseteq U$,*
(1) R is transitive; (2) $\underline{apr}'(X) \subseteq \underline{apr}'(\underline{apr}'(X))$; (3) $\overline{apr}'(\overline{apr}'(X)) \subseteq X$.

The basic concepts of topology have been widely used in many areas.

Definition 2.7 ([14, 17])**.** Let U be a non-empty set. T is a family of subsets of U, which satisfies the three conditions:
(T1) $\varnothing, U \in T$;
(T2) If $A, B \in T$, then $A \cap B \in T$;
(T3) If $\mathcal{A} \subseteq T$, then $\bigcup_{A\in\mathcal{A}} A \in T$;
then we call T is a topology on U.
The pair (U, T), or briefly, U is called a topological space.

Definition 2.8 ([14, 17])**.** An operator I: $\mathcal{P}(U) \to \mathcal{P}(U)$ is called an interior operator on U if it satisfies the following conditions: $\forall A, B \subseteq U$,
(I1) $I(U) = U$;
(I2) $I(A) \subseteq A$;
(I3) $I(I(A)) = I(A)$;
(I4) $I(A \cap B) = I(A) \cap I(B)$.

Definition 2.9 ([14, 17])**.** An operator C: $\mathcal{P}(U) \to \mathcal{P}(U)$ is called a closure operator on U if it satisfies the following conditions: $\forall A, B \subseteq U$,
(C1) $C(\varnothing) = \varnothing$;
(C2) $A \subseteq C(A)$;
(C3) $C(C(A)) = C(A)$;
(C4) $C(A \cup B) = C(A) \cup C(B)$.

3. Major section

In this section, we will investigate the connection between four types of topological structures and two classes of rough set models based on arbitrary binary relations.

According to Definition 2.1 and the formula (1.1), we can define eight sets as follow.

$$\begin{cases} T_1' = \{X | X \subseteq \underline{apr}'(X)\}, \\ T_2' = \{X | X \supseteq \overline{apr}'(X)\}. \end{cases} \tag{3.1}$$

$$\begin{cases} T_3' = \{X | X = \underline{apr}'(X)\}, \\ T_4' = \{X | X = \overline{apr}'(X)\}. \end{cases} \tag{3.2}$$

$$\begin{cases} T_1'' = \{X | X \subseteq \underline{apr}''(X)\}, \\ T_2'' = \{X | X \supseteq \overline{apr}''(X)\}. \end{cases} \tag{3.3}$$

$$\begin{cases} T_3'' = \{X | X = \underline{apr}''(X)\}, \\ T_4'' = \{X | X = \overline{apr}''(X)\}. \end{cases} \tag{3.4}$$

T_3' was demonstrated in [4]. Obviously, we have the following relations: $T_3' \subseteq T_1'$, $T_4' \subseteq T_2'$, $T_3'' \subseteq T_1''$, $T_4'' \subseteq T_2''$.

3.1. **The conditions under which T_1', T_2', T_3' and T_4' are topologies respectively.**

In this subsection, we will discuss the relationship between topology T_i' (for $i = 1, 2, 3, 4$) and binary relations. The following proposition illustrates that T_1' and T_2' are topologies for arbitrary binary relations.

Proposition 3.1. *Let (U, R) be a generalized approximation space. Then T_1' and T_2' are topologies respectively.*

Proof. In order to prove T_1' is a topology on U, we only need to show that T_1' which is in the formula (3.1) satisfies the conditions of Definition 2.7.

(T1) It is easy to verify that $\varnothing \in T_1'$ and $U \in T_1'$.

(T2) Suppose that $X, Y \in T_1'$, that is, $X \subseteq \underline{apr}'(X)$, $Y \subseteq \underline{apr}'(Y)$. By Proposition 2.2, we have $\underline{apr}'(X \cap Y) = \underline{apr}'(X) \cap \underline{apr}'(Y)$. Thus $X \cap Y \subseteq \underline{apr}'(X \cap Y)$, that is, $X \cap Y \in T_1'$.

(T3) $\forall \mathcal{A} \subseteq T_1'$, we shall show that $\bigcup_{A \in \mathcal{A}} A \in T_1'$. By the formula (3.1), we only need to prove $\bigcup_{A \in \mathcal{A}} A \subseteq \underline{apr}'(\bigcup_{A \in \mathcal{A}} A)$. $\forall x \in \bigcup_{A \in \mathcal{A}} A$, $\exists A \in \mathcal{A}$ such that $x \in A$. By $\mathcal{A} \subseteq T_1'$, then $A \in T_1'$, which implies $x \in A \subseteq \underline{apr}'(A)$. Thus $R_s(x) \subseteq A \subseteq \bigcup_{A \in \mathcal{A}} A$. So $x \in \underline{apr}'(\bigcup_{A \in \mathcal{A}} A)$. This shows that $\bigcup_{A \in \mathcal{A}} A \subseteq \underline{apr}'(\bigcup_{A \in \mathcal{A}} A)$, that is, $\bigcup_{A \in \mathcal{A}} A \in T_1'$.

In summary, T_1' is a topology.

We can prove that T_2' is also a topology on U in the same way. □

However, for a binary relation R, T_3' or T_4' may not be a topology. When R is a reflexive relation, T_3' and T_4' are topologies [4]. The following example illustrates that reflexive relation is only a sufficient condition.

Example 3.2. Let $U = \{a, b, c\}$ and $R = \{(a, c), (b, a), (b, b), (b, c), (c, c)\}$. Then R is not a reflexive relation, and $R_s(a) = \{c\}, R_s(b) = \{a, b, c\}, R_s(c) = \{c\}$. Hence $T_3' = \{\varnothing, \{a, c\}, U\}$, $T_4' = \{\varnothing, \{b\}, U\}$. Clearly, both T_3' and T_4' are topologies on U.

Now we will give a necessary condition under which T_3' and T_4' are topologies respectively.

Proposition 3.3. *Let (U, R) be a generalized approximation space. The following results hold.*

(1) If T_3' is a topology, then R is a serial relation;

(2) If T_4' is a topology, then R is a serial relation.

Proof. (1) Since T_3' is a topology, then $\varnothing \in T_3'$. By the formula (3.2), $\underline{apr}'(\varnothing) = \varnothing$. If R is not serial, then $\exists x \in U$ such that $R_s(x) = \varnothing \subseteq \varnothing$. Hence $x \in \underline{apr}'(\varnothing)$ and this implies a contradiction.

(2) Since T_4' is a topology, then $U \in T_4'$. By the formula (3.2), $\overline{apr}'(U) = U$. If R is not serial , then $\exists x \in U$ such that $R_s(x) = \varnothing$. So $R_s(x) \cap U = \varnothing$. Hence $x \notin \overline{apr}'(U)$ and this implies a contradiction. □

Proposition 3.3 demonstrates that serial relation is a necessary condition under which T_3' and T_4' are topologies. Whether serial relation is a sufficient condition or not? The following example gives a negative answer.

Example 3.4. Let $U = \{a, b, c\}$ and $R = \{(a, a), (b, b), (c, a), (c, b)\}$. Then R is a serial relation, and $R_s(a) = \{a\}, R_s(b) = \{b\}, R_s(c) = \{a, b\}$. Hence $T'_3 = \{\varnothing, \{a\}, \{b\}, U\}$ and $T'_4 = \{\varnothing, \{a, c\}, \{b, c\}, U\}$. Clearly, both T'_3 and T'_4 are not topologies on U.

In the next section, we shall give a sufficient and necessary condition under which both of them are topologies.

Let $T \subseteq \mathcal{P}(U)$, where $\mathcal{P}(U)$ is the power set of U. We define the set T^C as follows: $T^C = \{\sim X | X \in T\}$.

Proposition 3.5. *Let (U, R) be a generalized approximation space. T'_3 is a topology if and only if T'_4 is a topology.*

Proof. If $X \in T'_3$, by the formula (3.2), $\underline{apr}'(X) = X$. By Proposition 2.2, we have $\overline{apr}'(\sim X) =\sim \underline{apr}'(X) =\sim X$. So $\sim X \in T'_4$. Hence $T'_4 = T'^C_3$. In the same way, we have $T'_3 = T'^C_4$.

For the necessity, it is need to prove that T'_4 satisfies the conditions of Definition 2.7.

(T1) Since T'_3 is a topology, then $\varnothing, U \in T'_3$. Hence $\varnothing, U \in T'_4$.

(T2) If $X, Y \in T'_4$, then $\sim X, \sim Y \in T'_3$. By the formula (3.2), $\overline{apr}'(X \cap Y) =\sim \underline{apr}'(\sim X \cup \sim Y) =\sim (\sim X \cup \sim Y) = X \cap Y$. Hence $X \cap Y \in T'_4$.

(T3) $\forall X, Y \in T'_4$, then $\sim X, \sim Y \in T'_3$. By the formula (3.2), $\overline{apr}'(X \cup Y) =\sim \underline{apr}'(\sim X \cap \sim Y) =\sim (\sim X \cap \sim Y) = X \cup Y$. Since U is a finite universe, then T'_4 distributes over arbitrary unions of subsets.

In summary, T'_4 is a topology.

We can prove the sufficient condition in the same way. □

From the above propositions, we have investigated connections between topology T'_i (for $i = 1, 2, 3, 4$) and the first pair of approximation operators $\overline{apr}'$ and $\underline{apr}'$. In the following subsection, we will study the topological structures induced by the second pair of approximation operators $\overline{apr}''$ and $\underline{apr}''$.

3.2. The conditions under which T''_1, T''_2, T''_3 and T''_4 are topologies respectively.

Proposition 3.6. *Let (U, R) be a generalized approximation space and R be a reflexive and transitive relation on U. Then $\underline{apr}'(X) = \underline{apr}''(X)$ for each $X \subseteq U$. In this case, $T''_1 = T'_1$ and thus T''_1 is a topology.*

Proof. $\forall x \in \underline{apr}'(X)$, we have $R_s(x) \subseteq X$. Since R is reflexive, then $x \in R_s(x)$, which implies $x \in \underline{apr}''(X)$. On the other hand, $\forall x \in \underline{apr}''(X)$, $\exists y \in U$ such that $x \in R_s(y)$ and $R_s(y) \subseteq X$. Since R is transitive, then $R_s(x) \subseteq X$ and $x \in \underline{apr}'(X)$. Therefore, we have $\underline{apr}'(X) = \underline{apr}''(X)$. According to Proposition 3.1, $T''_1 = T'_1$ and thus T''_1 is a topology. □

This result illustrates that T''_1 is a topology when R is a reflexive and transitive relation. However, it is not a necessary condition. Now we shall give a necessary condition.

Proposition 3.7. *Let (U, R) be a generalized approximation space. Then R is inverse serial if and only if $\underline{apr}''(U) = U$.*

Proof. R is inverse serial $\Leftrightarrow \bigcup_{x\in U} R_s(x) = U \Leftrightarrow \bigcup\{R_s(x) \mid R_s(x) \subseteq U\} = U \Leftrightarrow \underline{apr}''(U) = U$, by Proposition 2.4. □

Proposition 3.8. *Let (U, R) be a generalized approximation space. If T_1'' is a topology, then R is a inverse serial relation.*

Proof. It is obvious by Propositions 2.2 and 3.7. □

However, inverse serial relation may not be a sufficient condition.

Example 3.9. Let $U = \{a, b, c, d\}$ and $R = \{(a, a), (a, b), (b, c), (b, d), (c, a), (c, c)\}$. Then R is a inverse serial relation, and $R_s(a) = \{a, b\}, R_s(b) = \{c, d\}, R_s(c) = \{a, c\}, R_s(d) = \varnothing$. Hence $T_1'' = \{\varnothing, \{a, b\}, \{a, c\}, \{c, d\}, \{a, c, d\}, U\}$. Clearly, T_1'' is not a topology on U.

According to Proposition 2.3(2), we have $T_3'' = T_1''$. Thus, we will get the following results: (1) A reflexive and transitive relation can induce topology T_3''. (2) Inverse serial relation is only a necessary condition under which T_3'' is a topology.

Proposition 3.10. *Let (U, R) be a generalized approximation space. Then T_2'' is a topology.*

Proof. We only need to show that T_2'' which is in the formula (3.3) satisfies the conditions of Definition 2.7.

(T1) Obviously, $\overline{apr}''(U) \subseteq U$ and $\overline{apr}''(\varnothing) = \varnothing \subseteq \varnothing$. Then $\varnothing, U \in T_2''$.

(T2) If $X, Y \in T_2''$, that is, $\overline{apr}''(X) \subseteq X$ and $\overline{apr}''(Y) \subseteq Y$. By Proposition 2.3, we have $\overline{apr}''(X \cap Y) \subseteq \overline{apr}''(X) \cap \overline{apr}''(Y) \subseteq X \cap Y$. Thus $X \cap Y \in T_2''$.

(T3) $\forall \mathcal{A} \subseteq T_2''$, we shall show that $\bigcup_{A\in\mathcal{A}} A \in T_2''$. By the formula (3.3), we only need to prove $\overline{apr}''(\bigcup_{A\in\mathcal{A}} A) \subseteq \bigcup_{A\in\mathcal{A}} A$. $\forall x \in \overline{apr}''(\bigcup_{A\in\mathcal{A}} A)$, $\exists y \in U$ such that $x \in R_s(y)$ and $R_s(y) \cap (\bigcup_{A\in\mathcal{A}} A) \neq \varnothing$. Thus $\exists A \in \mathcal{A}$ and $R_s(y) \cap A \neq \varnothing$. So $x \in \overline{apr}''(A)$. By $\mathcal{A} \subseteq T_2''$, then $A \in T_2''$, which implies $\overline{apr}''(A) \subseteq A$. So $x \in \bigcup_{A\in\mathcal{A}} A$. It follows that $\overline{apr}''(\bigcup_{A\in\mathcal{A}} A) \subseteq \bigcup_{A\in\mathcal{A}} A$. Hence $\bigcup_{A\in\mathcal{A}} A \in T_2''$.

In summary, T_2'' is a topology on U. □

However, for a binary relation R on U, T_4'' may not be a topology. The following example will demonstrates this conclusion.

Example 3.11. Let $U = \{a, b, c\}$ and $R = \{(a, a), (b, a), (c, b)\}$. Then R is a binary relation and $R_s(a) = \{a\}, R_s(b) = \{a\}, R_s(c) = \{b\}$. Hence, $T_4'' = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. Obviously, T_4'' is not a topology.

A natural question thus arises: What's the condition under which T_4'' is a topology? Next, we shall give a sufficient and necessary condition.

Proposition 3.12. *Let (U, R) be a generalized approximation space. Then $\bigcup\{R_s(x) \mid R_s(x) \cap U \neq \varnothing\} = \bigcup_{x\in U} R_s(x)$.*

Proof. Clearly, $\bigcup\{R_s(x) \mid R_s(x) \cap U \neq \varnothing\} \subseteq \bigcup_{x\in U} R_s(x)$. On the other hand, $\forall x \in U$ and $x \in \bigcup_{x\in U} R_s(x)$, then $\exists R_s(y)$ such that $x \in R_s(y)$ and $R_s(y) \cap U \neq \varnothing$. Thus $x \in \bigcup\{R_s(x) \mid R_s(x) \cap U \neq \varnothing\}$. Hence $\bigcup_{x\in U} R_s(x) \subseteq \bigcup\{R_s(x) \mid R_s(x) \cap U \neq \varnothing\}$. □

Proposition 3.13. *Let (U, R) be a generalized approximation space. T_4'' is a topology if and only if R is a inverse serial relation.*

Proof.
Since T_4'' is a topology, then $U \in T_4''$. By the formula (3.4), we have $\overline{apr}''(U) = U$. Then $\bigcup\{R_s(x) \mid R_s(x) \cap U \neq \varnothing\} = U$. By Proposition 3.12, we have $\bigcup_{x \in U} R_s(x) = U$. Hence R is inverse serial.

Conversely, we only need to prove that T_4'' satisfies the conditions of Definition 2.7.

(T1) Clearly, $\overline{apr}''(\varnothing) = \varnothing$. Hence $\varnothing \in T_4''$. Since R is inverse serial, by Proposition 3.13 and Definition 2.1, then $\overline{apr}''(U) = U$. Hence $U \in T_4''$.

(T2) If $X, Y \in T_4''$, we have $\overline{apr}''(X \cap Y) \subseteq \overline{apr}''(X) \cap \overline{apr}''(Y) = X \cap Y$, by Proposition 2.3. $\forall x \in X \cap Y$, that is, $x \in \overline{apr}''(X) \cap \overline{apr}''(Y)$. Since $x \in \overline{apr}''(X)$, $\exists y \in U$ such that $x \in R_s(y)$. So $R_s(y) \cap (X \cap Y) \neq \varnothing$, which implies $x \in \overline{apr}''(X \cap Y)$. Thus $X \cap Y \subseteq \overline{apr}''(X \cap Y)$. Hence $X \cap Y \in T_4''$.

(T3) If $\forall \mathcal{A} \subseteq T_4''$, we shall show that $\bigcup_{A \in \mathcal{A}} A \in T_4''$. By the formula (3.4), we only need to prove $\overline{apr}''(\bigcup_{A \in \mathcal{A}} A) = \bigcup_{A \in \mathcal{A}} A$. $\forall x \in \overline{apr}''(\bigcup_{A \in \mathcal{A}} A)$, $\exists y \in U$ such that $x \in R_s(y)$ and $R_s(y) \cap (\bigcup_{A \in \mathcal{A}} A) \neq \varnothing$. Thus $\exists A \in \mathcal{A}$ and $R_s(y) \cap A \neq \varnothing$. So $x \in \overline{apr}''(A)$. By $\mathcal{A} \subseteq T_4''$, $A \in T_4''$, which implies $\overline{apr}''(A) = A$. Hence $x \in A$, and thus $x \in \bigcup_{A \in \mathcal{A}} A$. On the other hand, since R is inverse serial and Proposition 2.3, we have $\bigcup_{A \in \mathcal{A}} A \subseteq \overline{apr}''(\bigcup_{A \in \mathcal{A}} A)$. Therefore, $\overline{apr}''(\bigcup_{A \in \mathcal{A}} A) = \bigcup_{A \in \mathcal{A}} A$. Hence $\bigcup_{A \in \mathcal{A}} A \in T_4''$.

In summary, T_4'' is a topology on U. □

3.3. **The relationships among these topologies.**

According to Definition 2.1, if R is an equivalent relation, the two forms of approximation operators are equivalent. Obviously, we have $T_1' = T_1'' = T_2' = T_2'' = T_3' = T_3'' = T_4' = T_4''$. This subsection discusses the relationship among the eight sets which are induced by arbitrary binary relations.

Proposition 3.14. *Let (U, R) be a generalized approximation space. Then*
(1) $T_1' = T_3'$ if and only if R is a reflexive relation;
(2) $T_2' = T_4'$ if and only if R is a reflexive relation.

Proof. It is obvious by Proposition 2.5. □

According to Proposition 2.3(2), we know that $T_1'' = T_3''$ for a binary relation on U. Whether T_2'' and T_4'' are equal for a binary relation? However, the following example shows that they are not necessary equal.

Example 3.15. In Example 3.11, we can get

$$T_2'' = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, U\}.$$

Obviously, $T_2'' \neq T_4''$.

So we have the following proposition.

Proposition 3.16. *Let (U, R) be a generalized approximation space. $T_2'' = T_4''$ if and only if R is a inverse serial relation.*

Proof. Since $T''_2 = T''_4$, we have $U \subseteq \overline{apr}''(U) \subseteq U$. Then $\overline{apr}''(U) = U$. So $\bigcup\{R_s(x) \mid R_s(x) \cap X \neq \varnothing\} = \bigcup_{x\in U} R_s(x) = U$. According to Proposition 2.4, R is a inverse serial relation.

Conversely, it is obvious that $T''_4 \subseteq T''_2$. $\forall X \in T''_2$, by the formula (3.3), we have $\overline{apr}''(X) \subseteq X$. If R is a inverse serial relation, then $X \subseteq \overline{apr}''(X)$, by Proposition 2.3. Thus $X \in T''_4$. Hence $T''_2 = T''_4$. □

For general binary relations, the two forms of generalized rough approximation pairs $(\underline{apr}', \overline{apr}')$ and $(\underline{apr}'', \overline{apr}'')$ are different, but they are closely related. If R is a reflexive and transitive relation on U, we have $T'_1 = T''_1 = T'_3 = T''_3$ by Propositions 3.6 and 3.14. The following proposition illustrates the relationship among T'_2, T''_2, T'_4, and T''_4.

Proposition 3.17. *Let (U, R) be a generalized approximation space. If R is a symmetric and transitive relation, then $T'_2 = T''_2$ and $T'_4 = T''_4$.*

Proof. In order to prove the result, we only need to show $\overline{apr}'(X) = \overline{apr}''(X)$.

$\forall x \in \overline{apr}'(X)$, then $R_s(x) \cap X \neq \varnothing$. Suppose that $x \notin \overline{apr}''(X)$, then $\forall y$ such that $x \in R_s(y)$ and $R_s(y) \cap X = \varnothing$. Since R is transitive, then $R_s(x) \subseteq R_s(y)$. Hence $R_s(x) \cap X = \varnothing$ and this implies a contradiction, which implies $x \in \overline{apr}''(X)$.

On the other hand, $\forall x \in \overline{apr}''(X)$, $\exists y$ such that $x \in R_s(y)$ and $R_s(y) \cap X \neq \varnothing$. Suppose that $x \notin \overline{apr}'(X)$, then $R_s(x) \cap X = \varnothing$. Since R is a symmetric and transitive relation, then $y \in R_s(x)$ and $R_s(y) \subseteq R_s(x)$. Hence $R_s(y) \cap X = \varnothing$ and this implies a contradiction, which implies $x \in \overline{apr}'(X)$.

Therefore, we have $\overline{apr}'(X) = \overline{apr}''(X)$. Hence $T'_2 = T''_2$, $T'_4 = T''_4$. □

In summary, we have two tables. The following table 1 gives a summary of relationships between generalized rough sets and topologies (T'_i and T''_i) (for $i = 1, 2, 3, 4$). Table 2 gives a summary of relationships among these topologies.

In the following tables, the abbreviation BR, SR, ISR, RR, RTR, STR and ER stand for binary relation, serial relation, inverse serial relation, reflexive relation, reflexive and transitive relation, symmetric and transitive relation and equivalence relation respectively.

Table 1
The relationship between relations and topologies

	R is a BR	R is a SR	R is a ISR	R is a RR	R is a RTR
T'_1	topology	topology	topology	topology	topology
T'_2	topology	topology	topology	topology	topology
T'_3	not topology	not topology	not topology	topology	topology
T'_4	not topology	not topology	not topology	topology	topology
T''_1	not topology	not topology	not topology	not topology	topology
T''_2	topology	topology	topology	topology	topology
T''_3	not topology	not topology	not topology	not topology	topology
T''_4	not topology	not topology	topology	topology	topology

Table 2
The relationship among topologies

R is a BR	R is a ISR	R is a RR	R is a RTR	R is a STR	R is a ER
$T_1'' = T_3''$	$T_2'' = T_4''$	$T_1' = T_3'$ $T_2' = T_4'$ $T_1'' = T_3''$ $T_2'' = T_4''$	$T_1' = T_3'$ $T_2' = T_4'$ $T_1'' = T_3''$ $T_2'' = T_4''$ $T_1' = T_1''$ $T_3' = T_3''$	$T_2' = T_2''$ $T_4' = T_4''$	$T_i' = T_j'$ $T_i'' = T_j''$ $T_i' = T_j''$ $(i, j = 1, 2, 3, 4)$

3.4. **Different relations induce the same topology.**

Yu [20] studied the lower approximation $\underline{apr}'(X)$ and proposed some conclusions. In this section, we will show that there are some similar conclusions for the upper approximation $\overline{apr}'(X)$.

Definition 3.18 ([20]). Let R be a binary relation on U. The smallest transitive relation on U containing the relation R is called the transitive closure of R.

We denote the transitive closure of R by $t(R)$.

Proposition 3.19 ([12]). *Let R be a binary relation. Then $t(R) = R \cup R^2 \cup \cdots$.*

Proposition 3.20 ([20]). *Let R be a reflexive relation. Then $t(R)$ is also a reflexive relation.*

Let R be a reflexive relation. Denote: $T_4' = T_4'(R) = \{X \subseteq U \mid \overline{apr}'(X) = X\}$.

Proposition 3.21. *Let R be a reflexive relation. Then $T_4'(R) = T_4'(t(R))$.*

Proposition 3.22. *Let R be a reflexive relation and S be a binary relation on U. If $R \subseteq S \subseteq t(R)$, then $T_4'(R) = T_4'(S) = T_4'(t(R))$.*

Proof. By Proposition 3.21, it's obvious. □

Proposition 3.23 ([14]). *Let R be a binary relation. The following conditions are equivalent:*
(1) R is a reflexive and transitive relation;
(2) $\overline{apr}'$ is a closure operator on (U, T_3');
(3) $\underline{apr}'$ is an interior operator on (U, T_3').

Corollary 3.24. *If R is a reflexive and transitive relation, then $\underline{apr}'$ and $\overline{apr}'$ are interior operator and closure operator on $(U, T_i')(i = 1, 2, 4)$, respectively.*

Proof. By Propositions 2.2, 2.5, 2.6 and 3.14, it's obvious. □

Let (U, T) be a topological space and I and C be interior operator and closure operator, respectively. We may define the relation $R(T_4')$ on U as follows:

$$\forall x, y \in U, (x, y) \in R(T_4') \Leftrightarrow x \in C(\{y\}). \tag{3.5}$$

Proposition 3.25. *Let R be a reflexive and transitive relation, then $R(T_4'(R)) = R$.*

Proof. By Proposition 2.2 and Corollary 3.24, $\forall (x, y) \in R(T_4'(R)) \Leftrightarrow x \in \overline{apr}'\{y\} = R_p(y) \Leftrightarrow (x, y) \in R$. □

Proposition 3.26. *Let R be a reflexive relation on U and*

$$\Sigma_R = \{S|S \ is \ a \ reflexive \ relation \ on \ U \ and \ T_4'(S) = T_4'(R)\}.$$

Then we have
(1) $t(R) \in \Sigma_R$;
(2) $t(R)$ is the greatest element of Σ_R.

At the end of this section, we will explore relations between the granule-based approximation operators and interior and closure operators.

Proposition 3.27. *Let (U, R) be a generalized approximation space. If R is a reflexive and transitive relation, then $\underline{apr}''$ is an interior operator.*

Proof. By Propositions 3.6 and 3.23, it's obvious. □

Remark 3.28. If R is a reflexive and transitive relation, then the lower approximation $\underline{apr}''(X)$ of X satisfies: $\forall X, Y \subseteq U$, $\underline{apr}''(X \cap Y) = \underline{apr}''(X) \cap \underline{apr}''(Y)$.

However, for a reflexive and transitive relation, $\overline{apr}''$ may not be a closure operator. The following example illustrates this result.

Example 3.29. Let $U = \{a, b, c, d\}$ and let

$$R = (a, a), (a, d), (b, b), (b, c), (b, d), (c, c), (c, d), (d, d).$$

Then R is a reflexive and transitive relation, and $R_s(a) = \{a, d\}, R_s(b) = \{b, c, d\}, R_s(c) = \{c, d\}, R_s(d) = \{d\}$. Thus $\overline{apr}''(\{a\}) = \{a, d\}$, $\overline{apr}''(\{a, d\}) = U$. Hence $\overline{apr}''(\overline{apr}''(\{a\})) \neq \overline{apr}''(\{a\})$. So $\overline{apr}''$ does not satisfy (C3) in Proposition 2.9.

4. Conclusions

In this paper, we studied two types of generalized rough sets induced by arbitrary binary relations. The four classes of topological structures induced by the element and granule-based rough sets were introduced. Furthermore we investigated the relationship among them. Two distinct binary relations generating the same topology was shown in the last part. We believe that these results will be very useful for the research of covering rough sets and the promotion of rough set theory. In the future, we will do some research on extending rough set models by means of topological method.

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