Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 11, No. 1, (January 2016), pp. 63–74
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Maximizing a linear objective function subject to a system of min-t equations with a continuous archimedean t-conorm

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Received 28 February 2015; Revised 22 May 2015; Accepted 6 July 2015

Abstract. In this paper a new algorithm is proposed to solve a fuzzy linear programming problem with maximum linear objective function subject to asystem of fuzzy relation equations using min t-conorm operator. Using the new algorithm a numerical problem is solved.

2010 AMS Classification: 03E72, 90B05

Keywords: Fuzzy relation equations, t-conorm operator.

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1. Introduction

The notion of fuzzy relation equations was first introduced by Sanchez[11](1976) and it was developed by Zimmermann[14](1991), Higashi and Klir[8](1984), Yager[13] (1979) etc.,

Sanchez[11] studied the methods to resolve the fuzzy relations on fuzzy sets and some theorems were established to determine the solutions. Many researchers have tried to deal such problems and develop the procedures [2, 4, 5, 6]. S. Abbasbandy et al.,[1] formulated a condition for linear system of equations over max-min algebra to have not a unique solution by using fuzzy determinant. Masoud Allame, Benhnaz Vatankhahan [3] introduced iteration algorithms for solving the linear systems whose elements are from a Brouwerian lattice. Mohsen Hekmatnia et al., [7] investigated possibilisic linear programming and offered a new method to achieve optimal value of necessary degree of constaints for decision maker in fuzzy linear problem with fuzzy technological coefficients and solve problem by that value.

In this paper we propose a new method namely maximizing a linear objective function under min-T conorm method for finding an optimal solution to fuzzy relation equations(FRE) constraints problems.

Also a method is developed to solve a fuzzy linear programming problem with fuzzy relation equation and a numerical example is solved using the procedure developed.

2. Preliminaries

In this section we present basic definitions and some properties of min-t conorm.

Definition 2.1 ([7]). A fuzzy union / t-conorm is a binary operation on the unit interval that satisfies the following axioms : for all $a, b, d \in [0, 1]$,
1. $T(a, 0) = a$ (boundary condition),
2. $b \leq d$ implies $T(a, b) \leq j(a, d)$ (monotonicity),
3. $T(a, b) = T(b, a)$ (commutativity),
4. $T[a, T(b, d)] = T[T(a, b), d]$ (associativity).

Definition 2.2. Fuzzy LPP with fuzzy relation equation constraints

Maximize

$$Z(p) = \sum_{i=1}^{n} c_i p_i \tag{1}$$

subject to

$$p \circ Q = r \tag{2}$$

where $p \in [0, 1]^m, c_i \in R$ is the coefficient associated with variable p_i,

$Q = [q_{ij}]$ is an $m \times n$ matrix with $q_{ij} \leq 1$, r is an n dimensional vector with $0 \leq r_j \leq 1$ and the operation "o" represents the $min - T$ composition operator. where T is a continuous Archimedean triangular conorm.

Definition 2.3. Let $S(Q, r) = \{\ p \in [0, 1]^m | p \circ Q = r\}$ denote the solution set of (2) and let $I = \{\ 1, 2, 3, ..., m\}$ and $J = \{\ 1, 2, 3, ..., n\}$ be two index sets. Then, the solution vectors $[p \in [0, 1]^m$ of the given problem (2) is obtained by

$$\min_{i \in I}\{T(p_i, Q_{ij})\} = r_j, \forall j \in J. \tag{3}$$

3. Conditions for optimality

Proposition 3.1. *If $S(Q, r) \neq \phi$, then the minimal solutionof the problem (2) can be obtained by the following operation*

$$\underline{p} = \max_{j \in J}\{u_j(q_{ij}, r_j)\} \tag{4}$$

where $u_j(q_{ij}, r_j) = \inf\{\ p \in [0, 1] | T(p, q_{ij}) \geq r_j\}$.

Definition 3.2. If $S(Q, r) \neq \phi$ and $p = (p_1, p_2, p_3, ..., p_m)$ be any solution of (2), then p_i is said to be a locking variable if $T(p_i, q_{ij}) = r_j$ for some $j \in J$. The locking set of p_i is denoted by

$$J(p_i) = \{\ j \in J | T(p_i, q_{ij}) = r_j\}\ . \tag{5}$$

Lemma 3.3. *Let T be the continuous Archimedean t-conorm and if $q_{ij} > r_j$ for each $i \in I$ for any equation in (2) then the solution set $S(Q, r) = \phi$.*

Proof. Given that $q_{ij} > r_j$ for each $i \in I$. Then, by definion,

$$T(p_i, q_{ij}) \geq \max_{j \in J}(p_i, q_{ij}) > r_j, \forall i \in I.$$

Thus $\min_{i\in I}\{\ T(p_i, q_{ij})\}\ > r_j$ which is a contradiction to (3) and so there exists no variable p_i satisfying any equation in (2). Hence $S(Q, r) = \phi$. □

Note: From Lemma 3.3, $S(Q, r) \neq \phi$,then if the condition $q_{ij} \leq r_j$ is satisfied by one equation in (2) and in this case p_i is a locking variable.

Lemma 3.4. *Let T be the continuous Archimedean t-conorm and $\underline{p} = (\underline{p}_i)(i \in I)$ be the minimal solution and $p = (p_i)_(i \in I)$ be any solution of (3). If p_i is locking in the j^{th} equation, then $\underline{p}_i$ is also locking. If $\underline{p}_i$ is not a locking variable, then for p_i is also nonlocking.*

Proof. Let $p = (p_i)_{i\in I}$ be any solution of (2). Then $\min_{i\in I}\{\ T(p_i, q_{ij})\}\ = r_j \forall j \in J$. Since $\underline{p} = (\underline{p}_i)_{i\in I}$ is the minimal solution, from Definition 3.1,

$$T(\underline{p}_i, q_{ij}) \geq r_j, j \in J.$$

Suppose p_i is locking variable in the j^{th} equation. Then $j \in J(p_i)$ and $T(p_i, q_{ij}) = r_j, \forall j \in J$.

By the monotonicity of T-conorm, we obtain

$$r_j = T(p_i, q_{ij}) \geq T(\underline{p}_i, q_{ij}) \geq r_j.$$

Thus $T(\underline{p}_i, q_{ij}) = r_j$. So $\underline{p}_i$ is a locking variable in the j^{th} equation.

Again if $\underline{p}_i$ is not a locking for any $j \in J$, then $T(\underline{p}_i, q_{ij}) \geq r_j, j \in J$. Since $T(p_i, q_{ij}) \geq T(\underline{p}_i, q_{ij}) > r_j$, $T(p_i, q_{ij}) > r_j, j \in J$. And so p_i is also not a locking variable. □

Lemma 3.5. *Let T be a continuous t-conorm, and $S(Q, r) \neq \phi$ in (2). If r_j=1 for some $j \in J$, then all variables $p_i, \forall i \in I$ are locking in the j^{th} equation.*

Proof. Let $p = (p_i)_{i\in I}$ be any solution of (2) Then $\min_{i\in I}\{\ T(p_i, q_{ij})\}\ = r_j$,where $r_j \in$ [0,1].

Suppose $r_j = 1$ for some $j \in J$. Then $\min_{i\in I}\{\ T(p_i, q_{ij})\}$ =1. Since $T(p_i, q_{ij}) \in [0, 1]$, we get $T(p_i, q_{ij}) = 1, \forall i \in I$. Hence $T(p_i, q_{ij}) = r_j i \in I$ and p_i is the locking variable. □

Lemma 3.6. *Let T be a continuous Archemeadian t-conorm and $p = (p_i)_{i\in I}$ be any solution of (2). If p_i is only locking in equations with $r_j = 1$, then p_i can take any value in $[\underline{p}_i, 1]$.*

Proof. Assume p_i is locking in equation (2) with $r_j = 1$. By definition, $T(p_i, q_{ij}) = 1, j \in J(p_i)$. If $\underline{p} = (\underline{p}_i)_{i\in I}$ is the minimal solution, then, By lemma 3.4, $\underline{p}_i$ is also locking in equations with $r_j = 1$. Thus $T(\underline{p}_i, q_{ij}) = 1, \forall j \in J(\underline{p}_i)$.

If $1 \geq p_i \geq \underline{p}_i$, then, Since T is monotonic,

$$1 = T(1, q_{ij}) \geq T(p_i, q_{ij}) \geq T(\underline{p}_i, q_{ij}) = 1.$$

Hence $T(p_i, q_{ij})$=1 for all $p_i \in [\underline{p}_i, 1]$. □

Lemma 3.7. *Let T be a continuous t-conorm and $\underline{p} = (\underline{p}_i)_{i\in I}$ be the minimal solution. If $c_i \leq 0, \forall i \in I$ the cost coefficient in the objective function , then $\underline{p}_i$ is an optimal solution .*

Proof. Assume $\underline{p} = (\underline{p}_i)_{i\in I} \in S(Q, r)$ to be the minimal solution and $p = (p_i)_{i\in I}$ be any solution, then $\underline{p}_i \leq p_i, \forall i$. If $c_i \leq 0, \forall i \in I$, then $\sum_{i=1}^{m} c_i\underline{p}_i \geq \sum_{i=1}^{m} c_i p_i$. Thus $\underline{p}$ is an optimal solution of the given problem. □

Lemma 3.8. *Let T be a continuous t-conorm. If $c_i \geq 0, \forall i \in I$ and $\bar{S}(Q, r)$ be the set of all maximal solutions of (2). Then any one of the solutions in $\bar{S}(Q, r)$ is an optimal solution of the given problem.*

Proof. We know that if $S(Q, r) \neq \phi$, then

$$S(Q, r) = \bigcup_{\bar{p}\in \bar{S}(Q,r)}\{\ p \in [0, 1]^m | \bar{p} \geq p \geq \underline{p}\}.$$

Consider the set $\{\ p|\bar{p} \geq p \geq \underline{p}\}$, we have $\sum_{i=1}^{m} c_i\bar{p}_i \geq \sum_{i=1}^{m} c_i p_i \geq \sum_{i=1}^{m} c_i\underline{p}_i$. Therefore if $\bar{p} = (\bar{p}_i)_{i\in I}$ is the maximal solution, then it gives the greatest objective value. Since there are only finite number of maximal solution, the maximal solution which gives the greatest objective function is the optimal solution. □

4. Properties of continuous Archimedean t-conorm

Theorem 4.1 (Schweizer and Sklar,1963 [12]; Ling 1965 [10])**.** *Let T be a binary operation on the unite interval. Then T is an Archimedean t- conorm iff there exists an increasing generator such that $T(a, b) = g^{(-1)}(g(a) + g(b))$ where $g^{(-1)}$ is the pseudo inverse of an increasing generator g is a function from R to [0,1] defined by*

$$g^{(-1)}(a) = \begin{cases} g^{(-1)}(a), & \text{for } a \leq g(1); \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 4.2. *Let T be a continuous Archimedean t-conorm, $r = (r_j)_{j\in J}$ be a vector with $0 \leq r_j < 1$ with $S(Q, r) \neq \phi$ and $\underline{p} = (\underline{p}_i)_{i\in I} \in S(Q, r)$ be the minimal solution. If p_i is a locking variable, for any solution $p = (p_i)_{i\in I}$, then $p_i = \underline{p}_i$.*

Proof. Since $p = (p_i)_{i\in I}$ is any solution of the given problem, by definition,

$$\min_{i\in I}\{\ T(p_i, q_{ij}) = r_i\}\ , \forall j \in J.$$

Also p_i is a locking variable. Thus, by the definition,

$$T(p_i, q_{ij}) = r_j for some j \in J. \tag{6}$$

By Lemma 3.4, $\underline{p}_i$ is also locking in the j^{th} equation. So we have

$$T(\underline{p}_i, q_{ij}) = r_j. \tag{7}$$

Suppose that $p_i > \underline{p}_i$. Since $r_j < 1$, by Theorem 3.9, we have

$$g(p_i) + g(q_{ij}) \leq g(1), T(p_i, q_{ij}) = g^{(-1)}(g(p_i) + g(q_{ij})).$$

Since $p_i > \underline{p}_i$ and g is an increasing generator $g(p_i) > g(\underline{p}_i)$,

$$g(p_i) + g(q_{ij}) > g(\underline{p}_i) + g(q_{ij}) \text{ and } T(\underline{p}_i, q_{ij}) = g^{(-1)}(g(\underline{p}_i) + g(q_{ij})). \quad (8)$$

From equations (6), (7) and (8), we have

$1 > r_j = T(p_i, q_{ij})$
$= g^{(-1)}(g(p_i) + g(q_{ij}))$.
$> g^{(-1)}(g(\underline{p}_i) + g(q_{ij}))$
$= T(\underline{p}_i, q_{ij})$
$= r_j$,
i.e., $r_j = T(p_i, q_{ij}) > T(\underline{p}_i, q_{ij}) = r_j$
which is impossible therefore $p_i = \underline{p}_i$. □

Theorem 4.3. *Let T be a continuous Archimedean t-conorm , $0 \leq r_j \leq 1$ and $\underline{p} = (\underline{p}_i)_{i\in I} \in S(Q, r)$ be the minimal solution. If $\bar{p} = (\bar{p}_i)_{i\in I}$ is a maximal solution, then either $\bar{p}_i = 1$ or $\bar{p}_i = \underline{p}_i$ for each $i \in I$.*

Proof. Let $\bar{p} = (\bar{p}_i)_{i\in I}$ be amaximal solution of (2). Each variable $\bar{p}_i \in \bar{p}$ is either a locking or not a locking variable.

Assume that $\underline{p}_i$ is not a locking variable and $\bar{p}_i < 1$. Let $p = (p_i)_{i\in I}$ be any solution in $S(Q, r)$ such that $p_i = 1$ and $p_k = \bar{p}_k$ for all $k \in I$ and $k \neq i$. Then we can construct a solution p such that $p \geq \bar{p}$. Thus $\bar{p}$ is not a maximal solution. This is a contradiction. So if $\bar{p}_i$ is not a locking variable, then $\bar{p}_i = 1$.

Suppose that $\bar{p}_i$ is a locking variable. Since $0 \leq r_j \leq 1$ and $\underline{p} = (\underline{p}_i)_{i\in I}$ is a minimalsolution,

If $r_j < 1$, then, by Theorem 3.10, the maximal solution $\bar{p} = (\bar{p}_i)_{i\in I}$ exists such that $\bar{p}_i$ is a locking variable then $\bar{p}_i = \underline{p}_i$.

If $r_j = 1$, then, by Lemma 3.6, if p_i is only locking in equations with $r_j = 1$, then p_i can take any value in $[\underline{p}_i, 1]$.

Thus, in the maximal solution $\bar{p} = (\bar{p}_i)_{i\in I}, \bar{p}_i$ is only locking in equations with $r_j = 1$. So $\bar{p}_i = 1$. □

Theorem 4.4. *Let T be a continuous Archimedean t-conorm, r be a vector with $0 \leq r_j \leq 1$ for all $j \in J$ and $\underline{p} = (\underline{p}_i)_{i\in I}$ be the minimal solution. If $\bar{S}(Q, r)$ be the set of all maximal solutions of (2) then any one of the solutions in $\bar{S}(Q, r)$ with $\bar{p}_i = 1$ or $\bar{p}_i = \underline{p}_i$ for each $i \in I$ is an optimal solution of the given problem.*

Proof. If $c_i \geq 0, \forall i \in I$, then, by Lemma 3.8, any one of the solutions in $\bar{S}(Q, r)$ is an optimal solution of the given problem. Also by Theorem 3.11, if $\bar{p} = (\bar{p}_i)_{i\in I}$ is a maximal solution, then either $\bar{p}_i = 1$ or $\bar{p}_i = \underline{p}_i$ for each $i \in I$.

One of the maximal solution $\bar{p} = (\bar{p}_i)_{i\in I} \in \bar{S}(Q, r)$ with $\bar{p}_i = 1$ or $\bar{p}_i = \underline{p}_i$ for each $i \in I$ is an optimal solution of the given problem. □

5. Method of finding an optimal solution

Let us split the fuzzy LPP into two sub problems

$$\left.\begin{array}{l}\text{(i) Maximize } Z^1(p) = \sum_{i=1}^{m} c^1 p_i \\ \text{subject to } p \circ Q = r\end{array}\right\} \tag{9}$$

$$\left.\begin{array}{l}\text{(ii) Maximize } Z^2(p) = \sum_{i=1}^{m} c^2 p_i \\ \text{subject to } p \circ Q = r\end{array}\right\} \tag{10}$$

$$\text{where } c_i^1 = \begin{cases} 0 & \text{if } c_i < 0 \\ c_i & \text{if } c_i \geq 0 \end{cases} \text{ and } c_i^2 = \begin{cases} 0 & \text{if } c_i < 0 \\ c_i & \text{if } c_i \geq 0 \end{cases} \quad \forall j \in J.$$

The constraint equations of the problems (9) and (10) are the same as in the problem (2).

In the original problem $c_i = c_i^1 + c_i^2, \forall i \in I$ by Lemma 3.7, the minimal solution $\underline{p} \in S(Q,r)$ is an optimal solution for the problem (9), because $c_i < 0$ in (9). Then the optimal value is $Z^1(\underline{p})$.

By Lemma 3.8, if $c_i \geq 0\ \forall i \in I$, then one of the solutions in $\bar{S}(Q,r)$ is an optimal solution of the problem (10), therefore the optimal value is $Z^2(\bar{p})$.

Now, let us define

$$p^1 = \begin{cases} \underline{p}_i & \text{if } c_i < 0 \\ \bar{p}_i & \text{if } c_i \geq 0. \end{cases}$$

Hence, p^1 is a solution of the problem (2) with objective value $Z(p^1) = Z^1(\underline{p}) + Z^2(\bar{p})$.

Also, in the given problem

$$\begin{aligned} Z(p) &= \sum_{i=1}^{m} c_i p_i \\ &= \sum_{i=1}^{m} (c_i^1 + c_i^2) p_i \\ &= \sum_{i=1}^{m} c_i^1 p_i + \sum_{i=1}^{m} c_i^2 p_i \\ &\leq \sum_{i=1}^{m} c_i^1 \underline{p}_i + \sum_{i=1}^{m} c_i^2 p_i \\ &\leq \sum_{i=1}^{m} c_i^1 \underline{p}_i + \sum_{i=1}^{m} c_i^2 \bar{p}_i \\ &= Z^1(\underline{p} + Z^2(\bar{p})) \\ &= Z(p^1). \end{aligned}$$

Therefore p^1 is an optimal solution of the original problem with optimal value $Z(p^1)$.

6. Rules to reduce the problem size

For the givenmatrix Q, we define, the following index sets.

$J_i(Q) = \{\ j \in J | p_i \circ q_{ij} = r_j\}\ , \forall i \in I$ and $I_j = \{\ i \in I | p_i \circ q_{ij} = r_j\}\ , \forall j \in J$ the index set $j_i(Q)$ is nothing but the locking set $J(\bar{p}_i)$ of (2).

Rule 1 If $c_i \leq 0$, then any optimal solution $\bar{p} = (\bar{p}_i)_{i\in I}$ has $\bar{p}_i = \underline{p}_i$.

Proof. By Lemma 3.7, the proof is trivial.

Based on Rule 1, the row corresponding to p_i in Q can be deleted, Also if p_i is locking in some equations $j \in J$, then the columns corresponding to these equations can also be deleted from the matrix Q. □

Rule 2 If $I_j(Q)$ is singheton set for some $j \in J$, ie., $I_j(Q) = t$, then $\bar{p}_i = \underline{p}_t$ where $\bar{p}_t \in (\bar{p}_i)_{i\in I}$.

Proof. Since $I_j(Q) = \{\ t\}$. $\therefore$ theJthequation can be satisfied by the variable p_t.

If $p = (p_i)_{i\in I}$ be any solution of (2). Then the tthcomponent of this solution must be locking in the j^{th} equation. By Lemma 3.5.

If $r_j = 1$ for some $j \in J$, then all variables $p_i \forall i \in I$ are locking in the j^{th} equation. Thus $I_j(Q)$ is not a singleton set.

If $r_j < 1$, then by Theorem 3.10 we have $\bar{p}_t = \underline{p}_t$.

If we apply Rule 2, the j^{th} column of Q with $j \in J_t(Q)$ can be deleted. The row corresponding to p_t can also be deleted from matrix Q. □

Rule 3 If $I_p(Q) \supseteq I_q(Q)$ for some $p, q \in J$ in the matrix Q, then the q^{th} column of Q can be deleted.

Rule 4 If $J_s(Q) \neq \phi$ and $J_s(Q) \subseteq J_t(Q)$ for some s and t in set I with $c_s\underline{p}_s < c_t\underline{p}_t < 1$ and $c_s > c_t > 0$, then any optimal solution $\bar{p} = (\bar{p}_i)_{i\in I}$ has $\bar{p}_s = 1$.

Proof. Since $J_s(Q) \subseteq J_t(Q)$, $J(\underline{p}_s) \subseteq J(\underline{p}_t)$.

If $\bar{p}_s = 1$, then, since $\bar{p} = (\bar{p}_i)_{i\in I}$ is an optimal solution, there is nothing to prove. If $\bar{p}_s \neq 1$, i.e., $\bar{p}_s < 1$, then, by Theorem 3.10, $\bar{p}_s$ is a locking variable with $\bar{p}_s = \underline{p}_s$ and $J(\underline{p}_s, q_{sj}) = r_j \forall j \in J(\underline{p}_s)$.

Case (i) Assume that the optimal solution $\bar{p} = (\bar{p}_i)_{i\in I}$ contains $\bar{p}_s = \underline{p}_s < 1$ and $\bar{p}_t = 1$. Then the vector $p' = p$ can be constructed except that $p'_s = 1$ and $p'_t = \underline{p}_t$.

Since $T(\underline{p}_t, q_{tj}) = r_j$ for all $j \in J(\underline{p}_t)$.

Since $J(\underline{p}_s) \subseteq J(\underline{p}_t)$, the constraints satisfied by $p'_s = \underline{p}_s$ can be sustained by $p'_t = \underline{p}_t$.

Therefore p' is a solution of the problem.

Also

$$\begin{aligned} Z(\bar{p}) - Z(p') &= \sum_{i=1}^{m} c_i p_i + \sum_{i=1}^{m} c_i p_i \\ &= (c_s\underline{p}_s + c_t) - (c_s + c_t\underline{p}_t) \\ &= (c_s\underline{p}_s - c_t\underline{p}_t) + (c_t - c_s) < 0. \end{aligned}$$

Thus

$$\begin{aligned} Z(\bar{p}) - Z(p') &< 0 \\ Z(\bar{p}) &< Z(p') \end{aligned}$$

This is a contradiction to our assumption that $\bar{p} = (\bar{p}_i)_{i\in I}$ is an optimal solution. Therefore $1 > c_t\underline{p}_t > c_s\underline{p}_s$ and $c_s > c_t > 0$ then $\bar{p}$ is an optimal solution with $\bar{p}_s = 1$.

Case (ii)

Suppose the optimal solution $\bar{p} = (\bar{p}_i)_{i\in I}$ contains $\bar{p}_s = \underline{p}_s < 1$ and $\bar{p}_t = \underline{p}_t < 1$ then the vector $p' = p$ can be constructed except that $p'_s = 1$ and $p' = \underline{p}_t < 1$. Also $T(\underline{p}_t, q_{tj}) = r_j$ for all $j \in J(\underline{p}_t)$.

Since $J(\underline{p}_s)J(\underline{p}_t)$, the constraints satisfied by $p'_s = \underline{p}_s$ can be sustained by $p'_t = \underline{p}_t$. Since p' is a solution of the problem

$$\begin{aligned} Z(\bar{p}) - Z(p') &= \sum_{i=1}^{m} c_i p_i + \sum_{i=1}^{m} c_i p_i \\ &= c_s\underline{p}_s - c_t\underline{p}_t - c_s - c_t\underline{p}_t \\ &= c_s\underline{p}_s - c_s < 0 \text{ since } 0 < \underline{p}_s < 1 \\ Z(\bar{p}) - Z(p') &< 0 \\ Z(\bar{p}) &< Z(p') \end{aligned}$$

This is a contradiction. Therefore $\bar{p}$ is an optimal solution with $\bar{p}_s = 1$ □

Rule 4.1 Let $I' \subseteq I, \bigcup_i \in IJ_i(Q) = J'$ and $r \in I, r \notin I'$ then the following three results an true from rule 4.

[i] If $J_r(Q) \subseteq J', 1 > \sum_{i\in I'} > c_r\underline{p}_r$ and $c_r > \sum_{i\in I'} c_i$ then any optimal solution $\bar{p} = (\bar{p}_i)_{i\in I}$ has $\bar{p}_r = 1$.

[ii] If $J_r(Q) \supseteq J', 1 > c_r\underline{p}_r > \sum_{i\in I'} > c_i\underline{p}_i$ and $\sum_{i\in I'} c_i > c_r$ then any optimal solution $\bar{p} = (\bar{p}_i)_{i\in I}$ has $\bar{p}_i = 1$ for all $i \in I'$.

[iii] If $J_r(Q) = J', 1 > c_r\underline{p}_r = \sum_{i\in I'} > c_i\underline{p}_i$ and $\sum_{i\in I'} c_i = c_r$ then any optimal solution $\bar{p} = (\bar{p}_i)_{i\in I}$ has either $\bar{p}_i = 1$ for all $i \in I'$ or $\bar{p}_r = 1$.

Rule 5 If $J_k(Q) = \phi$ for some $k \in I$, then any optimal solution $\bar{p} = (\bar{p}_i)_{i\in I}$ has $\bar{p}_k = 1$.

Proof. since $J_k(Q) = \phi$ for some $k \in I$, p_k cannot be a locking in any equation. Also since $c_k \geq 0$ due to get a maximal solution we assigned $p_k = 1$ in any optimal solution.

If we apply Rule 5, then we delete the rows corresponding to the variable p_k. □

7. Algorithm

Step 1 Find the minimal solution $\underline{p} = (\underline{p}_i)_{i\in I}$ of the given problem by (4)

Step 2 If $\underline{p} \circ Q = r$, then go to step 3, otherwise stop the process. The given problem has no solution. i.e., $S(Q, r) = \phi$.

Step 3 Form two sub problems as (9) and (10).

Step 4 Compute index sets I_j and $J_i(or J(\underline{p}_i))$ for the given matrix Q. Find the optimal solution of the problem (9) by rule 1. Delete the corresponding row and columns in Q.

Step 5 Consider the remaining matrix Q, apply Rules 2-5 to determine the optimal solution of the problem (10).

Step 6 From optimal solutions of the problems (9) and (10) we find optimal solutions of the original problem by (11).

8. NumericalL example

Consider the following optimization problem with continuous Archimedean t-conorm fuzzy relational equations constraint.

Maximize $Z(p) = 6p_1 + 2p_2 + 2p_3 + 3p_4 + 4p_5 + 4p_6 + 2p_7 - 2p_8 + p_9$

Subject to $p \circ Q = r$

where $T(p, q) = \min(1, p + q)$, a continuous Archimedean t-conorm.

$$Q = \begin{pmatrix} 0.98 & 0.42 & 0.38 & 0.35 & 0.49 & 0.58 & 0.30 & 026 & 0.44 \\ 0.80 & 0.30 & 0.01 & 0.29 & 0.31 & 0.40 & 0.40 & 0.42 & 0.26 \\ 0.95 & 0.35 & 0.15 & 0.25 & 0.46 & 0.56 & 0.37 & 0.35 & 0.40 \\ 0.78 & 0.48 & 0.48 & 0.45 & 0.52 & 0.38 & 0.32 & 0.28 & 0.45 \\ 0.88 & 0.58 & 0.60 & 0.18 & 0.62 & 0.48 & 0.45 & 0.16 & 0.52 \\ 0.90 & 0.38 & 0.10 & 0.30 & 0.40 & 0.50 & 0.307 & 0.18 & 0.36 \\ 0.96 & 0.45 & 0.16 & 0.26 & 0.46 & 0.58 & 0.28 & 0.24 & 0.42 \\ 0.93 & 0.33 & 0.13 & 0.32 & 0.44 & 0.54 & 0.25 & 0.22 & 0.40 \\ 0.85 & 0.27 & 0.15 & 0.28 & 0.42 & 0.55 & 0.20 & 0.18 & 0.42 \end{pmatrix}$$

$$r = \begin{pmatrix} 1.00 & 0.40 & 0.20 & 0.30 & 0.50 & 0.60 & 0.32 & 0.28 & 0.45 \end{pmatrix}$$

Step 1: Find the minimal solution by (4)

$$i.e., \underline{p} = \begin{pmatrix} 0.02 & 0.20 & 0.05 & 0.22 & 0.12 & 0.10 & 0.04 & 0.07 & 0.15 \end{pmatrix}$$

Step 2: Since $p \circ Q = r$, go to the next step.

Step 3: Form two sub problems

(1)Maximize $Z^1(p) = -2p_8$

subject to $p \circ Q = r$

(2) Maximize $Z^2(p) = 6p_1 + 2p_2 + 2p_3 + 3p_4 + 4p_5 + 4p_6 + 2p_7 + 0p_8 + p_9$

subject to $p \circ Q = r$

Step 4: Compute index sets I_j and J_i (or $J(\underline{p}_i)$) for the given matrix Q and find the optimal solution of the problem 1 by Rule 1.

$J(\underline{p}_1) = 1, 6, 7, 8$ $\quad I_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$J(\underline{p}_2) = \{1, 6\}$ $\quad I_2 = \{3, 8\}$

$J(\underline{p}_3) = \{1, 2, 3, 4, 9\}$ $\quad I_3 = \{3, 6, 7, 8\}$

$J(\underline{p}_4) = \{1, 6\}$ $\quad I_4 = \{3, 5, 7\}$

$J(\underline{p}_5) = \{1, 4, 6, 8\}$ $\quad I_5 = \{6, 7\}$

$J(\underline{p}_6) = \{1, 3, 5, 6, 8\}$ $\quad I_6 = \{1, 2, 4, 5, 6\}$

$J(\underline{p}_7) = \{1, 3, 4, 5, 7, 8\}$ $\quad I_7 = \{1, 7, 8\}$

$J(\underline{p}_8) = \{1, 2, 3, 7\}$ $\quad I_8 = \{1, 5, 6, 7\}$

$J(\underline{p}_9) = \{1\}$ $\quad I_9 = \{3\}$

In the given problem $c_8 \leq 0$, then by rule1, the optimal solution $\underline{p} = (\bar{p}_i)_{i \in I}$ has $\bar{p}_8 = \underline{p}_8$. p_8 is also locking in equations 1,2,3,7. Hence these columns and the corresponding row of p_8 can be deleted from matrix Q.

Step 5: Consider the remaining matrix Q, apply Rules 2-5 to find the optimal solution of the problem 2.

$$Q = \begin{matrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_9 \end{matrix} \begin{pmatrix} 0.35 & 0.49 & 0.58 & 0.26 & 0.44 \\ 0.29 & 0.31 & 0.40 & 0.42 & 0.26 \\ 0.25 & 0.46 & 0.56 & 0.35 & 0.40 \\ 0.45 & 0.52 & 0.38 & 0.28 & 0.45 \\ 0.18 & 0.62 & 0.48 & 0.16 & 0.52 \\ 0.30 & 0.40 & 0.50 & 0.18 & 0.36 \\ 0.26 & 0.46 & 0.58 & 0.24 & 0.42 \\ 0.28 & 0.42 & 0.55 & 0.18 & 0.42 \end{pmatrix}$$

Index sets for the reduced matrix Q are

$$\begin{array}{ll} J(\underline{p}_1) = \{6, 8\} & I_4 = \{3, 5, 7\} \\ J(\underline{p}_2) = \{6\} & I_5 = \{6, 7\} \\ J(\underline{p}_3) = \{4, 9\} & I_6 = \{1, 2, 4, 5, 6\} \\ J(\underline{p}_4) = \{6\} & I_8 = \{1, 5, 6, 7\} \\ J(\underline{p}_5) = \{4, 6, 8\} & I_9 = \{3\} \\ J(\underline{p}_6) = \{5, 6, 8\} & \\ J(\underline{p}_7) = \{4, 5, 8\} & \\ J(\underline{p}_9) = \{\phi\} & \end{array}$$

Since $J(\underline{p}_9) = \{\phi\}$, then by rule 5 any optimal solution has $\underline{p}_9 = 1$, the corresponding row of variable p_9 can be deleted from matrix Q. Also $I_9 = \{3\}$ imlies that the variable p_3 is the only locking variable in the 9^{th} equation. By rule 2 any optimal solution has $\bar{p}_3 = \underline{p}_3$. $\underline{p}_3$ is also locking in equation 4. Hence these columns and the corresponding row of p_9 and p_3 can be deleted from matrix Q.

The reduced matrix Q becomes

$$Q = \begin{matrix} p_1 \\ p_2 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{matrix} \begin{pmatrix} 0.49 & 0.58 & 0.26 \\ 0.31 & 0.40 & 0.42 \\ 0.52 & 0.38 & 0.28 \\ 0.62 & 0.48 & 0.16 \\ 0.40 & 0.50 & 0.18 \\ 0.46 & 0.58 & 0.24 \end{pmatrix}$$

The index sets of Q are

$$\begin{array}{ll} J(\underline{p}_1) = \{6,8\} & \\ J(\underline{p}_2) = \{6\} & \\ J(\underline{p}_4) = \{6\} & I_5 = \{6,7\} \\ J(\underline{p}_5) = \{6,8\} & I_6 = \{1,2,4,5,6\} \\ J(\underline{p}_6) = \{5,6,8\} & I_8 = \{1,5,6,7\} \\ J(\underline{p}_7) = \{5,8\} & \end{array}$$

Since $I_5 \subseteq I_8$, column 5 of Q can be deleted by Rule 3.

Also $J(\underline{p}_1) = J(\underline{p}_5)$ with $c_1\underline{p}_1 < c_5\underline{p}_5$

$J(\underline{p}_2) = J(\underline{p}_4)$ with $c_4\underline{p}_4 < c_2\underline{p}_2$

Therefore we set $\bar{p}_1 = 1, \bar{p}_4 = 1$ in any optimal solution by Rule 4.
Rows corresponding to p_1 and p_4 can be deleted from matrix Q.

The reduced matrix Q becomes

$$Q = \begin{matrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{matrix} \begin{pmatrix} 0.40 & 0.42 \\ 0.48 & 0.16 \\ 0.50 & 0.18 \\ 0.58 & 0.24 \end{pmatrix}$$

The index sets are

$$\begin{array}{ll} J(\underline{p}_2) = \{6\} & \\ J(\underline{p}_5) = \{6,8\} & I_6 = \{2,5,6\} \\ J(\underline{p}_6) = \{6,8\} & I_8 = \{5,6,7\} \\ J(\underline{p}_7) = \{8\} & \end{array}$$

Since $J(\underline{p}_5) = J(\underline{p}_6)$ with $c_6\underline{p}_6 < c_5\underline{p}_5$. Therefore we set $\bar{p}_6 = 1$ by rule 4.

The row corresponding to p_6 can be deleted from Q.

The reduced matrix becomes

$$Q = \begin{matrix} p_2 \\ p_5 \\ p_7 \end{matrix} \begin{pmatrix} 0.40 & 0.42 \\ 0.48 & 0.16 \\ 0.58 & 0.24 \end{pmatrix}$$

The index sets are

$$\begin{array}{ll} J(\underline{p}_2) = \{6\} & I_6 = \{2,5,6\} \\ J(\underline{p}_5) = \{6,8\} & I_8 = \{5,6,7\} \\ J(\underline{p}_7) = \{8\} & \end{array}$$

Since $J(\underline{p}_5) = J(\underline{p}_2) \cup J(\underline{p}_7)$ and also $c_5\underline{p}_5 = c_2\underline{p}_2 + c_7\underline{p}_7$. Therefore we set $\underline{p}_5 = 1$, or $\underline{p}_2 = 1, \underline{p}_7 = 1$ by Rule 4.1.

If $\bar{p}_5 = 1$ then in order to satisfy 6^{th} and 8^{th} equation we set $\bar{p}_2 = \underline{p}_2$ and $\bar{p}_7 = \underline{p}_7$.

If $\underline{p}_2 = 1, \underline{p}_7 = 1$ then p_5 is the only variable to satisfy 6^{th} and 8^{th} equation.

$\therefore$ we set $\bar{p}_5 = \underline{p}_5$.

From the above discussion, all the decision variables have been determined goto the next step.

Step 6: Since we get two optimal solutions p^1 and p^2 as follows

$$p^1 = \begin{pmatrix}1, & 0.2, & 0.05, & 1, & 1, & 1, & 0.04, & 0.07, & 1\end{pmatrix} \text{ and}$$

$$p^2 = \begin{pmatrix}1, & 1, & 0.05, & 1, & 0.12, & 1, & 1, & 0.07, & 1\end{pmatrix}$$

The optimal value is $Z(p^1) = Z(p^2) = 18.44$

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