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$(\tilde{\alpha}, \alpha)$-Cubic new ideal of PU-algebra

Samy M. Mostafa, F. I. Sidky, Alaa Eldin I. Elkabany

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Abstract. The notions of $(\tilde{\alpha}, \alpha)$-cubic new-ideal of PU-algebras are introduced, and several related properties are investigated. Characterizations of $(\tilde{\alpha}, \alpha)$-cubic new-ideal on PU-algebras are established. The relations between $(\tilde{\alpha}, \alpha)$-cubic subalgebras and $(\tilde{\alpha}, \alpha)$-cubic new-ideal of PU-algebras are investigated. Moreover, the homomorphic image (pre image) of $(\tilde{\alpha}, \alpha)$-cubic new-ideal of a PU-algebra under homomorprhism of a PU-algebras is discussed.

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Corresponding Author: Samy M. Mostafa (samymostafa@yahoo.com)

1. Introduction

In 1966, Imai and Iseki [3, 4, 5] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2], Hu and Li introduced a wide class of abstract algebras BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Neggers et al.[11] introduced the notion of Q-algebras, which is a generalization of BCH/BCI/BCK-algebras. Megalai and Tamilarasi [8] introduced the notion of a TM-algebra which is a generalization of BCK/BCI/BCH-algebras and several results are presented. Mostafa et al.[9] introduced a new algebraic structure called PU-algebra, which is a dual for TM-algebra and investigated several basic properties. Moreover they derived new view of several ideals on PU-algebra and studied some properties of them. The concept of fuzzy sets was introduced by Zadeh [13]. In 1991, Xi [12] applied the concept of fuzzy sets to BCI, BCK, MV-algebras. Since its inception, the theory of fuzzy sets , ideal theory and its fuzzification has been developed in many directions and applied to a wide variety of fields. Mostafa et al [10] introduced the notion of α-fuzzy new-ideal of PU-algebra. They discussed the homomorphic image (pre image) of α-fuzzy new-ideal of PU-algebra under homomorprhism of PU-algebras. Jun et al. [6] introduced the notion of cubic sub-algebras /ideals in BCK/BCI-algebras, and then they investigated several properties. They discussed the relationship between a cubic sub-algebra and a cubic ideal. Also, they provided characterizations of a cubic sub-algebra/ideal and considered a method to produce a new cubic subalgebra from an old one.

In this paper, we modify the ideas of Jun [7] in order to introduce the notion $(\tilde{\alpha}, \alpha)$-cubic new-ideal of PU-algebra. The homomorphic image (preimage) of $(\tilde{\alpha}, \alpha)$-cubic new-ideal of PU-algebra under homomorprhism of PU-algebras are discussed. Finally, many related results have been derived.

2. Preliminaries

Now, we will recall some known concepts related to PU-algebra from the literature, which will be helpful in further study of this article

Definition 2.1 ([9])**.** A PU-algebra is a non-empty set X with a constant $0 \in X$ and a binary operation $*$ satisfying the following conditions :

(I) $0 * x = x$,

(II) $(x * z) * (y * z) = y * x$ for any $x, y, z \in X$. On X we can define a binary relation " $\leq$ " by: $x \leq y$ if and only if $y * x = 0$.

Example 2.2 ([9])**.** Let $X = \{0, 1, 2, 3, 4\}$ in which $*$ is defined by

$*$	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

Then $(X, *, 0)$ is a PU-algebra.

Proposition 2.3 ([9])**.** *In a PU-algebra $(X, *, 0)$ the following hold for all $x, y, z \in X$,*

(a) $x * x = 0$.
(b) $(x * z) * z = x$.
(c) $x * (y * z) = y * (x * z)$.
(d) $x * (y * x) = y * 0$.
(e) $(x * y) * 0 = y * x$.
(f) If $x \leq y$, *then* $x * 0 = y * 0$.
(g) $(x * y) * 0 = (x * z) * (y * z)$.
(h) $x * y \leq z$ *if and only if* $z * y \leq x$.
(i) $x \leq y$ *if and only if* $y * z \leq x * z$.
(j) In a PU-algebra $(X, *, 0)$, *the following are equivalent:*
(1) $x = y$, *(2)* $x * z = y * z$, *(3)* $z * x = z * y$.
(k) The right and the left cancellation laws hold in X.
(l) $(z * x) * (z * y) = x * y$,
(m) $(x * y) * z = (z * y) * x$.
(n) $(x * y) * (z * u) = (x * z) * (y * u)$ *for all* x, y, z *and* $u \in X$.

Definition 2.4 ([9]). A non-empty subset I of a PU-algebra $(X, *, 0)$ is called a sub-algebra of X if $x * y \in I$ whenever $x, y \in I$.

Definition 2.5 ([9]). A non-empty subset I of a PU-algebra $(X, *, 0)$ is called a *new*-ideal of X if,
(i) $0 \in I$,
(ii) $(a * (b * x)) * x \in I$, for all $a, b \in I$ and $x \in X$.

Example 2.6 ([9]). Let $X = \{0, a, b, c\}$, in which $*$ is defined by the following table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then $(X, *, 0)$ is a PU-algebra. It is easy to show that $I_1 = \{0, a\}$, $I_2 = \{0, b\}$, $I_3 = \{0, c\}$ are *new*-ideals of X.

Lemma 2.7 ([9]). *If $(X, *, 0)$ is a PU-algebra, then $(x * (y * z)) * z = (y * 0) * x$ for all $x, y, z \in X$.*

Theorem 2.8. *Any sub-algebra S of a PU-algebra X is a new-ideal of X.*

Proof. Let S be a sub-algebra of a PU-algebra X. Let $x \in S$, it follows by the definition of sub-algebra and properties of PU-algebra that $x * x = 0 \in S$. Let $a, b \in S$ and $x \in X$. Since $0 \in S$, then $b * 0 \in S$. Hence $(b * 0) * a \in S$. It follows (by Lemma 2.7), that $(a * (b * x)) * x = (b * 0) * a$. Then we have $(a * (b * x)) * x \in S$. Therefore S is a *new*-ideal of X. □

Definition 2.9 ([9]). Let $(X, *, 0)$ and $(X', *', 0')$ be PU-algebras. A map $f : X \to X'$ is called a homomorphism if $f(x * y) = f(x) *' f(y)$ for all $x, y \in X$.

Proposition 2.10 ([9]). *Let $(X, *, 0)$ and $(X', *', 0')$ be PU-algebras and $f : X \to X'$ be a homomorphism, then ker f is a new-ideal of X.*

Definition 2.11 ([13]). Let X be a non-empty set, a fuzzy subset μ in X is a function $\mu : X \to [0, 1]$.

Remark 2.12. An interval-valued fuzzy subset (briefly $i - v$ fuzzy subset) A defined in the set X is given by $A = \{(x, [\mu_A^L(x), \mu_A^U(x)])\}$, for all $x \in X$. (briefly, it is denoted by $A = [\mu_A^L(x), \mu_A^U(x)]$ where $\mu_A^L(x)$ and $\mu_A^U(x)$ are any two fuzzy subsets in X such that $\mu_A^L(x) \leq \mu_A^U(x)$ for all $x \in X$. Let $\tilde{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$, for all $x \in X$ and $D[0, 1]$ be denotes the family of all closed sub-intervals of $[0, 1]$. It is clear that if $\mu_A^L(x) = \mu_A^U(x) = c$, where $0 \leq c \leq 1$, then $\tilde{\mu}_A(x) = [c, c]$ in $D[0, 1]$, then $\tilde{\mu}_A(x) \in D[0m1]$, for all $x \in X$. Therefore the interval-valued fuzzy subset A is given by: $A\{\langle x, \tilde{\mu}_A(x)\rangle : x \in X\}$ where $\tilde{\mu}_A X \to D[0, 1]$. Now we define the refined minimum (briefly r min) and order " $\leq$ " on elements $D_1 = [a_1, b_1]$ and $D_2[a_2, b_2]$ of $D[0, 1]$ as follows:

$$r \min(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}], D_1 \leq D_2 \Leftrightarrow a_1 \leq a_2 \text{ and } b_1 \leq b_2.$$

Similarly we can define $(\geq)$ and $(=)$. Also we can define $D_1+D_2=[a_1+a_2,b_1+b_2]$, and if $c\in[0,1]$, then $cD_1=[ca_1,cb_1]$. Also if $D_1=[a_i,b_i], i\in I$, then we define $r\sup(D_i)=[\sup a_i,\sup b_i]$ and $r\inf(D_i)=[\inf a_i,\inf b_i]$. We willconsider that $\tilde{1}=[1,1]$ and $\tilde{0}=[0,0]$.
Jun et al. [6], introduced the concept of cubic sets defined on a non-empty set X as objects having the form: $A=\{\langle x,\tilde{\mu}_A(x),\lambda_A(x)\rangle : x\in X\}$, which is briefly denoted by $A=\langle\tilde{\mu}_A,\lambda_A\rangle$, where the functions $\tilde{\mu}_A : X\to F[0,1]$ and $\lambda_A : X\to[0,1]$.
Denote $C(X)$ by family of all cubic sets in X.

In what follows, let X denotes a PU-algebra unless otherwise specified.

3. $(\tilde{\alpha},\alpha)$-cubic new-ideal of PU-algebra

In this section, we shall introduce a new notion called $(\tilde{\alpha},\alpha)$-cubic *new*-ideal of a PU-algebra and study several properties of it.

Definition 3.1. Let X be a PU-algebra. A cubic set $A=\langle\tilde{\mu}_A,\lambda_A\rangle$ in X is called a cubic sub-algebra of X if
$(SC_1)\tilde{\mu}_A(x*y)\geq r\min\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}$,
$(SC_2)\lambda_A(x*y)\leq\max\{\lambda_A(x),\lambda_A(y)\}$, where $\tilde{\mu}_A : X\to D[0,1]$ and $\lambda_A : X\to[0,1]$ for all $x,y\in X$.

Definition 3.2. Let $A=\langle\tilde{\mu}_A,\lambda_A\rangle$ be a cubic set of a PU-algebra X. Let $\tilde{\alpha}\in D[0,1]$ and $\alpha\in[0,1]$. Then the cubic set $A^{(\tilde{\alpha},\alpha)}=\langle\tilde{\mu}_A^{\tilde{\alpha}},\lambda_A^{\alpha}\rangle$ of X is called the $(\tilde{\alpha},\alpha)$-cubic subset of X (w.r.t. cubic set $A=\langle\tilde{\mu}_A,\lambda_A\rangle$) if
$(a)\tilde{\mu}_A^{\tilde{\alpha}}(x)=r\min\{\tilde{\mu}_A,(x),\tilde{\alpha}\}$ and
$(b)\lambda_A^{\alpha}(x)=\max\{\lambda_A(x),\alpha\}$, for all $x\in X$.

Remark 3.3. Clearly, $\langle\tilde{\mu}_A^{\tilde{1}},\lambda_A^{0}\rangle=\langle\tilde{\mu}_A,\lambda_A\rangle$.

Lemma 3.4. *$\langle\tilde{\mu}_A,\lambda_A\rangle$ is a cubic sub-algebra of a PU-algebra X such that $A^{(\tilde{\alpha},\alpha)}$ is a $(\tilde{\alpha},\alpha)$-cubic subset, where $\tilde{\alpha}\in D[0,1]$ and $\alpha\in[0,1]$, then $\tilde{\mu}_A^{\tilde{\alpha}}(x*y)\geq r\min\{\tilde{\mu}_A^{\tilde{\alpha}}(x),\tilde{\mu}_A^{\tilde{\alpha}}(y)\}$ and $\lambda_A^{\alpha}(x*y)\leq\max\{\lambda_A^{\alpha}(x),\lambda_A^{\alpha}(y)\}$, for all $x,y\in X$.*

Proof. Let X be a PU-algebra, $\tilde{\alpha}\in D[0,1]$ and $\alpha\in[0,1]$. Then by Definition 3.2. we have that

$$\begin{aligned}\tilde{\mu}_A^{\tilde{\alpha}}(x*y)=r\min\{\tilde{\mu}_A(x*y),\tilde{\alpha}\} &\geq r\min\{r\min\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\},\tilde{\alpha}\}\\ &= r\min\{r\min\{\tilde{\mu}_A(x),\tilde{\alpha}\},r\min\{\tilde{\mu}_A(y),\tilde{\alpha}\}\}\\ &= r\min\{\tilde{\mu}_A^{\tilde{\alpha}}(x),\tilde{\mu}_A^{\tilde{\alpha}}(y)\},\ \text{for all } x,y\in X\end{aligned}$$

And

$$\begin{aligned}\lambda_A^{\alpha}(x*y)=\max\{\lambda_A(x*y),\alpha\} &\leq \max\{\max\{\lambda_A(x),\lambda_A(y)\},\alpha\}\\ &= \max\{\max\{\lambda_A(x),\alpha\},\max\{\lambda_A(y),\alpha\}\}\\ &= \max\{\lambda_A^{\alpha}(x),\lambda_A^{\alpha}(y)\},\ \text{for all } x,y\in X.\end{aligned}$$

□

Definition 3.5. Let X be a PU-algebra. A cubic subset $A^{(\tilde{\alpha},\alpha)}=\langle\tilde{\mu}_A^{\tilde{\alpha}},\lambda_A^{\alpha}\rangle$ in X is called $(\tilde{\alpha},\alpha)$-cubic sub-algebra of X if
$(C_{\tilde{\alpha}})\tilde{\mu}_A^{\tilde{\alpha}}(x*y)\geq r\min\{\tilde{\mu}_A^{\tilde{\alpha}}(x),\tilde{\mu}_A^{\tilde{\alpha}}(y)\}$,

$(C_\alpha)\lambda_A^\alpha(x*y) \leq \max\{\lambda_A^\alpha(x), \lambda_A^\alpha(y)\}$, for all $x, y \in X$.

It is clear that $(\tilde{\alpha}, \alpha)$-cubic sub-algebra of a PU-algebra X is a generalization of a cubic sub-algebra of X and a cubic sub-algebra of X is a special case when $\tilde{\alpha} = \tilde{1}$ and $\alpha = 0$.

Definition 3.6. Let $(X, *, 0)$ be a PU-algebra, a cubic subset $A = \langle \tilde{\mu}_A, \lambda_A \rangle$ in X is called a cubic *new*-ideal of X if it satisfies the following conditions:
$(A_1)\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$,
$(A_2)\tilde{\mu}_A((x*(y*z)*z) \geq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and $\lambda_A((x*(y*z)*z) \leq \max\{\lambda_A(x), \lambda_A(y)\}$, for all $x, y, z \in X$.

Lemma 3.7. *$A = \langle \tilde{\mu}_A, \lambda_A \rangle$ is a cubic new-ideal of a PU-algebra X, such that $A^{(\tilde{\alpha},\alpha)}$ is a $(\tilde{\alpha}, \alpha)$-cubic subset, where $\tilde{\alpha} \in D[0,1]$ and $\alpha \in [0,1]$, then*
1. *$\tilde{\mu}_A^{\tilde{\alpha}}(0) \geq \tilde{\mu}_A^{\tilde{\alpha}}(x)$ and $\lambda_A^\alpha(0) \leq \lambda_A^\alpha(x)$,*
2. *$\tilde{\mu}_A^{\tilde{\alpha}}((x*(y*z)*z) \geq r\min\{\tilde{\mu}_A^{\tilde{\alpha}}(x), \tilde{\mu}_A^{\tilde{\alpha}}(y)\}$,*
3. *$\lambda_A^\alpha((x*(y*z)*z) \leq \max\{\lambda_A^\alpha(x), \lambda_A^\alpha(y)\}$, for all $x, y, z \in X$.*

Proof. Let X be a PU-algebra, $\tilde{\alpha} \in D[0,1]$ and $\alpha \in [0,1]$. Then we have that:
$\tilde{\mu}_A^{\tilde{\alpha}}(0) = r\min\{\tilde{\mu}_A^(0), \tilde{\alpha}\} \geq r\min\{\tilde{\mu}_A^(x), \tilde{\alpha}\} = \tilde{\mu}_A^{\tilde{\alpha}}(x)$,
$\lambda_A^\alpha(0) = \max\{\lambda_A(0), \alpha\} \leq \max\{\lambda_A(x), \alpha\} = \lambda_A^\alpha(x)$, for all $x \in X$. And

$$\begin{aligned}\tilde{\mu}_A^{\tilde{\alpha}}((x*(y*z)*z) &= r\min\{\tilde{\mu}_A((x*(y*z))*z), \tilde{\alpha}\} \geq r\min\{r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}, \tilde{\alpha}\} \\ &= r\min\{r\min\{\tilde{\mu}_A(x), \tilde{\alpha}\}, r\min\{\tilde{\mu}_A(y), \tilde{\alpha}\}\} = r\min\{\tilde{\mu}_A^{\tilde{\alpha}}(x).\tilde{\mu}_A^{\tilde{\alpha}}(y)\},\end{aligned}$$

$$\lambda_A^\alpha((x*(y*z))*z) = \max\{\lambda_A((x*(y*z))*z), \alpha\} \leq \max\{\max\{\lambda_A(x), \lambda_A(y)\}, \alpha\}$$
$$= \max\{\max\{\lambda_A(x), \alpha\}, \max\{\lambda_A(y), \alpha\}\} = \max\{\lambda_A^\alpha(x), \lambda_A^\alpha(y)\}, \text{ for all } x, y, z \in X.$$
□

Definition 3.8. Let $(X, *, 0)$ be a PU-algebra, $(\tilde{\alpha}, \alpha)$-cubic subset $A^{(\tilde{\alpha},\alpha)} = \langle \tilde{\mu}_A^{\tilde{\alpha}}, \lambda_A^\alpha \rangle$ in X is called $(\tilde{\alpha}, \alpha)$-cubic *new*-ideal of X if it satisfies the following conditions:
$(A_0^{(\tilde{\alpha},\alpha)})\tilde{\mu}_A^{\tilde{\alpha}}(0) \geq \tilde{\mu}_A^{\tilde{\alpha}}(x)$ and $\lambda_A^\alpha(0) \leq \lambda_A^\alpha(x)$,
$(A_1^{(\tilde{\alpha},\alpha)})\tilde{\mu}_A^{\tilde{\alpha}}((x*(y*z))*z) \geq r\min\{\tilde{\mu}_A^{\tilde{\alpha}}(x), \tilde{\mu}_A^{\tilde{\alpha}}(y)\}$ and
$(A_2^{(\tilde{\alpha},\alpha)})\lambda_A^\alpha((x*(y*z))*z) \leq \max\{\lambda_A^\alpha(x), \lambda_A^\alpha(y)\}$, for all $x, y, z \in X$.
It is clear that $(\tilde{\alpha}, \alpha)$-cubic *new*-ideal of a PU-algebra X is a generalization of a cubic *new*-ideal of X and a cubic *new*-ideal of X is special case, when $\tilde{\alpha} = \tilde{1}$ and $\alpha = 0$.

Example 3.9. Let $X = \{0, 1, 2, 3\}$ in which $*$ is defined by the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X, *, 0)$ is a PU-algebra. Define $(\tilde{\alpha}, \alpha)$-cubic subset $A^{(\tilde{\alpha},\alpha)} = \langle \tilde{\mu}_A^{\tilde{\alpha}}, \lambda_A^\alpha \rangle\rangle$ in X by

$$\tilde{\mu}_A^{\tilde{\alpha}}(x) = \begin{cases} r\min\{\tilde{\alpha}, [0.3, 0.9]\} & if\ x \in \{0, 1\} \\ r\min\{\tilde{\alpha}, [0.1, 0.6]\} & otherwise \end{cases}$$

and

$$\lambda_A^{\alpha}(x) = \begin{cases} \max\{\alpha, 0.3\} & if\ x \in \{0, 1\} \\ \max\{\alpha, 0.7\} & otherwise. \end{cases}$$

Routine calculation gives that $A^{(\tilde{\alpha},\alpha)} = \langle \tilde{\mu}_A^{\tilde{\alpha}}, \lambda_A^{\alpha}\rangle$ is an $(\tilde{\alpha},\alpha)$-cubic *new*-ideal of X.

Lemma 3.10. *Let $A^{(\tilde{\alpha},\alpha)} = \langle \tilde{\mu}_A^{\tilde{\alpha}}, \lambda_A^{\alpha}\rangle$ be $(\tilde{\alpha},\alpha)$-cubic new-ideal of a PU-algebra X. If the inequality $x * y \leq z$ holds in X, then $\tilde{\mu}_A^{\tilde{\alpha}}(y) \geq r\min\{\tilde{\mu}_A^{\tilde{\alpha}}(x), \tilde{\mu}_A^{\tilde{\alpha}}(z)\}$ and $\lambda_A^{\alpha}(y) \leq \max\{\lambda_A^{\alpha}(x), \lambda_A^{\alpha}(z)\}$.*

Proof. Assume that the inequality $x * y \leq z$ holds in X, then $z * (x * y) = 0$ and by $(A_1^{(\tilde{\alpha},\alpha)})$, we have

$$\tilde{\mu}_A^{\tilde{\alpha}} \ (\overbrace{((z * (x * y))}^{0} * y) \geq r\min\{\tilde{\mu}_A^{\tilde{\alpha}}(x), \tilde{\mu}_A^{\tilde{\alpha}}(z)\}.$$

Since $\tilde{\mu}_A^{\tilde{\alpha}}(y) = \tilde{\mu}_A^{\tilde{\alpha}}(0 * y)$, then we have that $\tilde{\mu}_A^{\tilde{\alpha}}(y) \geq r\min\{\tilde{\mu}_A^{\tilde{\alpha}}(x), \tilde{\mu}_A^{\tilde{\alpha}}(z)\}$.
By $(A_2^{(\tilde{\alpha},\alpha)})$, we obtain

$$\lambda_A^{\alpha} \ (\overbrace{((z * (x * y))}^{0} * y) \leq \max\{\lambda_A^{\alpha}(x), \lambda_A^{\alpha}(z)\}.$$

Since $\lambda_A^{\alpha}(y) = \lambda_A^{\alpha}(0 * y)$, therefore we have that $\lambda_A^{\alpha}(y) \leq \max\{\lambda_A^{\alpha}(x), \lambda_A^{\alpha}(z)\}$. □

Corollary 3.11. *Let $A = \langle \tilde{\mu}_A, \lambda_A\rangle$ be a cubic new-ideal of a PU-algebra X. If the inequality $x * y \leq z$ holds in X, then $\tilde{\mu}_A(y) \geq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(z)\}$. and $\lambda_A(y) \leq \max\{\lambda_A(x), \lambda_A(z)\}$.*

Lemma 3.12. *If $A^{(\tilde{\alpha},\alpha)} = \langle \tilde{\mu}_A^{\tilde{\alpha}}, \lambda_A^{\alpha}\rangle$ is $(\tilde{\alpha},\alpha)$-cubic subset of a PU-algebra X and $x \leq y$, then $\tilde{\mu}_A^{\tilde{\alpha}}(x) = \tilde{\mu}_A^{\tilde{\alpha}}(y)$ and $\lambda_A^{\alpha}(x) = \lambda_A^{\alpha}(y)$.*

Proof. If $x \leq y$, then $y * x = 0$. Hence by the definition of PU-algebra and its properties we have that $\tilde{\mu}_A^{\tilde{\alpha}}(x) = r\min\{\tilde{\mu}_A(x), \tilde{\alpha}\} = r\min\{\tilde{\mu}_A(0 * x), \tilde{\alpha}\} = r\min\{\tilde{\mu}_A((y * x) * x), \tilde{\alpha}\} = r\min\{\tilde{\mu}_A(y), \tilde{\alpha}\} = \tilde{\mu}_A^{\tilde{\alpha}}(y)$ and $\lambda_A^{\alpha}(x) = \max\{\lambda_A(x), \alpha\} = \max\{\lambda_A(0 * x), \alpha\} = \max\{\lambda_A(y), \alpha\} = \lambda_A^{\alpha}(y)$. □

Corollary 3.13. *If $A = \langle \tilde{\mu}_A, \lambda_A\rangle$ is a cubic subset of a PU-algebra X and if $x \leq y$, then $\tilde{\mu}_A(x) = \tilde{\mu}_A(y)$ and $\lambda_A(x) = \lambda_A(y)$.*

Definition 3.14. Let $A^{(\tilde{\alpha},\alpha)} = \langle \tilde{\mu}_A^{\tilde{\alpha}}, \lambda_A^{\alpha}\rangle$ and $B^{(\tilde{\alpha},\alpha)} = \langle \tilde{\mu}_B^{\tilde{\alpha}}, \lambda_B^{\alpha}\rangle$ be two $(\tilde{\alpha},\alpha)$-cubic sets in a PU-algebra X, then we define

$$\begin{aligned} A^{(\tilde{\alpha},\alpha)} \bigcap B^{(\tilde{\alpha},\alpha)} &= \{\langle x, r\min\{\tilde{\mu}_A^{\tilde{\alpha}}(x), \tilde{\mu}_B^{\tilde{\alpha}}(x)\}, \max\{\lambda_A^{\alpha}(x), \lambda_B^{\alpha}(x)\}\rangle : x \in X\} \\ &= \{\langle x, \tilde{\mu}_A^{\tilde{\alpha}}(x) \bigcap \tilde{\mu}_B^{\tilde{\alpha}}(x), \lambda_A^{\alpha}(x) \bigcup \lambda_B^{\alpha}(x)\rangle\rangle : x \in X\} \end{aligned}$$

Proposition 3.15. *If $\{A_i^{(\tilde{\alpha},\alpha)}\}_{i\in I}$ is a family of $(\tilde{\alpha},\alpha)$-cubic new-ideals of a PU-algebra X, then $\bigcap\limits_{i\in I} A_i^{(\tilde{\alpha},\alpha)}$ is $(\tilde{\alpha},\alpha)$-cubic new-ideal of X.*

Proof. Let $\{A_i^{(\tilde{\alpha},\alpha)}\}_{i\in I}$ be a family of $(\tilde{\alpha},\alpha)$-cubic *new*-ideals of a PU-algebra X, then for any $x, y, z \in X$,

$$(\bigcap_{i\in I} \tilde{\mu}_{A_i}^{\tilde{\alpha}})(0) = r\inf(\tilde{\mu}_{A_i}^{\tilde{\alpha}}(0))_{i\in I} \geq r\inf(\tilde{\mu}_{A_i}^{\tilde{\alpha}}(x))_{i\in I} = (\bigcap_{i\in I} \tilde{\mu}_{A_i}^{\tilde{\alpha}})(x)$$

and

$$\begin{aligned}
&(\bigcup_{i\in I} \lambda^{\alpha}_{A_i})(0) = \sup(\lambda^{\alpha}_{A_i}(0))_{i\in I} \leq \sup(\lambda^{\alpha}_{A_i}(x))_{i\in I} = (\bigcup_{i\in I} \lambda^{\alpha}_{A_i})(x),\\
&(\bigcap_{i\in I} \tilde{\mu}^{\tilde{\alpha}}_{A_i})((x*(y*z))*z) = r\inf(\tilde{\mu}^{\tilde{\alpha}}_{A_i}((x*(y*z)))*z)_{i\in I}\\
&\geq r\inf(r\min\{\tilde{\mu}^{\tilde{\alpha}}_{A_i}(x), \tilde{\mu}^{\tilde{\alpha}}_{A_i}(y)\})_{i\in I}\\
&= r\min\{r\inf(\tilde{\mu}^{\tilde{\alpha}}_{A_i}(x))_{i\in I}, r\inf(\tilde{\mu}^{\tilde{\alpha}}_{A_i}(y))_{i\in I}\}\\
&= r\min\{(\bigcap_{i\in I} \tilde{\mu}^{\tilde{\alpha}}_{A_i})(x), (\bigcap_{i\in I} \tilde{\mu}^{\tilde{\alpha}}_{A_i})(y)\}
\end{aligned}$$

and

$$\begin{aligned}
&(\bigcup_{i\in I} \lambda^{\alpha}_{A_i})((x*(y*z)*z) = \sup(\lambda^{\alpha}_{A_i}((x*(y*z)*z)_{i\in I} \leq \sup(\max\{\lambda^{\alpha}_{A_i}(x), \mu^{\alpha}_{A_i}(y)\})_{i\in I}\\
&= \max\{\sup(\lambda^{\alpha}_{A_i}(x))_{i\in I}, \sup(\lambda^{\alpha}_{A_i}(y))_{i\in I}\} = \max\{(\bigcap_{i\in I} \lambda^{\alpha}_{A_i})(x), (\bigcap_{i\in I} \lambda^{\alpha}_{A_i})(y)\}.
\end{aligned}$$

This completes the proof. □

Theorem 3.16. *If $A^{(\tilde{\alpha},\alpha)} = \langle \tilde{\mu}^{\tilde{\alpha}}_A, \lambda^{\alpha}_A \rangle$ is $(\tilde{\alpha}, \alpha)$-cubic subset of a PU-algebra X, then $A^{(\tilde{\alpha},\alpha)} = \langle \tilde{\mu}^{\tilde{\alpha}}_A, \lambda^{\alpha}_A \rangle$ is $(\tilde{\alpha}, \alpha)$-cubic new-ideal of X if and only if it satisfies:*
$(\forall \tilde{\varepsilon} \in D[0,1])(U(\tilde{\mu}^{\tilde{\alpha}}_A; \tilde{\varepsilon}) \neq \phi \Rightarrow U(\tilde{\mu}^{\tilde{\alpha}}_A; \tilde{\varepsilon})$ is a new-ideal of X),
where, $U(\tilde{\mu}^{\tilde{\alpha}}_A; \tilde{\varepsilon}) = \{x \in X : \tilde{\mu}^{\tilde{\alpha}}_A(x) \geq \tilde{\varepsilon}\}$ and $(\forall \varepsilon \in [0,1], (L(\lambda^{\alpha}_A; \varepsilon) \neq \phi \Rightarrow L(\lambda^{\alpha}_A; \varepsilon)$ is a new-ideal of X), where $L(\lambda^{\alpha}_A; \varepsilon) = \{x \in X : \lambda^{\alpha}_A(x) \leq \varepsilon\}$.

Proof. Assume that $A^{(\tilde{\alpha},\alpha)} = \langle \tilde{\mu}^{\tilde{\alpha}}_A, \lambda^{\alpha}_A \rangle$ is an $(\tilde{\alpha}, \alpha)$-cubic *new*-ideal of X. Let $\tilde{\varepsilon} \in D[0,1]$ be such that $U(\tilde{\mu}^{\tilde{\alpha}}_A; \tilde{\varepsilon}) \neq \phi$. Let $x \in U(\tilde{\mu}^{\tilde{\alpha}}_A; \tilde{\varepsilon})$, then $\tilde{\mu}^{\tilde{\alpha}}_A(x) \geq \tilde{\varepsilon}$. Since $\tilde{\mu}^{\tilde{\alpha}}_A(0) \geq \tilde{\mu}^{\tilde{\alpha}}_A(x)$ for all $x \in X$, then $\tilde{\mu}^{\tilde{\alpha}}_A(0) \geq \tilde{\varepsilon}$. Thus $0 \in U(\tilde{\mu}^{\tilde{\alpha}}_A; \tilde{\varepsilon})$. Let $x \in X$ and $a, b \in U(\tilde{\mu}^{\tilde{\alpha}}_A; \tilde{\varepsilon})$, then $\tilde{\mu}^{\tilde{\alpha}}_A(a) \geq \tilde{\varepsilon}$ and $\tilde{\mu}^{\tilde{\alpha}}_A(b) \geq \tilde{\varepsilon}$. It follows by the definition of $(\tilde{\alpha}\alpha)$-cubic *new*-ideal that $\tilde{\mu}^{\tilde{\alpha}}_A((a*(b*x))*x) \geq \min\{\tilde{\mu}^{\tilde{\alpha}}_A(a), \tilde{\mu}^{\tilde{\alpha}}_A(b)\} \geq \tilde{\varepsilon}$, so that $(a*(b*x))*x \in U(\tilde{\mu}^{\tilde{\alpha}}_A; \tilde{\varepsilon})$. Hence $U(\tilde{\mu}^{\tilde{\alpha}}_A; \tilde{\varepsilon})$ is a *new*-ideal of X. Let $\varepsilon \in [0,1]$ be such that $L(\lambda^{\alpha}_A; \varepsilon) \neq \phi$. Let $x \in L(\lambda^{\alpha}_A; \varepsilon)$, then $\lambda^{\alpha}_A(x) \leq \varepsilon$. Since $\lambda^{\alpha}_A(0) \leq \lambda^{\alpha}_A(x)$ for all $x \in X$, then $\lambda^{\alpha}_A(0) \leq \varepsilon$. Thus $0 \in L(\lambda^{\alpha}_A; \varepsilon)$. Let $x \in X$ and $a, b \in L(\lambda^{\alpha}_A; \varepsilon)$, then $\lambda^{\alpha}_A(a) \leq \varepsilon$ and $\lambda^{\alpha}_A(b) \leq \varepsilon$. It follows by the definition of $(\tilde{\alpha}, \alpha)$-cubic *new*-ideal that $\lambda^{\alpha}_A((a*(b*x))*x) \leq \max\{\lambda^{\alpha}_A(a), \lambda^{\alpha}_A(b)\} \leq \varepsilon$, so that $(a*(b*x))*x \in L(\lambda^{\alpha}_A; \varepsilon)$. Hence $L(\lambda^{\alpha}_A; \varepsilon)$ is a *new*-ideal of X.
Conversely, suppose that $(\forall \tilde{\varepsilon} \in D[0,1])(U(\tilde{\mu}^{\tilde{\alpha}}_A; \tilde{\varepsilon}) \neq \phi \Rightarrow U(\tilde{\mu}^{\tilde{\alpha}}_A; \tilde{\varepsilon})$ is a *new*-ideal of X), where $U(\tilde{\mu}^{\tilde{\alpha}}_A; \tilde{\varepsilon}) = \{x \in X : \tilde{\mu}^{\tilde{\alpha}}_A(x) \geq \tilde{\varepsilon}\}$ and $(\forall \varepsilon \in [0,1])(L(\lambda^{\alpha}_A; \varepsilon) \neq \phi \Rightarrow (L(\lambda^{\alpha}_A; \varepsilon)$ is a *new*-ideal of X), where $(\lambda^{\alpha}_A; \varepsilon) = \{x \in X : \lambda^{\alpha}_A(x) \leq \varepsilon\}$. If $\tilde{\mu}^{\tilde{\alpha}}_A(0) < \tilde{\mu}^{\tilde{\alpha}}_A(x)$ for some $x \in X$, then $\tilde{\mu}^{\tilde{\alpha}}_A(0) < \tilde{\varepsilon}_0 < \tilde{\mu}^{\tilde{\alpha}}_A(x)$ by taking $\tilde{\varepsilon}_0 = (\tilde{\mu}^{\tilde{\alpha}}_A(0) + \tilde{\mu}^{\tilde{\alpha}}_A(x))/2$. Hence $0 \notin U(\tilde{\mu}^{\tilde{\alpha}}_A; \tilde{\varepsilon}_0)$, which is a contradiction. If $\lambda^{\alpha}_A(0) > \lambda^{\alpha}_A(x)$ for some $x \in X$, then $\lambda^{\alpha}_A(0) > \varepsilon_0 > \lambda^{\alpha}_A(x)$ be taking $\varepsilon_0 = (\lambda^{\alpha}_A(0) + \lambda^{\alpha}_A(x))/2$. Hence $0 \notin L(\lambda^{\alpha}_A; \varepsilon)$, which is a contradiction.
Let $a, b, c \in X$ be such that $\tilde{\mu}^{\tilde{\alpha}}_A((a*(b*c))*c) < r\min\{\tilde{\mu}^{\tilde{\alpha}}_A(a), \tilde{\mu}^{\tilde{\alpha}}_A(b)\}$. Taking $\tilde{\varepsilon}_1 = (\tilde{\mu}^{\tilde{\alpha}}_A(a*(b*c)*c) + r\min\{\tilde{\mu}^{\tilde{\alpha}}_A(a), \tilde{\mu}^{\tilde{\alpha}}_A(b)\})/2$, we have $\tilde{\varepsilon}_1 \in D[0,1]$ and $\tilde{\mu}^{\tilde{\alpha}}_A((a*(b*c))*c) < \tilde{\varepsilon}_1 < r\min\{\tilde{\mu}^{\tilde{\alpha}}_A(a), \tilde{\mu}^{\tilde{\alpha}}_A(b)\}$. It follows that $a, b \in U(\tilde{\mu}^{\tilde{\alpha}}_A; \tilde{\varepsilon_1})$ and $(a*(b*c))*c \notin U(\tilde{\mu}^{\tilde{\alpha}}_A; \tilde{\varepsilon_1})$. This is a contradiction.
Let $a, b, c \in X$ be such that $\lambda^{\alpha}_A((a*(b*c))*c) > \max\{\lambda^{\alpha}_A(a), \lambda^{\alpha}_A(b)\}$. Taking $\varepsilon_{=}(\lambda^{\alpha}_A(a*(b*c)*c) + \max\{\lambda^{\alpha}_A(a), \lambda^{\alpha}_A(b)\})/2$, we have $\varepsilon + 1 \in [0,1]$ and $\lambda^{\alpha}_A((a*(b*c))*c) > \varepsilon_1 > \max\{\lambda^{\alpha}_A(a), \lambda^{\alpha}_A(b)\}$. It follows that $a, b \in L(\lambda^{\alpha}_A; \varepsilon + 1)$ and

$(a*(b*c))*c \notin L(\lambda_A^{\alpha};\varepsilon_1)$, which is a contradiction. Therefore $A^{(\tilde{\alpha},\alpha)} = \langle \tilde{\mu}_A^{\tilde{\alpha}}, \lambda_A^{\alpha}\rangle$ is $(\tilde{\alpha},\alpha)$-cubic *new*-ideal of X. □

Corollary 3.17. *Let $A = \langle \tilde{\mu}_A, \lambda_A\rangle$ be a cubic subset of a PU-algebra X. Then $A = \langle \tilde{\mu}_A, \lambda_A\rangle$ is a cubic new-ideal of X if and only if it satisfies:*
$(\forall \tilde{\varepsilon} \in D[0,1])(U(\tilde{\mu}_A, \tilde{\varepsilon}) \neq \phi \Rightarrow U(\tilde{\mu}_A;\tilde{\varepsilon})$ is a new-ideal of X),
where, $U(\tilde{\mu}_A, \tilde{\varepsilon}) = \{x \in X : \tilde{\mu}(x) \geq \tilde{\varepsilon}\}$; and $(\forall \varepsilon \in [0,1](L(\lambda_A;\varepsilon) \neq \phi \Rightarrow L(\lambda_A;\varepsilon)$ is a new-ideal of X), where $(\lambda_A;\varepsilon) = \{x \in X : \lambda_A(x) \leq \varepsilon\}$.

Definition 3.18. Let f be a mapping from X to Y. If $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha}\rangle$ is an $(\tilde{\alpha},\alpha)$-cubic subset of X, then the $(\tilde{\alpha},\alpha)$-cubic subset $\langle \tilde{\beta}^{\tilde{\alpha}}, \gamma^{\alpha}\rangle$ of Y defined by

$$f(\tilde{\mu}^{\tilde{\alpha}})(y) = \tilde{\beta}^{\tilde{\alpha}}(y) = \begin{cases} r\sup\limits_{x\in f^{-1}(y)} \tilde{\mu}^{\tilde{\alpha}}(x) & if\ \ f^{-1}(y) \neq \phi \\ \tilde{0} & otherwise \end{cases}$$

and

$$f(\lambda^{\alpha})(y) = \gamma^{\alpha}(y) = \begin{cases} \inf\limits_{x\in f^{-1}(y)} \lambda^{\alpha}(x) & if\ \ f^{-1}(y) \neq \phi \\ 1 & otherwise \end{cases}$$

is said to be the image of $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha}\rangle$ under f.
Similarly if $\langle \tilde{\beta}^{\tilde{\alpha}}, \gamma^{\alpha}\rangle$ is an $(\tilde{\alpha},\alpha)$-cubic subset of Y, then the $(\tilde{\alpha},\alpha)$-cubic subset $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha}\rangle = \langle \tilde{\beta}^{\tilde{\alpha}}\circ f, \gamma^{\alpha}\circ f\rangle$ of X, where $(\tilde{\beta}^{\tilde{\alpha}}\circ f)(x) = \tilde{\beta}^{\tilde{\alpha}}(f(x)), (\gamma^{\alpha}\circ f)(x) = \gamma^{\alpha}(f(x))$ for all $x \in X$) is called the pre-image of $\langle \tilde{\beta}^{\tilde{\alpha}}, \gamma^{\alpha}\rangle$ under f.

Theorem 3.19. *Let $(X,*,0)$ and $(X',*',0')$ be PU-algebras and $f : X \to X'$ be a homomorphism. If $\langle \tilde{\beta}^{\tilde{\alpha}}, \gamma^{\alpha}\rangle$ is an $(\tilde{\alpha},\alpha)$-cubic new-ideal of X' and $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha}\rangle$ is the pre-image of $\langle \tilde{\beta}^{\tilde{\alpha}}, \gamma^{\alpha}\rangle$ under f, then $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha}\rangle$ is an $(\tilde{\alpha},\alpha)$-cubic new-ideal of X.*

Proof. Since $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha}\rangle$ is the pre-image of $\langle \tilde{\beta}^{\tilde{\alpha}}, \gamma^{\alpha}\rangle$ under f, then for all $x, y, z \in X$, $\tilde{\mu}^{\tilde{\alpha}}(0) = \tilde{\beta}^{\tilde{\alpha}}(f(0)) \geq \tilde{\beta}^{\tilde{\alpha}}(f(x)) = \tilde{\mu}^{\tilde{\alpha}}(x)$ and $\lambda^{\alpha}(0) = \gamma^{\alpha}(f(0)) \leq \gamma^{\alpha}(f(x)) = \lambda^{\alpha}(x)$,

$$\begin{aligned} \tilde{\mu}^{\tilde{\alpha}}((x*(y*z))*z) &= \tilde{\beta}^{\tilde{\alpha}}(f((x*z))*z) \\ &= \tilde{\beta}^{\tilde{\alpha}}(f(x*(y*z)) *' f(z)) \\ &= \tilde{\beta}^{\tilde{\alpha}}((f(x) *' f(y*z)) *' f(z)) \\ &= \tilde{\beta}^{\tilde{\alpha}}((f(x) *' (f(y) *' f(z))) *' f(z)) \\ &\geq r\min\{\tilde{\beta}^{\tilde{\alpha}}(f(x)), \tilde{\beta}^{\tilde{\alpha}}(f(y))\} = r\min\{\tilde{\mu}^{\tilde{\alpha}}(x), \tilde{\mu}^{\tilde{\alpha}}(y)\} \end{aligned}$$

and

$$\begin{aligned} \lambda^{\alpha}((x*(y*z))*z) &= \gamma^{\alpha}(f((x*(y*z))*z)) \\ &= \gamma^{\alpha}(f(x*(y*z)) *' f(z)) \\ &= \gamma^{\alpha}((f(x) *' f(y*z)) *' f(z)) \\ &= \gamma^{\alpha}((f(x) *' (f(y) *' f(z))) *' f(z)) \\ &\leq \max\{\gamma^{\alpha}(f(x)), \gamma^{\alpha}(f(y))\} = \max\{\lambda^{\alpha}(x), \lambda^{\alpha}(y)\}. \end{aligned}$$

Therefore $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha}\rangle$ is $(\tilde{\alpha},\alpha)$-cubic *new*-ideal of X. □

Theorem 3.20. *Let $(X,*,0)$ and $(Y,*',0')$ be PU-algebras, $f : X \to Y$ be a homomorphism, $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha}\rangle$ be an $(\tilde{\alpha},\alpha)$-cubic subset of X, $\langle \tilde{\beta}^{\tilde{\alpha}}, \gamma^{\alpha}\rangle$ be the image of $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha}\rangle$ under f and $\tilde{\mu}^{\tilde{\alpha}}(x) = \tilde{\beta}^{\tilde{\alpha}}(f(x)), \lambda^{\alpha}(x) = \gamma^{\alpha}(f(x))$ for all $x \in X$. If $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha}\rangle$ is an $(\tilde{\alpha},\alpha)$-cubic new-ideal of Y.*

Proof. Since $0 \in f^{-1}(0')$, then $f^{-1}(0') \neq \phi$. It follows that

$$\tilde{\beta}^{\tilde{\alpha}}(0') = r \sup_{t \in f^{-1}(0')} \tilde{\mu}^{\tilde{\alpha}}(t) = \tilde{\mu}^{\tilde{\alpha}}(0) \geq \tilde{\mu}^{\tilde{\alpha}}(x),$$

for all $x \in X$. Thus $\tilde{\beta}^{\tilde{\alpha}}(0') = r \sup_{t \in f^{-1}(x')} \tilde{\mu}^{\tilde{\alpha}}(t)$ for all $x' \in Y$. So $\tilde{\beta}^{\tilde{\alpha}}(0') \geq \tilde{\beta}^{\tilde{\alpha}}(x')$ for all $x' \in Y$.
Since $0 \in f^{-1}(0')$, $f^{-1}(0') \neq \phi$. It follows that

$$\gamma^{\alpha}(0') = \inf_{t \in f^{-1}(0')} \lambda^{\alpha}(t) = \lambda^{\alpha}(0) \leq \lambda^{\alpha}(x),$$

for all $x \in X$. Thus $\gamma^{\alpha}(0') = \inf_{t \in f^{-1}(x')} \lambda^{\alpha}(t)$ for all $x' \in Y$. So $\gamma^{\alpha}(0') \leq \gamma^{\alpha}(x')$ for all $x' \in Y$.
For any $x', y', z' \in Y$. If $f^{-1}(x') = \phi$ or $f^{-1}(y') = \phi$, then $\tilde{\beta}^{\tilde{\alpha}}(x') = \tilde{0}$ or $\tilde{\beta}^{\tilde{\alpha}}(y') = \tilde{0}$. It follows that $r\min\{\tilde{\beta}^{\tilde{\alpha}}(x'), \tilde{\beta}^{\tilde{\alpha}}(y')\} = \tilde{0}$. Thus

$$\tilde{\beta}^{\tilde{\alpha}}((x' *' (y' *' z')) *' z') \geq r\min\{\tilde{\beta}^{\tilde{\alpha}}(x'), \tilde{\beta}^{\tilde{\alpha}}(y')\}.$$

Suppose $f^{-1}(x') \neq \phi$ and $f^{-1}(y') \neq \phi$. Let $x_0 \in f^{-1}(x'), y_0 \in f^{-1}(y')$, be such that $\tilde{\mu}^{\tilde{\alpha}}(x_0) = r \sup_{t \in f^{-1}(x')} \tilde{\mu}^{\tilde{\alpha}}(t)$ and $\tilde{\mu}^{\tilde{\alpha}}(y_0) = r \sup_{t \in f^{-1}(y')} \tilde{\mu}^{\tilde{\alpha}}(t)$. It follows by given and properties of PU-algebra that

$$\begin{aligned}
&\tilde{\beta}^{\tilde{\alpha}}((x' *' (y' *' z')) *' z') = \tilde{\beta}^{\tilde{\alpha}}((z' *' (y' *' z')) *' x') \\
&= \tilde{\beta}^{\tilde{\alpha}}((y' *' (z' *' z')) *' x') = \tilde{\beta}^{\tilde{\alpha}}((y' *' 0') *' x') \\
&= \tilde{\beta}^{\tilde{\alpha}}((f(y_0) *' f(0)) *' f(x_0)) = \tilde{\beta}^{\tilde{\alpha}}(f(y_0 * 0) * x_0)) \\
&= \tilde{\mu}^{\tilde{\alpha}}((y_0 * 0) * x_0) = \tilde{\mu}^{\tilde{\alpha}}((y_0 * (z_0 * z_0)) * x_0) \\
&= \tilde{\mu}^{\tilde{\alpha}}((z_0 * ((z_0 * z_0)) * x_0) = \tilde{\mu}^{\tilde{\alpha}}((x_0 * (y_0 * z_0)) * z_0) \\
&\geq r\min\{\tilde{\mu}^{\tilde{\alpha}}(x_0), \tilde{\mu}^{\tilde{\alpha}}(y_0)\} \\
&= r\min\{\sup_{t \in f^{-1}(x')} \tilde{\mu}^{\tilde{\alpha}}(t), \sup_{t \in f^{-1}(y')} \tilde{\mu}^{\tilde{\alpha}}(t)\} = r\min\{\tilde{\beta}^{\tilde{\alpha}}(x'), \tilde{\beta}^{\tilde{\alpha}}(y')\}.
\end{aligned}$$

For any $x', y', z' \in Y$, if $f^{-1}(x') = \phi$ or $f^{-1}(y') = \phi$, then $\gamma^{\alpha}(x') = 1$ or $\gamma^{\alpha}(y') = 1$. It follows that $\max\{\gamma^{\alpha}(x'), \gamma^{\alpha}(y')\} = 1$. Thus $\gamma^{\alpha}((x' *' (y' *' z')) *' z') \leq \max\{\gamma^{\alpha}(x'), \gamma^{\alpha}(y')\}$.
suppose $f^{-1}(x') \neq \phi$ and $f^{-1}(y') \neq \phi$. Let $x_0 \in f^{-1}(x'), y_0 \in f^{-1}(y')$ be such that $\lambda^{\alpha}(x_0) = \inf_{t \in f^{-1}(x')} \lambda^{\alpha}(t)$ and $\lambda^{\alpha}(y_0) = \inf_{t \in f^{-1}(y')} \lambda^{\alpha}(t)$. It follows by given and properties of PU-algebra that

$$\begin{aligned}
&\gamma^{\alpha}((x' *' (y' *' z')) *' z') = \gamma^{\alpha}((z' *' (y' *' z')) *' x') \\
&= \gamma^{\alpha}((y' *' (z' *' z')) *' z') = \gamma^{\alpha}((y' *' 0')/'x') = \gamma^{\alpha}((f(y_0) *' f(0)) *' f(x_0)) \\
&= \gamma^{\alpha}(f((y_0 * 0) * x_0)) = \lambda^{\alpha}((Y_0 * 0) * x_0) = \lambda^{\alpha}((y_0 * (z_0 * z_0)) * x_0) \\
&= \lambda^{\alpha}((z_0 * (y_0 * z_0)) * x_0) = \lambda^{\alpha}((x_0 * (y_0 * z_))) * z_0) \\
&\leq \max\{\lambda^{\alpha}(x_0), \lambda^{\alpha}(y_0)\} \\
&= \max\{\inf_{t \in f^{-1}(x')} \lambda^{\alpha}(t), \inf_{t \in f^{-1}(y')} \lambda^{\alpha}(t)\} = \max\{\gamma^{\alpha}(x'), \gamma^{\alpha}(y')\}.
\end{aligned}$$

Therefore $\langle \tilde{\beta}^{\tilde{\alpha}}, \gamma^{\alpha} \rangle$ is an $(\tilde{\alpha}, \alpha)$-cubic *new*-ideal of Y.

□

Corollary 3.21. *Let $(X, *, 0)$ and $(y, *', 0')$ be PU-algebras, $f : X \to Y$ be a homomorphism, $\langle \tilde{\mu}, \lambda \rangle$ be a cubic subset of X, $\langle \tilde{\beta}, \gamma \rangle$ be the image of $\langle \tilde{\mu}, \lambda \rangle$ under f, and $\tilde{\mu}(x) = \tilde{\beta}(f(x)), \lambda(x) = \gamma(f(x))$ for all $x \in X$. If $\langle \tilde{\mu}, \lambda \rangle$ is a cubic new-ideal of X, then $\langle \tilde{\beta}, \gamma \rangle$ is a cubic new-ideal of Y.*

4. Cartesian product of $(\tilde{\alpha}, \alpha)$-cubic *new*-ideal of PU-algebra

In this section, we introduce the concept of Cartesian product of an $(\tilde{\alpha}, \alpha)$-cubic *new*-ideal of a PU-algebra.

Definition 4.1. Let S be a non-empty set. The cubic set $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha} \rangle$ is called an $(\tilde{\alpha}, \alpha)$-cubic relation on S if $\tilde{\mu}^{\tilde{\alpha}}$ and λ^{α} are two functions defined by $\tilde{\mu}^{\tilde{\alpha}} : S \times S \to D[0, 1]$ and $\lambda^{\alpha} : S \times S \to [0, 1]$.

Definition 4.2. If $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha} \rangle$ is an $(\tilde{\alpha}, \alpha)$-cubic relation on a set of S and $\langle \tilde{\beta}^{\tilde{\alpha}}, \gamma^{\alpha} \rangle$ is an $(\tilde{\alpha}, \alpha)$-cubic subset of S, then $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha} \rangle$ is an $(\tilde{\alpha}, \alpha)$-cubic relation on $\langle \tilde{\beta}^{\tilde{\alpha}}, \gamma^{\alpha} \rangle$ if $\tilde{\mu}^{\tilde{\alpha}}(x, y) \leq r\min\{\tilde{\beta}^{\tilde{\alpha}}(x), \tilde{\beta}^{\tilde{\alpha}}(y)\}$, and $\lambda^{\alpha}(x, y) \geq \max\{\gamma^{\alpha}(x)m\gamma^{\alpha}(y)\}$ for all $x, y \in S$.

Definition 4.3. If $\langle \tilde{\beta}^{\tilde{\alpha}}, \gamma^{\alpha} \rangle$ is an $(\tilde{\alpha}, \alpha)$-cubic subset of a set S, the strongest $(\tilde{\alpha}, \alpha)$-cubic relation on S that is an $(\tilde{\alpha}^{\alpha})$-cubic relation on $\langle \tilde{\beta}^{\tilde{\alpha}}, \gamma^{\alpha} \rangle$ is $\langle \tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}, \lambda^{\alpha}_{\gamma^{\alpha}} \rangle$ given by $\tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(x, y) = r\min\{\tilde{\beta}^{\tilde{\alpha}}(x), \tilde{\beta}^{\tilde{\alpha}}(y)\}$ and $\lambda^{\alpha}_{\gamma^{\alpha}}(x, y) = \max\{\gamma^{\alpha}(x), \gamma^{\alpha}(y)\}$ for all $x, y \in S$

Definition 4.4. We define the binary operation $*$ on the Cartesian product $X \times X$ as follows: $(x_1, x_2) * (y_1, y_2) = (x_1 * y_1, x_2 * y_2)$ for all $(x_1, x_2), (y_1, y_2) \in X \times X$.

Lemma 4.5. *If $(X, *, 0)$ is a PU-algebra, then $(X \times X, *, (0, 0))$ is a PU-algebra, where $(x_1, x_2) * (y_1, y_2) = (x_1 * y_1, x_2 * y_2)$ for all $(x_1, x_2), (y_1, y_2) \in X \times X$.*

Proof. Clear. □

Theorem 4.6. *Let $\langle \tilde{\beta}^{\tilde{\alpha}}, \gamma^{\alpha} \rangle$ be an $(\tilde{\alpha}, \alpha)$-cubic subset of a PU-algebra X and $\langle \tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}, \lambda^{\alpha}_{\gamma^{\alpha}} \rangle$ be the strongest $(\tilde{\alpha}, \alpha)$-cubic relation on X, then $\langle \tilde{\beta}^{\tilde{\alpha}}, \gamma^{\alpha} \rangle$ is an $(\tilde{\alpha}, \alpha)$-cubic new-ideal of X if and only if $\langle \tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}, \lambda^{\alpha}_{\gamma^{\alpha}} \rangle$ is an $(\tilde{\alpha}, \alpha)$-cubic new-ideal of $X \times X$.*

Proof. ($\Rightarrow$): Assume that $\langle \tilde{\beta}^{\tilde{\alpha}}, \gamma^{\alpha} \rangle$ is an $(\tilde{\alpha}, \alpha)$-cubic *new*-ideal of X, we note that:
$\tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(0, 0) = r\min\{\tilde{\beta}^{\tilde{\alpha}}(0), \tilde{\beta}^{\tilde{\alpha}}(0)\} \geq r\min\{\tilde{\beta}^{\tilde{\alpha}}(x), \tilde{\beta}^{\tilde{\alpha}}(y)\} = \tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(x, y)$ and
$\lambda^{\alpha}_{\gamma^{\alpha}}(0, 0) = \max\{\gamma^{\alpha}(x), \gamma^{\alpha}(y)\} = \lambda^{\alpha}_{\gamma^{\alpha}}(x, y)$ for all $x, y \in X$.
Now, for any $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, we have:

$$
\begin{aligned}
&\tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(((x_1,x_2)*((y_1,y_2)*(z_1,z_2)))*(z_1,z_2))\\
&=\tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(((x_1,x_2)*(y_1*z_1,y_2*z_2))*(z_1,z_2))\\
&=\tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}((x_1*(y_1*z_1),x_2*(y_2*z_2))*(z_1,z_2))\\
&=\tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}((x_1*(y_1*z_1))*z_1,(x_2*(y_2*z_2))*z_2)\\
&=r\min\{\tilde{\beta}^{\tilde{\alpha}}((x_1*(y_1*z_1))*z_1),\tilde{\beta}^{\tilde{\alpha}}((x_2*(y_2*z_2))*z_2)\}\\
&\geq r\min\{r\min\{\tilde{\beta}^{\tilde{\alpha}}(x_1),\tilde{\beta}^{\alpha(\tilde{y_1})}\},r\min\{\tilde{\beta}^{\tilde{\alpha}}(x_2),\tilde{\beta}^{\tilde{\alpha}}(y_2)\}\}\\
&=\min\{r\min\{\tilde{\beta}^{\tilde{\alpha}}(x_1),\tilde{\beta}^{\tilde{\alpha}}(x_2)\},r\min\{\tilde{\beta}^{\tilde{\alpha}}(y_1),\tilde{\beta}^{\tilde{\alpha}}(y_2)\}\}\\
&=r\min\{\tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(x_1,x_2),\tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(y_1,y_2)\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\lambda^{\alpha}_{\gamma^{\alpha}}(((x_1,x_2)*((y_1,y_2)*(z_1,z_2)))*(z_1,z_2))\\
&=\lambda^{\alpha}_{\gamma^{\alpha}}(((x_1,x_2)*(y_1*z_1,y_2*z_2))*(z_1,z_2))\\
&=\lambda^{\alpha}_{\gamma^{\alpha}}((x_1*(y_1*z_1),x_2*(y_2,z_2))*(z_1,z_2))\\
&=\lambda^{\alpha}_{\gamma^{\alpha}}((x_1*(y_1*z_1))*z_1,(x_2*(y_2*z_2))*z_2)\\
&=\max\{\gamma^{\alpha}((x_1*(y_1*z_1))*z_1),\gamma^{\alpha}((x_2*(y_2*z_2))*z_2)\}\\
&\geq\max\{\max\{\gamma^{\alpha}(x_1),\gamma^{\alpha}(y_1)\},\max\{\gamma^{\alpha}(x_2),\gamma^{\alpha}(y_2)\}\}\\
&=\max\{\max\{\gamma^{\alpha}(x_1),\gamma^{\alpha}(x_2)\},\max\{\gamma^{\alpha}(y_1),\gamma^{\alpha}(y_2)\}\}\\
&=\max\{\lambda^{\alpha}_{\gamma^{\alpha}}(x_1,x_2),\lambda^{\alpha}_{\gamma^{\alpha}}(y_1,y_2)\}.
\end{aligned}
$$

Hence $\langle\tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}},\lambda^{\alpha}_{\gamma^{\alpha}}\rangle$ is an $(\tilde{\alpha},\alpha)$-cubic *new*-ideal of $X\times X$.
($\Leftarrow$): For all $(x,x)\in X\times X$, we have

$$\tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(0,0)=r\min\{\tilde{\beta}^{\tilde{\alpha}}(0),\tilde{\beta}^{\tilde{\alpha}}(0)\}\geq\tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(x,x)$$

and

$$\lambda^{\alpha}_{\gamma^{\alpha}}(0,0)=\max\{\gamma^{\alpha}(0),\gamma^{\alpha}(0)\}\leq\lambda^{\alpha}_{\gamma^{\alpha}}(x,x).$$

Then

$$\tilde{\beta}^{\tilde{\alpha}}(0)=r\min\{\tilde{\beta}^{\tilde{\alpha}}(0),\tilde{\beta}^{\tilde{\alpha}}(0)\}\geq r\min\{\tilde{\beta}^{\tilde{\alpha}}(x),\tilde{\beta}^{\tilde{\alpha}}(x)\}=\tilde{\beta}^{\tilde{\alpha}}(x)$$

and

$$\gamma^{\alpha}(0)=\max\{\gamma^{\alpha}(0),\gamma^{\alpha}(0)\}\leq\max\{\gamma^{\alpha}(x),\gamma^{\alpha}(x)\}=\gamma^{\alpha}(x),$$

for all $x \in X$. Now, for all $x, y, z \in X$, we have

$$\begin{aligned}
\tilde{\beta}^{\tilde{\alpha}}((x*(y*z))*z) &= r\min\{\tilde{\beta}^{\tilde{\alpha}}((x*(y*z))*z), \tilde{\beta}^{\tilde{\alpha}}((x*(y*z))*z)\} \\
&= \tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}((x*(y*z))*z, (x*(y*z))*z) \\
&= \tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}((x*(y*z), x*(y*z))*(z,z)) \\
&= \tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(((x,x)((y*z),(y*z)))*(z,z)) \\
&= \tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(((x,x)*((y,y)*(z,z)))*(z,z)) \\
&\geq r\min\{\tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(x,x), \tilde{\mu}^{\tilde{\alpha}}_{\tilde{\beta}^{\tilde{\alpha}}}(y,y)\} \\
&= r\min\{r\min\{\tilde{\beta}^{\tilde{\alpha}}(x), \tilde{\beta}^{\tilde{\alpha}}(x)\}, r\min\{\tilde{\beta}^{\tilde{\alpha}}(y), \tilde{\beta}^{\tilde{\alpha}}(y)\}\} \\
&= r\min\{\tilde{\beta}^{\tilde{\alpha}}(x), \tilde{\beta}^{\tilde{\alpha}}(y)\}.
\end{aligned}$$

and,

$$\begin{aligned}
\gamma^{\alpha}((x*(y*z_*z) &= \max\{\gamma^{\alpha}((x*(y*z)*z)), \gamma^{\alpha}((x*(y*z))*z)\} \\
&= \lambda^{\alpha}_{\gamma^{\alpha}}((x*(y*z))*z, (x*(y*z))*z) \\
&= \lambda^{\alpha}_{\gamma^{\alpha}}((x*(y*z), x*(y*z))*(z,z)) \\
&= \lambda^{\alpha}_{\gamma^{\alpha}}(((x,x)*((y*z),(y*z)))*(z,z)) \\
&= \lambda^{\alpha}_{\gamma^{\alpha}}(((x,x)*((y,y)*(z,z)))*(z,z)) \\
&\leq \max\{\lambda^{\alpha}_{\gamma^{\alpha}}(x,x), \lambda^{\alpha}_{\gamma^{\alpha}}(y,y)\} \\
&= \max\{\max\{\gamma^{\alpha}(x), \gamma^{\alpha}(x)\}, \max\{\gamma^{\alpha}(y), \gamma^{\alpha}(y)\}\} \\
&= \max\{\gamma^{\alpha}(x), \gamma^{\alpha}(y)\}.
\end{aligned}$$

Hence $\langle \tilde{\beta}^{\tilde{\alpha}}, \gamma^{\alpha} \rangle$ is an $(\tilde{\alpha}, \alpha)$-cubic *new*-ideal of X. □

Definition 4.7. Let $\langle \tilde{\mu}, \lambda \rangle$ and $\langle \tilde{\delta}, \psi \rangle$ be cubic subsets in X. The Cartesian product $\langle \tilde{\mu}, \lambda \rangle \times \langle \tilde{\delta}, \psi \rangle$ is defined by $(\tilde{\mu} \times \tilde{\delta}) : X \times X \to D[0,1]$ and $(\lambda \times \psi) : X \times X \to [0,1]$, where $(\tilde{\mu} \times \tilde{\delta})(x,y) = r\min\{\tilde{\mu}(x), \tilde{\delta}(y)\}$ and $(\lambda \times \psi)(x,y) = \max\{\lambda(x), \psi(y)\}$ for all $x, y \in X$.

Definition 4.8. Let $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha} \rangle$ and $\langle \tilde{\delta}^{\tilde{\alpha}}, \psi^{\alpha}$ be $(\tilde{\alpha}, \alpha)$-cubic subsets in X. The Cartesian product $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha} \rangle \times \langle \tilde{\delta}^{\tilde{\alpha}}, \psi^{\alpha} \rangle$ is defined by $(\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}}) : X \times X \to D[0,1]$ and $(\lambda^{\alpha} \times \psi^{\alpha}) : X \times X \to [0,1]$, where $(\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}})(x,y) = r\min\{\tilde{\mu}^{\tilde{\alpha}}(x), \tilde{\delta}^{\tilde{\alpha}}(y)\}$ and $(\lambda^{\alpha} \times \psi^{\alpha})(x,y) = \max\{\lambda^{\alpha}(x), \psi^{\alpha}(y)\}$ for all $x, y \in X$.

Theorem 4.9. *If $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha} \rangle$ and $\langle \tilde{\delta}^{\tilde{\alpha}}, \psi^{\alpha} \rangle$ are $(\tilde{\alpha}, \alpha)$-cubic new-ideals in a PU-algebra X, then $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha} \rangle \times \langle \tilde{\delta}^{\tilde{\alpha}}, \psi^{\alpha} \rangle$ is an $(\tilde{\alpha}, \alpha)$-cubic new-ideal in $X \times X$.*

Proof.

$$\begin{aligned}
&(\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}})(0,0) = r\min\{\tilde{\mu}^{\tilde{\alpha}}(0), \tilde{\delta}^{\tilde{\alpha}}(0)\} \\
&\geq r\min\{\tilde{\mu}^{\tilde{\alpha}}(x_1), \tilde{\delta}^{\tilde{\alpha}}(x_2)\} = (\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}})(x_1, x_2)
\end{aligned}$$

and

$$\begin{aligned}
&(\lambda^{\alpha} \times \psi^{\alpha})(0,0) = \max\{\lambda^{\alpha}(0), \psi^{\alpha}(0)\} \\
&\leq \max\{\lambda^{\alpha}(x_1), \psi^{\alpha}(x_2)\} = (\lambda^{\alpha} \times \psi^{\alpha})(x_1, x_2).
\end{aligned}$$

for all $(x_1, x_2) \in X \times X$. Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$. Then we have that

$$\begin{aligned}
&(\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}})(((x_1, x_2) * ((y_1, y_2) * (z_1, z_2))) * (z_1, z_2)) \\
&= (\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}})(((x_1, x_2) * (y_1 * z_1, y_2 * z_2)) * (z_1, z_2)) \\
&= (\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}})((x_1 * (y_1 * z_1), x_2 * (y_2 * z_2)) * (z_1, z_2)) \\
&= (\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha}})((x_1 * (y_1 * z_1)) * z_1, (x_2 * (y_2 * z_2)) * z_2) \\
&= r\min\{\tilde{\mu}^{\tilde{\alpha}}(x_1 * (y_1 * z_1)) * z_1), \tilde{\delta}^{\tilde{\alpha}}(x_2 * (y_2 * z_2)) * z_2)\} \\
&\geq r\min\{r\min\{\tilde{\mu}^{\tilde{\alpha}}(x_1), \tilde{\mu}^{\tilde{\alpha}}(y_1)\}, r\min\{\tilde{\delta}^{\tilde{\alpha}}(x_2), \tilde{\delta}^{\tilde{\alpha}}(y_2)\}\} \\
&= r\min\{(\tilde{\mu}^{\tilde{\alpha}} \times \tilde{\delta}^{\tilde{\alpha})}(x_1, x_2)
\end{aligned}$$

and

$$\begin{aligned}
&(\lambda^{\alpha} \times \psi^{\alpha})(((x_1, x_2) * ((y_1, y_2) * (z_1, z_2))) * (z_1, z_2)) \\
&= (\lambda^{\alpha} \times \psi^{\alpha})(((x_1, x_2) * (y_1 * z_1, y_2 * z_2))(z_1, z_2)) \\
&= (\lambda^{\alpha} \times \psi^{\alpha})((x_1 * (y_1 * z_1), x_2 * (y_2 * z_2)) * (z_1, z_2)) \\
&= (\lambda^{\alpha} \times \psi^{\alpha})((x_1 * (y_1 * z_1)) * z_1, (x_2 * (y_2 * z_2)) * z_2) \\
&= \max\{\lambda^{\alpha}(x_1 * (y_1 * z_1)) * z_1), \psi^{\alpha}(x_2 * (y_2 * z_2)) * z_2)\} \\
&\geq \max\{\max\{\lambda^{\alpha}(x_1), \lambda^{\alpha}(y_1)\}, \max\{\psi^{\alpha}(x_2), \psi^{\alpha}(y_2)\}\} \\
&= \max\{(\lambda^{\alpha} \times \psi^{\alpha})(x_1, x_2), (\lambda^{\alpha} \times \psi^{\alpha})(y_1, y_2)\}.
\end{aligned}$$

Therefore $\langle \tilde{\mu}^{\tilde{\alpha}}, \lambda^{\alpha} \rangle \times \langle \tilde{\delta}^{\tilde{\alpha}}, \psi^{\alpha} \rangle$ is an $(\tilde{\alpha}, \alpha)$-cubic *new*-ideal in $X \times X$. □

5. Conclusions

In the present paper, we have introduced the concept of $(\tilde{\alpha}, \alpha)$-cubic *new*-ideal of PU-algebras and investigated some of their useful properties. We believe that these results are very useful in developing algebraic structures also these definitions and main results can be similarly extended to some other algebraic structure such as PS-algebras, Q-algebras, SU-algebras , IS-algebras, β algebras and semirings. It is our hope that this work would other foundations for further study of the theory of BCI-algebras. In our future study of fuzzy structure of PU-algebras, may be the following topics should be considered:

(1) To establish the interval value, bipolar and intuitionistic α-fuzzy *new*-ideal in PU-algebras.

(2) To establish $(\tilde{\tau}, \tilde{\rho})$-interval-valued α-fuzzy *new*-ideal of PU-algebras.

(3) To get more results in $(\tilde{\alpha}, \alpha)$-cubic *new*-ideal of PU-algebras and it's application (Cubic soft sets with applications in PU-algebras).

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Algorithms for PU-algebra

Input (X: set with 0 element, $*$: Binary operation)
Output ("X is a PU-algebra or not")
If $X = \phi$ then;

Go to (1.)
End if
If $0 \notin X$ then go to (1.);
End If
Stop: = false
$i = 1$;
While $i \leq |X|$ and not (Stop) do
If $0 * x_i \neq x_i$, then
Stop: = true
End if
$j = 1$;
While $j \leq |X|$, and not (Stop) do
$k = 1$;
While $k \leq |X|$ and not (stop) do
If $(x_i * x_k) * (x_j * x_k) \neq x_j * x_i$, then
Stop: = true
End if
End while
End if
End while
If stop then
Output ("X is a PU-algebra")
Else
(1.) Output ("X is not a PU-algebra")
End if
End.

Algorithms for PU-ideal in PU-algebra
Input (X: PU-algebra, I: subset of X)
Output ("I is a PU-ideal of X or not")
If $I = \phi$ then
Go to (1.);
End if
If $0 \notin I$ then
Go to (1.);
End if
Stop: = false
$i = 1$;
While $i \leq |X|$ and not (stop) do
$j = 1$
While $j \leq |X|$ and not (stop) do
$k = 1$
While $k \leq |X|$ and not (stop) do
If $x_j * x_i \in I$, and $x_i * x_k \in I$ then
If $x_j * x_k \notin I$ then
Stop: = false

End if
End while
End while
End while
If stop then
Output ("I is a PU-ideal of X")
Else
(1.) Output ("I is not ("I is a PU-ideal of X")
End if
End.
Algorithm for fuzzy subsets
Input (X: PU-algebra, $A : X \to [0, 1]$);
Output ("A is a fuzzy subset of X or not")
Begin
Stop: =false;
$i := 1$;
While $i \leq |X|$ and not (Stop) do
If $(A(x_i) < 0)$ or $(A(x_i) > 1$ then
Stop: = true;
End If
End While
If Stop then
Output ("A is an anti fuzzy subset of X")

Else
Output ("A is not an anti fuzzy subset of X")

End If
End

Algorithm for new-ideal in PU-algebra
Input (X: PU-algebra, I: subset of X);
Output ("I is an *new*-ideal of X or not");
Begin
If $I = \phi$ then go to (1.);
End If
If $0 \notin I$ then go to (1.);
End If
Stop: =false;
$i := 1$;
While $i \leq |X|$ and not (Stop) do
$j := 1$
While $j \leq |X|$ and not (Stop) do
$k := 1$
While $k \leq |X|$ and not (Stop) do
If $x_i, x_j \in I$ and $x_k \in X$, then

If $(x_i * (x_j * x_k)) * x_k \notin I$ then

Stop: = true;
End If
End If
End While
End While
End While
If Stop then
Output ("$I$is is *new*-ideal of X")
Else (1.) Output ("I is not is *new*-ideal of X")

End If
End

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Samy M. Mostafa (samymostafa@yahoo.com)
Ain Shams University, Faculty of Education, Department of Mathematics, Roxy, Cairo, Egypt

F. I. Sidky
Department of Mathematic, Faculty of Science, University of Zagazig, Egypt

Alaa Eldin I. Elkabany (alaa512@hotmail.com)
Ain Shams University, Faculty of Education, Department of Mathematics, Roxy, Cairo, Egypt