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On IF-closure spaces vs IF-rough sets

Anupam K. Singh, S. P. Tiwari

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Abstract. The objective of this paper is to study the relationship between IF-rough sets, IF-closure spaces and IF-topology. We show that the bijective correspondence between the family of all IF-reflexive approximation spaces and the family of all quasi-discrete IF-closure spaces satisfying a certain extra condition. We also introduce and study the similar correspondence between the family of all IF-tolerance approximation spaces and the family of all symmetric quasi-discrete IF-closure spaces satisfying a certain extra condition.

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Corresponding Author: Anupam K. Singh (anupam09.bhu@gmail.com)

1. Introduction

Fuzzy rough set theory, firstly proposed by Dubois and Prade [8] as a fuzzy generalization of rough sets by replacing crisp binary relations to fuzzy relations, has now developed significantly [8, 13, 15, 18, 21, 22]. Simultaneously, the relationship between fuzzy rough sets and fuzzy topological spaces were studied by different researchers(cf., [3, 15, 18, 25]). Among these works in [18], it has shown that there is a bijective correspondence between the set of all fuzzy preorder approximation spaces and the set of all saturated topological spaces satisfying a certain extra condition. In order to find such correspondence for fuzzy reflexive/tolerance approximation spaces and some set of fuzzy topological spaces; recently, in [24], it has shown that there exists a bijective correspondence between the set of fuzzy reflexive/tolerance approximation spaces and the set of fuzzy closure spaces satisfying a certain extra condition.

After introduction of an intuitionistic fuzzy set by Atanassov [1], the researchers proposed a new hybrid model, namely intuitionistic fuzzy rough set theory, to describe the uncertain information by combining intuitionistic fuzzy set theory and rough set theory (cf., [4, 6, 7, 14, 16, 17]) [At this point, we mention that in [9], it has been argued, rather convincingly, that the use of the term intuitionistic for the concept introduced by Atanassov [1], is inappropriate. Accordingly, we use in this paper the prefix IF- in place of intuitionistic fuzzy, thus for example, an intuitionistic fuzzy set is renamed here as an IF-set. This terminology has already been used in [20, 22].]. Simultaneously, the relationship between IF-rough sets and IF-topological spaces were studied (cf., [22, 23, 27, 28, 29]).

Among these works, in [22, 26], it has been shown that there is a bijective correspondence between the family of all IF-preorders and the family of all saturated IF-topological spaces satisfying a 'certain extra condition', which is a consequence of the similar result introduced in [18] and [21] respectively. Still there is a silence on the bijective correspondence between the set of all IF-reflexive/tolerance approximation spaces and some set of IF-topological spaces. Throughout this paper, we try to fill this gap by using the concept of IF-closure spaces.

2. Preliminaries

In this section, we recall some basic concepts and results related to IF-set, IF-rough set and IF-binary relation, which will be used in the subsequent sections. Throughout, I stands for $[0,1]$. For a nonempty set X and $A \subseteq X$, I^X and $1_A : X \to I$ shall, respectively denote the set of all fuzzy sets in X and the characteristic function of A. Throughout, J is an index set.

We begin with the following.

Definition 2.1 ([1])**.** An IF-set A in X is a pair (μ_A, ν_A) of fuzzy sets in X, i.e., functions $\mu_A, \nu_A : X \to I$, such that $\mu_A(x) + \nu_A(x) \leq 1$; $\forall x \in X$.

($\mu_A(x)$ and $\nu_A(x)$, appearing in the above definition, are usually interpreted respectively as the degree of membership and the degree of non-membership of x in A).

Throughout, $IFS(X)$ will denote the family of all IF-sets in X and I^* means the set $\{(x_1, x_2) : (x_1, x_2) \in I \times I, x_1 + x_2 \leq 1\}$.

Remark 2.2. We shall usually denote the two parts of an IF-set A also as μ_A and ν_A and express A as (μ_A, ν_A). An IF-set $A = (\mu_A, \nu_A)$ in X will frequently be also viewed as a function $A : X \to I^*$, given by $A(x) = (\mu_A(x), \nu_A(x)), x \in X$. In particular, the IF-set $\widehat{(\alpha, \beta)}$ is given by $\widehat{(\alpha, \beta)}(x) = (\alpha, \beta)$, where α and β are respectively the α-valued and the β-valued constant fuzzy sets in X such that $\alpha + \beta \leq 1$. We shall denote the IF-set by $\hat{1}$, which is given by $\widehat{(1, 0)}(x) = (\mathbf{1}, \mathbf{0}), x \in X$, where $\mathbf{1}$ and $\mathbf{0}$ are respectively the 1-valued and the 0-valued constant fuzzy sets in X.

Definition 2.3 ([28])**.** For $y \in X$, an IF-singleton set $1_y = (\mu_{1_y}, \nu_{1_y})$, is defined as follows:

$$\mu_{1_y}(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \qquad \nu_{1_y}(x) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}.$$

Definition 2.4 ([26]). Let R be an IF-binary relation on X. Then R is called

(i) reflexive if $\mu_R(x,x) = 1$ and $\nu_R(x,x) = 0, \forall x \in X$;
(ii) symmetric if $\mu_R(x,y) = \mu_R(y,x)$ and $\nu_R(x,y) = \nu_R(y,x), \forall x,y \in X$;
(iii) transitive if $\forall x,y,z \in X, \mu_R(x,y) \wedge \mu_R(y,z) \le \mu_R(x,z)$ and $\nu_R(x,y) \vee \nu_R(y,z) \ge \nu_R(x,z)$.

Definition 2.5. Let R be an IF-binary relation on X. Then R is called an IF-tolerance relation if it is IF-reflexive and IF-symmetric.

Definition 2.6 ([29]). A pair (X,R) is called an IF-approximation space if X is a nonempty set and R is an IF-relation on X.

For an IF-reflexive relation R, we call the IF-approximation space (X,R), an IF-reflexive approximation space. Also, if R is an IF-tolerance relation, we call (X,R) an IF-tolerance approximation space.

Definition 2.7 ([29]). Let (X,R) be an IF-approximation space and $A \in IFS(X)$. The lower and upper approximation of A, denoted by $\underline{R}(A)$ and $\overline{R}(A)$, are two IF-sets and are respectively defined as follows:

$\underline{R}(A) = (\mu_{\underline{R}(A)}, \nu_{\underline{R}(A)})$, $\overline{R}(A) = (\mu_{\overline{R}(A)}, \nu_{\overline{R}(A)})$, where

$$\mu_{\underline{R}(A)}(x) = \wedge\{\mu_{R^c}(x,y) \vee \mu_A(y) : y \in X\},$$

$$\nu_{\underline{R}(A)}(x) = \vee\{\nu_{R^c}(x,y) \wedge \nu_A(y) : y \in X\},$$

$$\mu_{\overline{R}(A)}(x) = \vee\{\mu_R(x,y) \wedge \mu_A(y) : y \in X\},$$

$$\nu_{\overline{R}(A)}(x) = \wedge\{\nu_R(x,y) \vee \nu_A(y) : y \in X\}.$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called an IF-rough set.

Remark 2.8 ([28]). For an IF-binary relation R on X and $A \in IFS(X)$, the pair $\overline{R}$ and $\underline{R}$ are 'dual' i.e., $\overline{R}(A) = [\underline{R}(A^c)]^c$ and $\underline{R}(A) = [\overline{R}(A^c)]^c$, since $\forall x \in X, [\overline{R}(A^c)]^c(x) = 1 - \vee\{R(x,y) \wedge A^c(y) : y \in X\} = 1 - \vee\{R(x,y) \wedge (1 - A(y)) : y \in X\} = \wedge\{1 - R(x,y) \vee A(y) : y \in X\} = \wedge\{R^c(x,y) \vee A(y) : y \in X\}$.

Proposition 2.9 ([29]). *Let (X,R) be an IF-reflexive approximation space and $A, B \in IFS(X)$. Then*

(i) $\underline{R}(A \wedge B) = \underline{R}(A) \wedge \underline{R}(B)$;
(ii) $\overline{R}(A \vee B) = \overline{R}(A) \vee \overline{R}(B)$;
(iii) $\underline{R}(A) \le A,\ A \le \overline{R}(A)$.

Proposition 2.10 ([29]). *Let (X,R) be an IF-approximation space and $A \in IFS(X)$. Then R is an IF-transitive relation on X if and only if $\overline{R}(\overline{R}(A)) \le \overline{R}(A)$.*

The IF-topological concepts, we use here are fairly standard and based on [5].

Definition 2.11. An IF-topology on a nonempty set X is a family τ of IF-sets in X, such that

(i) $\widehat{(\alpha,\beta)} \in \tau, \forall (\alpha,\beta) \in I^*$;
(ii) $\{A_i : i \in J\} \in \tau \Rightarrow \vee\{A_i : i \in J\} \in \tau$;
(iii) $A, B \in \tau \Rightarrow A \wedge B \in \tau$.

The pair (X,τ) is called an IF-topological space. The IF-set in τ are called IF τ-open set and there complements are called IF τ-closed set. Further, an IF-topological space (X,τ) is called saturated if $\{A_j : j \in J\} \in \tau \Rightarrow \wedge\{A_j : j \in J\} \in \tau$.

Definition 2.12. A Kuratowski IF-closure operator on X is a map $k : IFS(X) \to IFS(X)$, such that $\forall A, B \in IFS(X)$ and $\forall (\alpha,\beta) \in I^*$, the following condition holds:

(i) $k(\widehat{(\alpha,\beta)}) = \widehat{(\alpha,\beta)}$,
(ii) $A \leq k(A)$,
(iii) $k(A \vee B) = k(A) \vee k(B)$,
(iv) $k(k(A)) = k(A)$.

Definition 2.13. A Kuratowski IF-closure space is a pair (X,k), where X is a nonempty set and $k : IFS(X) \to IFS(X)$ is a Kuratowski IF-closure operator on X.

Proposition 2.14 ([26])**.** *Let (X,R) be an IF-reflexive approximation space. Then $\tau_R = \{A \in IFS(X) : \underline{R}(A) = A\}$ is a saturated IF-topology (in the sense that arbitrary supremum of an IF $\tau_{\overline{c}}$-closed set is also IF $\tau_{\overline{c}}$-closed) on X.*

Proposition 2.15 ([26])**.** *Let k be Kuratowski IF-closure operator on X. Then there exists an IF-reflexive and IF-transitive relation $S_k = (\mu_{S_k}, \nu_{S_k})$ on X such that $\forall A \in IFS(X)$ and $\forall (\alpha,\beta) \in I^*$, $\overline{S_k}(A) = k(A)$ if and only if $k(A \wedge \widehat{(\alpha,\beta)}) = k(A) \wedge \widehat{(\alpha,\beta)}$.*

3. IF-closure space

The notion of fuzzy closure space was introduced in [12, 19]. In literature [2, 11, 26] several generalizations of fuzzy closure space have been studied. In this section, we introduce the concepts of IF-closure space and investigate their relationship with Kuratowski IF-closure operators. Lastly, we show that the Kuratowski IF-closure operator associated with a quasi-discrete IF-closure space induces a saturated IF-topology.

We begin by introducing the following concept of an IF-closure space.

Definition 3.1. An IF-closure space is a pair (X,c), where X is a nonempty set and $c : IFS(X) \to IFS(X)$ is a map such that

(i) $c(\widehat{(\alpha,\beta)}) = \widehat{(\alpha,\beta)}$,
(ii) $A \leq c(A)$,
(iii) $c(A \vee B) = c(A) \vee c(B)$,

$\forall A, B \in IFS(X)$ and $\forall (\alpha,\beta) \in I^*$.

Remark 3.2. It is not necessary that every IF-closure space is a Kuratowski IF-closure space.

The following example shows that how a IF-closure space differ from a Kuratowski IF-closure space. Specifically, it is shown here that for an IF-closure space $c, c(c(A)) \neq c(A), \forall A \in IFS(X)$.

Example 3.3. Let X be any set containing at least two points. Define a map $c : IFS(X) \to IFS(X)$ as follows : for $A = (\mu_A, \nu_A) \in IFS(X)$ and $x \in X$, $c(A)(x) = (\mu_{c(A)}(x), \nu_{c(A)}(x))$, where

$$\mu_{c(A)}(x) = \begin{cases} 2\mu_A(x) & \text{if } \mu_A(x) < k_1/2, \\ & \text{where } k_1 = \vee\{\mu_A(z)\}_{z\in X} \\ \\ k_1 & \text{if } \mu_A(x) > k_1/2 \quad , \end{cases}$$

and

$$\nu_{c(A)}(x) = \begin{cases} 2\nu_A(x) & \text{if } \nu_A(x) > k_2/2, \\ & \text{where } k_2 = \wedge\{\nu_A(z)\}_{z\in X} \\ \\ k_2 & \text{if } \nu_A(x) < k_2/2 \ . \end{cases}$$

Then the conditions (i) $c(\widehat{(\alpha,\beta)}) = \widehat{(\alpha,\beta)}$ and (ii) $A \leq c(A)$ are obvious. To prove condition (iii) $c(A \vee B) = c(A) \vee c(B)$, let $\vee\{\mu_{A\cup B}(z)\}_{z\in X} = k_1$. Now, if $\mu_{A\cup B}(x) < k_1/2$, then $\mu_A(x) < k_1/2$ or $\mu_B(x) < k_1/2$. Thus $\mu_A(x) < k_1/2, \mu_B(x) < k_1/2$, or that $\mu_{c(A)}(x) = 2\mu_A(x)$ and $\mu_{c(B)}(x) = 2\mu_B(x)$. Therefore $\mu_{c(A\cup B)}(x) = 2\mu_{A\cup B}(x) = 2\mu_A(x) \cup 2\mu_B(x) = \mu_{c(A)}(x) \cup \mu_{c(B)}(x) = (\mu_{c(A)} \cup \mu_{c(B)})(x)$ and $\wedge\{\nu_{A\cap B}(z)\}_{z\in X} = k_2$. Then if $\nu_{A\cap B}(x) > k_2/2$, then $\nu_A(x) > k_2/2$ and $\nu_B(x) > k_2/2$. Thus $\nu_A(x) > k_2/2, \nu_B(x) > k_2/2$. Hence $\nu_{c(A)}(x) = 2\nu_A(x)$ and $\nu_{c(B)}(x) = 2\nu_B(x)$. Therefore $\nu_{c(A\cap B)}(x) = 2\nu_{A\cap B}(x) = 2\nu_A(x) \cap 2\nu_B(x) = \nu_{c(A)}(x) \cap \nu_{c(B)}(x) = (\nu_{c(A)} \cap \nu_{c(B)})(x)$. Hence in this case, $c(A \cup B) = c(A) \cup c(B)$.
Further, if $\mu_{A\cup B}(x) \geq k_1/2$, then $\mu_A(x) \geq k_1/2$ or $\mu_B(x) \geq k_1/2$. Thus $\mu_A(x) \geq k_1/2, \mu_B(x) \geq k_1/2$. Therefore $\mu_{c(A)}(x) = \mu_{c(B)}(x) = k_1$ and if $\nu_{A\cap B}(x) \leq k_2/2$, then $\nu_A(x) \leq k_2/2$ and $\nu_B(x) \leq k_2/2$. Thus $\nu_A(x) \leq k_2/2, \nu_B(x) \leq k_2/2$. Therefore $\nu_{c(A)}(x) = \nu_{c(B)}(x) = k_2$. Thus in this case also $c(A \cup B) = c(A) \cup c(B)$. Hence, c is an IF-closure space in the sense of Definition 3.1. On the other hand, it is not a Kuratowaski IF-closure operator, since if we take a non-constant IF-sets $A = (\mu_A, \nu_A) \in IFS(X)$ such that $A(x_0) = (\mu_A(x_0), \nu_A(x_0)) = k/16 = (k_1/16, k_2/16)$,i.e., $\mu_A(x_0) = k_1/16, \nu_A(x_0) = k_2/16$ for some $x_0 \in X$, where $\vee\{\mu_A(z)\}_{z\in X}$ and $k_2 = \wedge\{\nu_A(z)\}_{z\in X}$. Thus $\mu_{c(A)}(x_0) = k_1/8, \nu_{c(A)}(x_0) = k_2/8$ but $\mu_{c(c(A))}(x_0) = k_1/4, \nu_{c(c(A))}(x_0) = k_2/4$, showing that $c(c(A)) \neq c(A)$.

Definition 3.4. An IF-closure space (X, c) is called quasi-discrete if $c(\vee\{A_i : i \in J\}) = \vee\{c(A_i) : i \in J\}, \forall A \in IFS(X), i \in J$.

Definition 3.5. An IF-closure space (X, c) is called symmetric if $\mu_{c(1_y)(x)} = \mu_{c(1_x)(y)}$ and $\nu_{c(1_y)(x)} = \nu_{c(1_x)(y)}, \forall x, y \in X$.

Definition 3.6. Let (X, c) be an IF-closure space and $A \in IFS(X)$. Then the IF-interior $i(A) = (\mu_{i(A)}, \nu_{i(A)})$ of A is given by $\mu_{i(A)} = \mu_{[c(A^c)]^c}$ and $\nu_{i(A)} = \nu_{[c(A^c)]^c}$.

Definition 3.7. Let (X, c) be an IF-closure space and $A \in IFS(X)$. Then A is IF-closed if $\mu_{c(A)} = \mu_A$ and $\nu_{c(A)} = \nu_A$.

Proposition 3.8. *Let (X, c) be an IF-closure space. Then*
(i) $c(\wedge_{i\in J} A_i) \leq \wedge_{i\in J} c(A_i), \forall A_i \in IFS(X), i \in J$; *and*
(ii) *if* $A \leq B$ *then* $c(A) \leq c(B), \forall A, B \in IFS(X)$.

Proof. The proof is straightforward. □

Definition 3.9. For an IF-closure space $(X, \bar{c})$, a Kuratowski IF-closure operator $\bar{c} : IFS(X) \to IFS(X)$ is defined by $\bar{c}(A) = \cap\{B \in IFS(X) : A \leq B$ and $c(B) = B\}$.

It follows immediately from the above definition that $A \leq \bar{c}(A)$.

We shall denote by $\tau_{\bar{c}}$, the IF-topology induced by $\bar{c}$, which is given by $\tau_{\bar{c}} = \{A \in IFS(X) : \bar{c}(A^c) = A^c\}$.

Proposition 3.10. *Let (X, c) be an IF-closure space. Then* $\forall A \in IFS(X)$.
(i) $c(\bar{c}(A)) = \bar{c}(A)$, *(i.e., $\bar{c}(A)$ is IF $\tau_{\bar{c}}$-closed),*
(ii) $c(A) \leq \bar{c}(A)$,
(iii) $c(A) = A$ *if and only if* $\bar{c}(A) = A$.

Proof. (i) Let $A \in IFS(X)$. Then from Proposition 3.8 (i), we have

$$\begin{aligned}\mu_{c(\bar{c}(A))} &= \mu_{c(\wedge\{B:A\leq B \text{ and } c(B)=B\})} \leq \wedge\, \mu_{\{c(B):A\leq B \text{ and } c(B)=B\}} \\ &= \wedge\, \mu_{\{B:A\leq B \text{ and } c(B)=B\}} = \mu_{\bar{c}(A)}\end{aligned}$$

and

$$\begin{aligned}\nu_{c(\bar{c}(A))} &= \nu_{c(\vee\{B:A\geq B \text{ and } c(B)=B\})} \geq \vee\, \nu_{\{c(B):A\geq B \text{ and } c(B)=B\}} \\ &= \vee\, \nu_{\{B:A\geq B \text{ and } c(B)=B\}} = \nu_{\bar{c}(A)}.\end{aligned}$$

Thus $c(\bar{c}(A)) \leq \bar{c}(A)$. Also, $\bar{c}(A) \leq c(\bar{c}(A))$ obviously. So $c(\bar{c}(A)) = \bar{c}(A)$.

(ii) Since $\mu_A \leq \mu_{\bar{c}(A)}$ and $\nu_A \geq \nu_{\bar{c}(A)}$, $\mu_{c(A)} \leq \mu_{c(\bar{c}(A))} = \mu_{\bar{c}(A)}$ and $\nu_{c(A)} \geq \nu_{c(\bar{c}(A))} = \nu_{\bar{c}(A)}$. Thus $c(A) \leq \bar{c}(A)$.

(iii) Let $\mu_{c(A)} = \mu_A$ and $\nu_{c(A)} = \nu_A, \forall A \in IFS(X)$. Then A is an IF $\tau_{\bar{c}}$-closed. Thus, $\mu_{\bar{c}(A)} \leq \mu_A$ and $\nu_{\bar{c}(A)} \geq \nu_A$, (cf., Definition 3.9). Also, $\mu_A \leq \mu_{\bar{c}(A)}$ and $\nu_A \geq \nu_{\bar{c}(A)}$. So $\mu_{\bar{c}(A)} = \mu_A$ and $\nu_{\bar{c}(A)} = \nu_A$. Conversly, let $\mu_{\bar{c}(A)} = \mu_A$ and $\nu_{\bar{c}(A)} = \nu_A$. Then from (ii), we have $\mu_A \leq \mu_{c(A)} \leq \mu_{\bar{c}(A)}$ and $\nu_A \geq \nu_{c(A)} \geq \nu_{\bar{c}(A)}$. Now, if $\mu_{\bar{c}(A)} = \mu_A$ and $\nu_{\bar{c}(A)} = \nu_A$. Then, we have $\mu_{c(A)} = \mu_A$ and $\nu_{c(A)} = \nu_A$. Thus $c(A) = A$. □

Proposition 3.11. *Let (X, c) be an IF-closure space. Then* $\forall A \in IFS(X), c(A) = \bar{c}(A)$ *if and only if* $c(c(A)) = c(A)$.

Proof. Let $c(A) = \bar{c}(A)$, i.e., $\mu_{c(A)} = \mu_{\bar{c}(A)}$ and $\nu_{c(A)} = \nu_{\bar{c}(A)}, \forall A \in IFS(X)$. Then $\mu_{c(c(A))} = \mu_{c(\bar{c}(A))} = \mu_{\bar{c}(A)} = \mu_{c(A)}$ and $\nu_{c(c(A))} = \nu_{c(\bar{c}(A))} = \nu_{\bar{c}(A)} = \nu_{c(A)}$. Thus $c(c(A)) = c(A)$. Conversly, let $c(c(A)) = c(A)$, i.e., $\mu_{c(c(A))} = \mu_{c(A)}$ and $\nu_{c(c(A))} = \nu_{c(A)}$. Then $c(A)$ is an IF $\tau_{\bar{c}}$-closed. Thus by Proposition 3.10 (iii), $\mu_{\bar{c}(A)} = \mu_{c(A)}$ and $\nu_{\bar{c}(A)} = \nu_{c(A)}$. So $c(A) = \bar{c}(A)$. □

Proposition 3.12. *Let (X, c) be a quasi-discrete IF-closure space. Then the IF-topology $\tau_{\bar{c}}$ induced by $\bar{c}$ on X is a saturated IF-topology.*

Proof. Follows from Definitions 3.4, 3.9 and Proposition 3.11. □

4. IF-closure spaces and IF-rough sets

In this section, we show that there exists a bijective correspondence between the family of all IF-reflexive/IF-tolerance approximation spaces and the family of all quasi-discrete IF-closure spaces satisfying certain extra conditions.

We begin by introducing the following.

Proposition 4.1. *Let (X, R) be an IF-reflexive approximation space. Then $(X, \overline{R})$ is a quasi-discrete IF-closure space such that $\overline{R}(A \wedge \widehat{(\alpha,\beta)}) = \overline{R}(A) \wedge \widehat{(\alpha,\beta)}, \forall A \in IFS(X)$ and $\forall (\alpha,\beta) \in I^*$.*

Proof. From Proposition 2.9, it follows that $(X, \overline{R})$ is an IF-closure space. Now, let $A_i = (\mu_{A_i}, \nu_{A_i}) \in IFS(X), i \in J$ and $x \in X$. Then

$$\begin{aligned}\mu_{\overline{R}(\vee A_i : i \in J)}(x) &= \vee\{\mu_R(x,y) \wedge (\vee\{\mu_{A_i} : i \in J\})(y) : y \in X\} \\ &= \vee\{\vee\{\mu_R(x,y) \wedge \mu_{A_i}(y) : y \in X\} : i \in J\} \\ &= \vee\{\mu_{\overline{R}(A_i)} : i \in J\}\end{aligned}$$

and

$$\begin{aligned}\nu_{\overline{R}(\wedge A_i : i \in J)}(x) &= \wedge\{\nu_R(x,y) \vee (\wedge\{\nu_{A_i} : i \in J\})(y) : y \in X\} \\ &= \wedge\{\wedge\{\nu_R(x,y) \vee \nu_{A_i}(y) : y \in X\} : i \in J\} \\ &= \wedge\{\nu_{\overline{R}(A_i)} : i \in J\},\end{aligned}$$

whereby $\overline{R}(\vee\{A_i : i \in J\}) = \vee\{\overline{R}(A_i) : i \in J\}$. Hence $(X, \overline{R})$ is a quasi-discrete IF-closure space. Also, for each $A \in IFS(X)$ and $(\alpha,\beta) \in I^*, \overline{R}(A \wedge \widehat{(\alpha,\beta)}) = \overline{R}(A) \wedge \widehat{(\alpha,\beta)}$ follows trivially. □

Proposition 4.2. *For a quasi-discrete IF-closure space $(X, \overline{R}), \underline{R}$ is an IF-interior operator on X.*

Proof. Follows from Remark 2.8 and Definition 3.12.

□

Before stating next, recall the following from [10].

Definition 4.3. For $y \in X$ and $(\alpha,\beta) \in I^*$, with $\alpha + \beta \leq 1$. The IF-subset $(1_y \wedge (\alpha,\beta)) = (\mu_{1_y} \wedge \alpha, \nu_{1_y} \vee \beta)$ of X is called an IF-point in X, and is denoted as $y_{(\alpha,\beta)}$, where

$$y_{(\alpha,\beta)}(x) = \begin{cases} (\alpha,\beta) & \text{if } x = y \\ (0,1) & \text{if } x \neq y \ . \end{cases}$$

Proposition 4.4. *Let (X, c) be a quasi-discrete IF-closure space such that $c(A \wedge \widehat{(\alpha,\beta)}) = c(A) \wedge \widehat{(\alpha,\beta)}, \forall A \in IFS(X)$ and $\forall (\alpha,\beta) \in I^*$. Then there exists a unique IF-reflexive relation $R_c = (\mu_{R_c}, \nu_{R_c})$ on X such that $\overline{R}_c(A) = c(A), \forall A \in IFS(X)$.*

Proof. Let (X, c) be a quasi-discrete IF-closure space such that $c(A \wedge \widehat{(\alpha,\beta)}) = c(A) \wedge \widehat{(\alpha,\beta)}$, $\forall A \in IFS(X)$ and $\forall(\alpha,\beta) \in I^*$. Also, let R_c be an IF-relation on X given by $\mu_{R_c}(x,y) = \mu_{c(1_y)}(x)$ and $\nu_{R_c}(x,y) = \nu_{c(1_y)}(x), \forall x,y \in X$. Then $\hat{1} = \widehat{(1,0)} = (\mu_{1_x}(x), \nu_{1_x}(x)) \leq (\mu_{c(1_x)}(x), \nu_{c(1_x)}(x))$. Thus $c(1_x)(x) = \hat{1}$, i.e., $(\mu_{c(1_x)}(x), \nu_{c(1_x)}(x)) = \widehat{(1,0)} = \hat{1}$, whereby R_c is an IF-reflexive relation on X. Now, for $x \in X$,

$$\mu_{\overline{R}_c(A)}(x) = \mu_{\overline{R}_c(\cup_{y\in X} y_{\widehat{\alpha}})}(x) = \cup_{y\in X}\mu_{\overline{R}_c(y_{\widehat{\alpha}})}(x)$$

and

$$\nu_{\overline{R}_c(A)}(x) = \nu_{\overline{R}_c(\cap_{y\in X} y_{\widehat{\beta}})}(x) = \cap_{y\in X}\nu_{\overline{R}_c(y_{\widehat{\beta}})}(x).$$

Also,

$$\mu_{\overline{R}_c(y_{\widehat{\alpha}})}(x) = \vee\{\mu_{R_c}(x,u) \wedge \mu_{y_{\widehat{\alpha}}}(u) : u \in X\} = \mu_{R_c}(x,y) \wedge \alpha = \mu_{c(1_y)}(x) \wedge \alpha$$
$$= \mu_{(c(1_y)\wedge\widehat{\alpha})}(x) = \mu_{c(1_y\wedge\widehat{\alpha})}(x)$$

and

$$\nu_{\overline{R}_c(y_{\widehat{\beta}})}(x) = \wedge\{\nu_{R_c}(x,u) \vee \nu_{y_{\widehat{\beta}}}(u) : u \in X\} = \nu_{R_c}(x,y) \vee \beta = \nu_{c(1_y)}(x) \vee \beta$$
$$=\nu_{(c(1_y)\vee\hat{\beta})}(x) = \nu_{c(1_y\vee\hat{\beta})}(x).$$

Thus $\overline{R}_c(y_{\widehat{(\alpha,\beta)}})(x) = c(1_y \wedge \widehat{(\alpha,\beta)})(x) = c(y_{\widehat{(\alpha,\beta)}})(x)$. Further,

$$\mu_{\overline{R}_c(A)}(x) = \cup_{y\in X}\mu_{c(y_{\widehat{(\alpha,\beta)}})}(x) = \mu_{c(\cup_{y\in X} y_{(\alpha,\beta)})}(x) = \mu_{c(A)}(x)$$

and

$$\nu_{\overline{R}_c(A)}(x) = \cap_{y\in X}\nu_{c(y_{(\alpha,\beta)})}(x) = \nu_{c(\cap_{y\in X} y_{(\alpha,\beta)})}(x) = \nu_{c(A)}(x)$$

and

$$\mu_{\overline{R}_c(A)}(x) = \cup_{y\in X}\mu_{c(y_{\widehat{(\alpha,\beta)}})}(x) = \mu_{c(\cup_{y\in X} y_{\widehat{(\alpha,\beta)}})}(x) = \mu_{c(A)}(x)$$

and

$$\nu_{\overline{R}_c(A)}(x) = \cap_{y\in X}\nu_{c(y_{\widehat{(\alpha,\beta)}})}(x) = \nu_{c(\cap_{y\in X} y_{\widehat{(\alpha,\beta)}})}(x) = \nu_{c(A)}(x).$$

Thus, $\overline{R}_c(A) = c(A), \forall A \in IFS(X)$. To show the uniqueness of IF-relation R_c, let $R' = (\mu_{R'}, \nu_{R'})$ be another IF-reflexive relation on X such that $\mu_{\overline{R'}(A)}(x) = \mu_{c(A)}$ and $\nu_{\overline{R'}(A)}(x) = \nu_{c(A)}, \forall A \in IFS(X)$. Then

$$\mu_{R_c}(x,y) = \mu_{c(1_y)}(x) = \mu_{\overline{R'}(1_y)}(x) = \vee\{\mu_{R'}(x,u) \wedge \mu_{1_y}(u) : u \in X\} = \mu_{R'}(x,y)$$

and

$$\nu_{R_c}(x,y) = \nu_{c(1_y)}(x) = \nu_{\overline{R'}(1_y)}(x) = \wedge\{\mu_{R'}(x,u) \vee \nu_{1_y}(u) : u \in X\} = \nu_{R'}(x,y).$$

Thus $R_c = R'$. Hence the IF-relation R_c on X is unique. □

Proposition 4.5. *Let* **F** *be the set of all IF-reflexive approximation space and* **T** *be the set of all quasi-discrete IF-closure spaces satisfying* $c(A \wedge \widehat{(\alpha,\beta)}) = c(A) \wedge \widehat{(\alpha,\beta)}, \forall A \in IFS(X)$ *and* $\forall(\alpha,\beta) \in I^*$. *Then there exists a bijective correspondence between* **F** *and* **T**.

Proof. Follows from Propositions 4.1 and 4.4. □

Proposition 4.6. *Let* (X, c) *be a quasi-discrete IF-closure space satisfying* $c(A \wedge \widehat{(\alpha,\beta)}) = c(A) \wedge \widehat{(\alpha,\beta)}, \forall A \in IFS(X)$ *and* $\forall(\alpha,\beta) \in I^*$ *and* R_c *be an IF-reflexive relation on* X *such that* $\overline{R}_c(A) = c(A), \forall A \in IFS(X)$. *Then* $\tau_{R_c} = \tau_{\bar{c}}$.

Proof. Let $A \in \tau_{\bar{c}}$. Then $\bar{c}(A^c) = A^c$, i.e., $\mu_{\bar{c}(A^c)} = \mu_{A^c}$ and $\nu_{\bar{c}(A^c)} = \nu_{A^c}$. As from Proposition 3.10, $\mu_{c(A)} \leq \mu_{\bar{c}(A)}$ and $\nu_{c(A)} \geq \nu_{\bar{c}(A)}, \forall A \in IFS(X)$, we have $\mu_{c(A^c)} \leq \mu_{A^c}$ and $\nu_{c(A^c)} \geq \nu_{A^c}$, or that $\mu_A \leq \mu_{[c(A^c)]^c}$ and $\nu_A \geq \nu_{[c(A^c)]^c}$, showing that $\mu_A \leq \mu_{\underline{R}_c(A)}$ and $\nu_A \geq \nu_{\underline{R}_c(A)}$. Also $\mu_{\underline{R}_c(A)} \leq \mu_A$ and $\nu_{\underline{R}_c(A)} \geq \nu_A$, whereby $\underline{R}_c(A) = A$. Thus $\tau_{\bar{c}} \leq \tau_{R_c}$. Conversly, let $A \in \tau_{R_c}$. Then $\mu_{\underline{R}_c(A)} = \mu_A$ and $\nu_{\underline{R}_c(A)} = \nu_A$, or that $\mu_{[\overline{R}_c(A^c)]^c} = \mu_A$ and $\nu_{[\overline{R}_c(A^c)]^c} = \nu_A$, i.e., $\mu_{\overline{R}_c(A^c)} = \mu_{A^c}$ and $\nu_{\overline{R}_c(A^c)} = \nu_{A^c}$, whereby $\mu_{c(A^c)} = \mu_{A^c}$ and $\nu_{c(A^c)} = \nu_{A^c}$. Thus from Proposition 3.10, $\mu_{\bar{c}(A^c)} = \mu_{A^c}$ and $\nu_{\bar{c}(A^c)} = \nu_{A^c}$, whereby $A \in \tau_{\bar{c}}$, or that $\tau_{R_c} \leq \tau_{\bar{c}}$. Hence $\tau_{R_c} = \tau_{\bar{c}}$. □

For a given quasi-discrete IF-closure space (X, c) satisfying $c(A \wedge \widehat{(\alpha,\beta)}) = c(A) \wedge \widehat{(\alpha,\beta)}, \forall A \in IFS(X)$, $\forall(\alpha,\beta) \in I^*$ and its associated Kuratowski IF-closure operator $\bar{c}, (X, \bar{c})$ is obviously a quasi-discrete IF-closure space such that $\bar{c}(A \wedge \widehat{(\alpha,\beta)}) = \bar{c}(A) \wedge \widehat{(\alpha,\beta)}, \forall A \in IFS(X)$, $\forall(\alpha,\beta) \in I^*$ and hence Proposition 4.4, will induce an IF-reflexive relation, say, $S_{\bar{c}} = (\mu_{S_{\bar{c}}}, \nu_{S_{\bar{c}}})$ on X, given by $S_{\bar{c}}(x,y) = \bar{c}(1_y)(x)$, i.e., $\mu_{S_{\bar{c}}}(x,y) = \mu_{\bar{c}(1_y)}(x), \nu_{S_{\bar{c}}}(x,y) = \nu_{\bar{c}(1_y)}(x), \forall x, y \in X$.

Before stating next, we recall the following concept of IF-transitive closure.

Definition 4.7. Let R and T be two IF-relations on X. Then T is called IF-transitive closure of R if

- (i) T is an IF-transitive,
- (ii) $R \leq T$, and
- (iii) if S is an IF-transitive with $R \leq S$, then $T \leq S$, i.e., T is the smallest IF-transitive relation containing R.

Before stating the next proposition we need to prove the following lemma.

Lemma 4.8. *Let* (X, R) *and* (X, S) *be two IF-approximation spaces. Then* $R \leq S$ *if and only if* $\overline{R}(A) = (\mu_{\overline{R}(A)}, \nu_{\overline{R}(A)}) \leq (\mu_{\overline{S}(A)}, \nu_{\overline{S}(A)}) = \overline{S}(A), \forall A \in IFS(X)$.

Proof. Let $\mu_{\overline{R}(A)} \leq \mu_{\overline{S}(A)}$ and $\nu_{\overline{R}(A)} \geq \nu_{\overline{S}(A)}, \forall A \in IFS(X)$. Then
$\mu_{\overline{R}(A)}(x) \leq \mu_{\overline{S}(A)}(x), \forall x \in X$, whereby

$$\vee\{\mu_R(x,y) \wedge \mu_A(y) : y \in X\} \leq \vee\{\mu_S(x,y) \wedge \mu_A(y) : y \in X\}$$
$$\Rightarrow \mu_R(x,y) \leq \mu_S(x,y), \forall x, y \in X$$

and
$\nu_{\overline{R}(A)}(x) \geq \nu_{\overline{S}(A)}(x), \forall x \in X$, whereby

$$\wedge\{\nu_R(x,y)\vee\nu_A(y):y\in X\}\geq\wedge\{\nu_S(x,y)\vee\nu_A(y):y\in X\}$$

$$\Rightarrow\nu_R(x,y)\geq\nu_S(x,y),\forall x,y\in X.$$

Hence $R\leq S$.
Conversly, let $R\leq S$, i.e., $\mu_R\leq\mu_S$ and $\nu_R\geq\nu_S$. We have to show that $\mu_{\overline{R}(A)}\leq\mu_{\overline{S}(A)}$ and $\nu_{\overline{R}(A)}\geq\nu_{\overline{S}(A)},\forall A\in IFS(X)$. Now,

$$\mu_{\overline{R}(A)}(x)=\vee\{\mu_R(x,y)\wedge\mu_A(y):y\in X\}\leq\vee\{\mu_S(x,y)\wedge\mu_A(y):y\in X\}$$
$$=\mu_{\overline{S}(A)}(x),\forall x\in X$$

and

$$\nu_{\overline{R}(A)}(x)=\wedge\{\nu_R(x,y)\vee\nu_A(y):y\in X\}\geq\wedge\{\nu_S(x,y)\vee\nu_A(y):y\in X\}$$
$$=\nu_{\overline{S}(A)}(x),\forall x\in X.$$

Hence $\mu_{\overline{R}(A)}\leq\mu_{\overline{S}(A)}$ and $\nu_{\overline{R}(A)}\geq\nu_{\overline{S}(A)},\forall A\in IFS(X)$, i.e., $\overline{R}(A)\leq\overline{S}(A),\forall A\in IFS(X)$. □

Proposition 4.9. *Let (X,c) be a quasi-discrete IF-closure space such that $c(A\wedge\widehat{(\alpha,\beta)})=c(A)\wedge\widehat{(\alpha,\beta)},\forall A\in IFS(X)$ and $\forall(\alpha,\beta)\in I^*$ and $\bar{c}$ be the associated Kuratowski IF-closure operator. Then the IF-relation $S_{\bar{c}}=(\mu_{S_{\bar{c}}},\nu_{S_{\bar{c}}})$ is the IF-transitive closure of IF-relation $R_c=(\mu_{R_c},\nu_{R_c})$.*

Proof. Let $S_{\bar{c}}(x,y)=\bar{c}(1_y)(x)$, i.e., $(\mu_{S_{\bar{c}}}(x,y),\nu_{S_{\bar{c}}}(x,y))=(\mu_{\bar{c}(1_y)}(x),\nu_{\bar{c}(1_y)}(x)),\forall x$, $y\in X$. Then from Propositions 2.10 and 4.4, $S_{\bar{c}}$ is the IF-transitive relation on X. Also, from Proposition 3.10, it follows that $\mu_{R_c}\leq\mu_{S_{\bar{c}}}$ and $\nu_{R_c}\geq\nu_{S_{\bar{c}}}$. To show that relation $S_{\bar{c}}$ is the IF-transitive closure of IF-relation R_c, it only remains to show that $S_{\bar{c}}$ is the smallest IF-reflexive transitive relation containing R_c. So, let $T=(\mu_T,\nu_T)$ be another IF-reflexive transitive relation on X such that $\mu_{R_c}\leq\mu_T$ and $\nu_{R_c}\geq\nu_T$. Then from the reflexivity of $T,(X,\overline{T})$ is quasi-discrete IF-closure space. Now by using the fact that T is IF-transitive also, and by using Proposition 3.11 followed by Proposition 2.15,

$$\mu_{\overline{T}(A)}=\mu_{\wedge\{B\in IFS(X):A\leq B,\overline{T}(B)=B\}}$$

and

$$\nu_{\overline{T}(A)}=\nu_{\vee\{B\in IFS(X):A\geq B,\overline{T}(B)=B\}},$$

$\forall A\in IFS(X)$. Also, $S_{\bar{c}}$ being IF-reflexive and IF-transitive relation associated with Kuratowski IF-closure operator $\bar{c}$, from Proposition 2.15, it follows that $\mu_{\overline{S}_{\bar{c}}(A)}=\mu_{\bar{c}(A)}$ and $\nu_{\overline{S}_{\bar{c}}(A)}=\nu_{\bar{c}(A)},\forall A\in IFS(X)$. Finally, $\bar{c}$ being Kuratowski IF-closure operator associated with quasi-discrete IF-closure space (X,c),

$$\mu_{\bar{c}(A)}=\wedge\,\mu_{\{B\in IFS(X):A\leq B,c(B)=B\}}=\wedge\mu_{\{B\in IFS(X):A\leq B,\overline{R}_c(B)=B\}}$$

and

$$\nu_{\bar{c}(A)}=\vee\,\nu_{\{B\in IFS(X):A\geq B,c(B)=B\}}=\vee\,\nu_{\{B\in IFS(X):A\geq B,\overline{R}_c(B)=B\}},$$

$\forall A\in IFS(X)$ (cf., Proposition 4.4). Thus from Lemma 4.8,

$$\mu_{\overline{S_{\bar{c}}}(A)} = \wedge\, \mu_{\{B\in IFS(X):A\le B,\overline{R}_c(B)=B\}} \le \wedge\, \mu_{\{B\in IFS(X):A\le B,\overline{T}(B)=B\}} = \mu_{\overline{T}(A)}$$

and

$$\nu_{\overline{S_{\bar{c}}}(A)} = \vee\, \nu_{\{B\in IFS(X):A\ge B,\overline{R}_c(B)=B\}} \ge \vee\, \nu_{\{B\in IFS(X):A\ge B,\overline{T}(B)=B\}} = \nu_{\overline{T}(A)},$$

whereby $\mu_{\overline{S_{\bar{c}}}(A)} \le \mu_{\overline{T}(A)}$ and $\nu_{\overline{S_{\bar{c}}}(A)} \ge \nu_{\overline{T}(A)}$, showing that $S_{\bar{c}} \le T$. □

Now, we show that there is a bijective correspondence between the set of all IF-tolerance approximation spaces and the set of all symmetric quasi-discrete IF-closure spaces satisfying an extra condition.

Proposition 4.10. *Let (X, R) be an IF-tolerance approximation space. Then $(X, \overline{R})$ is a symmetric quasi-discrete IF-closure space such that $\overline{R}(A \wedge \widehat{(\alpha,\beta)}) = \overline{R}(A) \wedge \widehat{(\alpha,\beta)}$, $\forall A \in IFS(X)$ and $\forall(\alpha,\beta) \in I^*$.*

Proof. Similar to that of Proposition 4.1. □

Proposition 4.11. *Let (X, c) be a symmetric quasi-discrete IF-closure space such that $c(A\wedge\widehat{(\alpha,\beta)}) = c(A)\wedge\widehat{(\alpha,\beta)}$, $\forall A \in IFS(X)$ and $\forall(\alpha,\beta) \in I^*$. Then there exists a unique IF-tolerance relation R_c on X such that $\overline{R}_c(A) = c(A), \forall A \in IFS(X)$.*

Proof. Follows from Proposition 4.4 and the fact that (X, c) is a symmetric IF-closure space. □

Proposition 4.12. *Let* **F** *be the set of all IF-tolerance approximation spaces and* **T** *be the set of all symmetric quasi-discrete IF- closure spaces satisfying $c(A\wedge\widehat{(\alpha,\beta)}) = c(A) \wedge \widehat{(\alpha,\beta)}$, $\forall A \in IFS(X)$ and $\forall(\alpha,\beta) \in I^*$. Then there exists a bijective correspondence between* **F** *and* **T**.

Proof. Follows from Propositions 4.10 and 4.11. □

Proposition 4.13. *Let (X, c) be a symmetric quasi-discrete IF-closure space such that $c(A\wedge\widehat{(\alpha,\beta)}) = c(A)\wedge\widehat{(\alpha,\beta)}$, $\forall A \in IFS(X)$, $\forall(\alpha,\beta) \in I^*$ and $\bar{c}$ be the associated Kuratowski IF-closure operator. Then the IF-relation $S_{\bar{c}}$ is an IF-transitive closure of an IF-relation R_c.*

Proof. Follows from Proposition 4.9. □

5. Conclusions

We have tried to introduce the concept of IF-closure spaces and establish their relationship with IF-approximation spaces. The notable results we have shown here are the bijective correspondence between the family of all IF-reflexive approximation spaces and the family of all quasi-discrete IF-closure spaces satisfying certain extra conditions as well as the bijective correspondence between the family of all IF-tolerance approximation spaces and the family of all symmetric quasi-discrete IF-closure spaces satisfying certain extra conditions.

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Anupam K. Singh (anupam09.bhu@gmail.com)
Amity Institute of Applied Science (AIAS), Amity University, Sector-125, Noida Uttar Pradesh-201313, India

S. P. Tiwari (sptiwarimaths@gmail.com)
Department of Applied Mathematics, Indian School of Mines, Dhanbad-826004, India