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Random fuzzy integral equations of Urysohn-Volterra type

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Abstract. In this paper, the existence and uniqueness results for the random fuzzy integral equations of Urysohn- Volterra type is first proven. The continuity of solutions with respect to the coefficients of the equations is investigated. Moreover, we provide examples to illustrate the results.

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1. Introduction

The fuzzy differential and integral equation (see e.g. [2, 3, 4, 6, 7, 10, 11, 19, 20, 22, 23, 24] and references therein) are a very important topic of fuzzy analytic theory as well as on their applications (see e.g.[1, 12, 21]). The study of fuzzy differential equation has been extended to the random fuzzy differential equations (see e.g. [13, 14, 15, 16, 25, 26, 27]). The random fuzzy differential equations (RFDEs) can provide good models of the dynamics of real phenomena which are subjected to two kinds of uncertainty: randomness and fuzziness, simultaneously. Here, it is called a fuzzy random variable. We can find various definitions of fuzzy random variables in [17, 18, 28]. The relations between different concepts of measurability for fuzzy random variables are contained in the paper of Kim [28]. Malinowski [13] studied the existence and uniqueness of solution to RFDEs under classical Hukuhara derivative. In [14, 15] the different types of solution to RFDEs with kinds of two different concepts of fuzzy derivative is studied. For the existence and uniqueness of solution to RFDEs in [13, 14, 15, 16], author used the method of successive approximations. From the idea of the paper [13, 14, 15, 16], Park and Jeong [9] discussed the existence and uniqueness of solution to random functional fuzzy differential equations.

In this paper, inspired and motivated by Fei (see [25, 26]) and Malinowski (see e.g. [13, 14, 15, 16]). We consider the random fuzzy integral equations of Urysohn-

Volterra type. The paper will be organized as follows: In section 2 we collect the fundamental notions and facts about fuzzy set space. We recall the notions of fuzzy random variable and fuzzy stochastic process. In section 3, we prove the existence and uniqueness results for the random fuzzy integral equations of Urysohn- Volterra type. The continuity of solutions with respect to the coefficients of the equations is discussed. In Section 4, we provide examples to illustrate the results.

2. Preliminaries

Let $K_c(\mathbb{R}^d)$ denote the collection of all nonempty compact and convex subsets of $\mathbb{R}^d$. The addition and scalar multiplication in $K_c(\mathbb{R}^d)$, we define as usual, i.e. $A, B \in K_c(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$, then we have

$$A + B = \{a + b \,|\, a \in A, b \in B\}, \qquad \lambda A = \{\lambda a \,|\, a \in A\}.$$

The Hausdorff distance d_H in $K_c(\mathbb{R}^d)$ is defined as follows

$$d_H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|_{R^d}, \sup_{b\in B} \inf_{a\in A} \|a - b\|_{R^d}\},$$

where $A, B \in K_c(\mathbb{R}^d)$, $\|.\|_{R^n}$ denotes the Euclidean norm in $\mathbb{R}^d$. It is known that $(K_c(\mathbb{R}^d), d_H)$ is a complete a metric space.

Denote $E^d = \{u : \mathbb{R}^d \to [0, 1]$ such that $u(z)$ satisfies (i)-(iv) below$\}$

(i) u is normal, that is, there exists an $z_0 \in \mathbb{R}^d$ such that $u(z_0) = 1$;
(ii) u is fuzzy convex, that is, for $0 \leq \lambda \leq 1$, $u(\lambda z_1+(1-\lambda)z_2) \geq \min\{u(z_1), u(z_2)\}$, for any $z_1, z_2 \in \mathbb{R}^d$;
(iii) u is upper semicontinuous;
(iv) $[u]^0 = cl\{z \in \mathbb{R}^d : u(z) > 0\}$ is compact.

Although elements of E^d are often called the fuzzy numbers, we shall just call them the fuzzy sets. For $\alpha \in (0, 1]$, denote $[u]^\alpha = \{z \in \mathbb{R}^d \,|\, u(z) \geq \alpha\}$. We will call this set an $\alpha-$cut ($\alpha-$ level set) of the fuzzy set u. For $u \in E^d$ one has that $[u]^\alpha \in K_c(\mathbb{R}^d)$ for every $\alpha \in [0, 1]$. For two number fuzzys $u_1, u_2 \in \mathbb{E}^d$, we denote $u_1 \leq u_2$ if and only if $[u_1]^\alpha \subset [u_2]^\alpha$.

If $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a function then, according to Zadeh's extension principle, one can extend g to $E^d \times E^d \to E^d$ by the formula

$$g(u_1, u_2)(z) = \sup_{z=g(z_1,z_2)} \min\{u_1(z_1), u_2(z_2)\}.$$

It is well known that if g is continuous then $[g(u_1, u_2)]^\alpha = g([u_1]^\alpha, [u_2]^\alpha)$ for all $u_1, u_2 \in E^d, \alpha \in [0, 1]$. Especially, for addition and scalar multiplication in fuzzy set space E^d, we have

$$[u_1 + u_2]^\alpha = [u_1]^\alpha + [u_2]^\alpha, [\lambda u_1]^\alpha = \lambda[u_1]^\alpha.$$

where $u_1, u_2 \in E^d$, $\lambda \in \mathbb{R} \setminus \{0\}$ and $\alpha \in [0, 1]$.

Let us denote

$$D(u_1, u_2) = \sup\{d_H([u_1]^\alpha, [u_2]^\alpha) : 0 \leq \alpha \leq 1\}$$

the distance between u_1 and u_2 in E^d, where $d_H([u_1]^\alpha, [u_2]^\alpha)$ is Hausdorff distance between two set $[u_1]^\alpha, [u_2]^\alpha$ of $K_c(\mathbb{R}^d)$. It is easy to see that D is a metric in E^d.

Some properties of metric D are as follows:

$$\begin{aligned} D(u_1+u_3,u_2+u_3) &= D(u_1,u_2),\\ D(\lambda u_1,\lambda u_2) &= |\lambda| D(u_1,u_2),\\ D(u_1,u_2) &\le D(u_1,u_3)+D(u_3,u_2), \end{aligned}$$

for all $u_1, u_2, u_3 \in E^d$ and $\lambda \in \mathbb{R}$.

Let $u_1, u_2 \in E^d$, if there exists $u_3 \in E^d$ such that $u_1 = u_2 + u_3$ then u_3 is called the H-difference of u_1, u_2 and it is denoted $u_1 \ominus u_2$. Let us remark that $u_1 \ominus u_2 \neq u_1 + (-1)u_2$.

We define $\hat{0} \in E^d$ as $\hat{0}(x) = 1$ if $x = 0$, and $\hat{0}(x) = 0$ if $x \neq 0$.

The following remarks can be verified (see [4, 14, 15]).

Remark 2.1. If for fuzzy sets $u_1, u_2, u_3 \in E^d$ there exist H-difference of $u_1 \ominus u_2$ and $u_1 \ominus u_3$ then $D(u_1 \ominus u_2, \hat{0}) \le D(u_1, u_2)$ and $D(u_1 \ominus u_2, u_1 \ominus u_3) \le D(u_1, u_3)$.

Remark 2.2. If for fuzzy sets $u_1, u_2, u_3, u_4 \in E^d$ there exist H-difference of $u_1 \ominus u_2$ and $u_3 \ominus u_4$ then $D(u_1 \ominus u_2, u_3 \ominus u_4) \le D(u_1 + u_3, u_2 + u_4)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A function $x : \Omega \to E^d$ is called fuzzy random variable, if the set-valued mapping $[x(\cdot)]^\alpha : \Omega \to K_c(\mathbb{R}^d)$ is a measurable multiplication for all $\alpha \in [0,1]$, i.e. $\{\omega \in \Omega |\, [x(\omega)]^\alpha \cap B \neq \varnothing\} \in \mathcal{F}$ for every closed set $B \subset \mathbb{R}^n$. A mapping $x : [a,b] \times \Omega \to E^d$ is said to be a fuzzy stochastic process if $x(\cdot,\omega)$ is a fuzzy set-valued function with any fixed $\omega \in \Omega$, and $x(t,\cdot)$ is a fuzzy random variable for any fixed $t \in \mathbb{R}^+$. In [13], $x(\cdot,\omega)$ function is called a trajectory. Beside concepts above, a fuzzy stochastic process $x(t,\omega) \in E^d$ is called continuous if for almost all $\omega \in \Omega$ the trajectory $x(\cdot,\omega)$ is a continuous function on $[a,b]$ with respect to the metric D.

Assume that $u : [0,b] \times \Omega \times E^d \to E^d$ satisfies:

(c1) $u_{\cdot}(t,x) : \Omega \to E^d$ is a fuzzy random variable for every $t \in [0,b]$ and every $x \in E^d$;

(c2) with $\mathbb{P}.1$, the function $u_\omega(\cdot,\cdot) : [0,b] \times E^d \to E^d$ is a continuous fuzzy mapping at every point $(t_0, x_0) \in [0,b] \times E^d$ i.e. there exists $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ and such that for every $\omega \in \Omega_0$ the following is true: for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $t \in [0,b]$, and every $x \in E^d$ it holds

$$\max\{|t-t_0|, D(x,x_0)\} < \delta \Longrightarrow D(u_\omega(t,x), u_\omega(t_0,x_0)) < \epsilon.$$

3. Main result

In this paper, we consider the two kinds of random fuzzy integral equations of Urysohn- Volterra type as follows:

$$x(t,\omega) \overset{[0,b],\,\mathbb{P}.1}{=} x_0(\omega) + \int_0^t g_\omega(t,s,u_\omega(s,x(s,\omega)))ds \tag{3.1}$$

and

$$x_0(\omega) \overset{[0,b],\,\mathbb{P}.1}{=} x(t,\omega) \ominus (-1)\int_0^t g_\omega(t,s,u_\omega(s,x(s,\omega)))ds \tag{3.2}$$

Where $g : \Omega \times [0,b] \times [0,b] \times E^d \to E^d$, $x_0 : \Omega \to E^d$ is fuzzy random variable and $u : \Omega \times [0,b] \times E^d \to E^d$ are a fuzzy stochastic process.

Denote $\mathscr{B}_\rho(x_0) = \{x \in E^d : D(x, x_0) \leq \rho\}$, $\rho \geq 0$. Assume that $g : \Omega \times [0,b] \times [0,b] \times \mathscr{B}_\rho(x_0) \to E^d$ and $u : \Omega \times [0,b] \times \mathscr{B}_\rho(x_0) \to E^d$ and satisfy the conditions:

(g1) there exists a constant $M_g > 0$ such that $D(g_\omega(t,s,x), \hat{0}) \leq M_g$, for every $(t,s) \in [0,b] \times [0,b]$, every $x \in \mathscr{B}_\rho(x_0)$.

(g2) there exists a stochastic process $L_g : [0,b] \times \Omega \to \mathbb{R}^+$, $L_g(\cdot, \omega)$ is continuous with $\mathbb{P}.1$ such that $D(g_\omega(t,s,x), g_\omega(t,s,y)) \leq L_g(t,\omega) D(x,y)$, for every $(t,s) \in [0,b] \times [0,b]$, every $x, y \in \mathscr{B}_\rho(x_0)$.

(u1) there exists a stochastic process $L_u : [0,b] \times \Omega \to \mathbb{R}^+$, $L_u(\cdot,\omega)$ is continuous with $\mathbb{P}.1$, such that $D(u_\omega(t,x), u_\omega(t,y)) \leq L_u(t,\omega) D(x,y)$, for every $t \in [0,b]$, every $x, y \in \mathscr{B}_\rho(x_0)$.

Theorem 3.1. Let $g : \Omega \times [0,b] \times [0,b] \times \mathscr{B}_\rho(x_0) \to E^d$ and $u : \Omega \times [0,b] \times \mathscr{B}_\rho(x_0) \to E^d$ satisfy assumptions (g1) - (g2), (u1) and (c1) - (c2). Assume that there exist some positive constant M_u such that $D(u_\omega(t, x_0(t,\omega)), \hat{0}) \overset{[0,b],\, \mathbb{P}.1}{\leq} M_u$. Then (3.1) has unique solution $x(t,\omega)$ on $[0,b]$.

Proof. The proof of Theorem 3.1 is similarly the proof of Theorem 3.2. Therefore, we shall prove Theorem 3.2. □

Theorem 3.2. Let $g : \Omega \times [0,b] \times [0,b] \times \mathscr{B}_\rho(x_0) \to E^d$ and $u : \Omega \times [0,b] \times \mathscr{B}_\rho(x_0) \to E^d$ satisfy assumptions (g1) - (g2), (u1) and (c1) - (c2). Assume that there exist some positive constant M_u such that $D(u_\omega(t, x_0(t,\omega)), \hat{0}) \overset{[0,b],\, \mathbb{P}.1}{\leq} M_u$. Assume that there exists $\eta > 0$ such that the sequence $\{x_n\}_{n=0}^\infty$ to (3.2) given by

$$\begin{aligned} x_0(t,\omega) &\overset{[0,\eta],\, \mathbb{P}.1}{=} x_0(\omega), \\ x_n(t,\omega) &\overset{[0,\eta],\, \mathbb{P}.1}{=} x_0(\omega) \ominus (-1) \int_0^t g_\omega(t,s,u_\omega(s, x_{n-1}(s,\omega)))ds, \quad n = 1, 2, \ldots. \end{aligned}$$

is well-defined, i.e. the foregoing Hukuhara differences do exist. Then (3.2) has unique solution $x(t,\omega)$ on $[0,r]$, $r \leq b$.

Proof. Define $\gamma := \{b, \rho M_g^{-1}\}$. Note that for every $t \in [0,\gamma]$, every $\omega \in \Omega$ we have

$$\begin{aligned} D(x_1(t,\omega), x_0(t,\omega)) &\overset{[0,\gamma],\, \mathbb{P}.1}{=} D\left(x_0(\omega) \ominus (-1) \int_0^t g_\omega(t,s,u_\omega(s, x_0(\omega)))ds, x_0(\omega)\right) \\ &\overset{[0,\gamma],\, \mathbb{P}.1}{\leq} D\left(\int_0^t g_\omega(t,s,u_\omega(s,x_0(\omega)))ds, \hat{0}\right) \overset{[0,\gamma],\, \mathbb{P}.1}{\leq} M_g \gamma. \end{aligned}$$

and for every $n \geq 2$

$$\begin{aligned} D(x_n(t,\omega), x_{n-1}(t,\omega)) &\overset{[0,\gamma],\,\mathbb{P}.1}{=} D\left(\int_0^t g_\omega(t,s,u_\omega(s,x_{n-1}(s,\omega)))ds, \int_0^t g_\omega(t,s,u_\omega(s,x_{n-2}(s,\omega)))ds \right) \\ &\overset{[0,\gamma],\,\mathbb{P}.1}{\leq} \int_0^t D(g_\omega(t,s,u_\omega(s,x_{n-1}(s,\omega))), g_\omega(t,s,u_\omega(s,x_{n-2}(\omega))))ds \\ &\overset{[0,\gamma],\,\mathbb{P}.1}{\leq} \int_0^t L_g(s,\omega) D(u_\omega(s,x_{n-1}(s,\omega)), u_\omega(s,x_{n-2}(s,\omega)))ds \\ &\overset{[0,\gamma],\,\mathbb{P}.1}{\leq} L_g(\omega) \int_0^t L_u(s,\omega) D(x_{n-1}(s,\omega), x_{n-2}(s,\omega))ds \\ &\overset{[0,\gamma],\,\mathbb{P}.1}{\leq} L_g(\omega) L_u(\omega) \int_0^t D(x_{n-1}(s,\omega), x_{n-2}(s,\omega))ds \end{aligned}$$

where $L_g(\omega) \overset{\mathbb{P}.1}{=} \sup\limits_{s\in[0,\gamma]} L_g(s,\omega)$, $L_u(\omega) \overset{\mathbb{P}.1}{=} \sup\limits_{s\in[0,\gamma]} L_u(s,\omega)$.

In particular,

$$\begin{aligned} D(x_2(t,\omega), x_1(t,\omega)) &\overset{[0,\gamma],\,\mathbb{P}.1}{=} D\left(\int_0^t g_\omega(t,s,u_\omega(s,x_1(s,\omega)))ds, \int_0^t g_\omega(t,s,u_\omega(s,x_0(s,\omega)))ds \right) \\ &\overset{[0,\gamma],\,\mathbb{P}.1}{\leq} \int_0^t D(g_\omega(t,s,u_\omega(s,x_1(\omega))), g_\omega(t,s,u_\omega(s,x_0(s,\omega))))ds \\ &\overset{[0,\gamma],\,\mathbb{P}.1}{\leq} \int_0^t L_g(s,\omega) D(u_\omega(s,x_1(s,\omega)), u_\omega(s,x_0(s,\omega)))ds \\ &\overset{[0,\gamma],\,\mathbb{P}.1}{\leq} L_g(\omega) L_u(\omega) \int_0^t D(x_1(s,\omega), x_0(s,\omega))ds \\ &\overset{[0,\gamma],\,\mathbb{P}.1}{\leq} \frac{M_g}{L_g(\omega)L_u(\omega)} \cdot \frac{[L_g(\omega)L_u(\omega)\cdot t)]^2}{2!}, \end{aligned}$$

where $L_g(\omega) \overset{\mathbb{P}.1}{=} \sup\limits_{s\in[0,\gamma]} L_g(s,\omega)$, $L_u(\omega) \overset{\mathbb{P}.1}{=} \sup\limits_{s\in[0,\gamma]} L_u(s,\omega)$.

Further, if we assume that

$$\begin{aligned} D(x_{n-1}(t,\omega), x_{n-2}(t,\omega)) &\overset{[0,\gamma],\,\mathbb{P}.1}{\leq} L_g(\omega) L_u(\omega) \int_0^t D(x_{n-2}(s,\omega), x_{n-1}(s,\omega))ds \\ &\overset{[0,\gamma],\,\mathbb{P}.1}{\leq} \frac{M_g}{L_g(\omega)L_u(\omega)} \cdot \frac{[L_g(\omega)L_u(\omega)\cdot t]^{n-1}}{(n-1)!}. \end{aligned}$$

then we have

$$\begin{aligned} D(x_n(t,\omega), x_{n-1}(t,\omega)]) &\overset{[0,\gamma],\,\mathbb{P}.1}{\leq} L_g(\omega) L_u(\omega) \int_0^t D(x_{n-1}(s,\omega), x_{n-2}(s,\omega))ds \\ &\overset{[0,\gamma],\,\mathbb{P}.1}{\leq} \frac{M_g}{L_g(\omega)L_u(\omega)} \cdot \frac{[L_g(\omega)L_u(\omega)\cdot t]^{n}}{n!}. \end{aligned}$$

Hence, for $n > m > 0$ we have

$$\sup_{t\in[0,\gamma]} D(x_n(t,\omega), x_m(t,\omega)) \overset{\mathbb{P}.1}{\leq} \frac{M_g}{L_g(\omega)L_u(\omega)} \cdot \sum_{k=m+1}^{n} \frac{[L_g(\omega)L_u(\omega)\cdot t]^k}{k!}.$$

The almost sure convergence of the series $\sum_{k=m+1}^{n} \frac{[L_g(\omega)L_u(\omega)\cdot t]^k}{k!}$ implies that for any $\epsilon > 0$ one can find $n \in \mathbb{N}$ large enough, such that for $n \geq m \geq n_0$,

$$\sup_{t\in[0,\gamma]} D(x_n(t,\omega), x_m(t,\omega)) \overset{\mathbb{P}.1}{\leq} \epsilon. \tag{3.3}$$

We can infer that there exists $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$ and for every $\omega \in \Omega_0$ the sequence $\{x_n(\cdot,\omega)\}$ is uniformly convergent. For $\omega \in \Omega_0$, let us define $x : [0,\gamma]\times\Omega \to E^d$ in the following way: $x(\cdot,\omega) = x^*(\cdot,\omega)$ if $\omega \in \Omega_0$, and in the case $\omega \in \Omega\backslash\Omega_0$, $x(\cdot,\omega)$ as freely chosen fuzzy function. Then $\sup_{t\in[0,\gamma]} D(x_n(t,\omega), x(t,\omega)) \to 0$, as $n \to \infty$ with $\mathbb{P}.1$ and $x(t,\omega)$ is a continuous fuzzy stochastic process.

We shall prove that $x(t,\omega)$ is a solution of the problem (3.2). Since for every $t \in [0,\gamma]$ one has

$$\sup_{t\in[0,\gamma]} D\left(x_0(\omega) \ominus (-1)\int_0^t g_\omega(t,s,u_\omega(s,x_n(s,\omega)))ds, x_0(\omega) \ominus (-1)\int_0^t g_\omega(t,s,u_\omega(s,x(s,\omega)))ds\right)$$
$$\overset{\mathbb{P}.1}{\leq} L_g(\omega)L_u(\omega) \sup_{t\in[0,\gamma]} D(x_n(t,\omega), x(t,\omega)).$$

Thus, by Lesbesgue dominated convergence theorem, as $n \to \infty$, we infer that

$$D\left(x_0(\omega) \ominus (-1)\int_0^t g_\omega(t,s,u_\omega(s,x_n(s,\omega)))ds, x_0(\omega) \ominus (-1)\int_0^t g_\omega(t,s,u_\omega(s,x(s,\omega)))ds\right) \overset{\mathbb{P}.1}{\to} 0.$$

Note that

$$\sup_{t\in[0,\gamma]} D\left(x(t,\omega), x_0(\omega) \ominus (-1)\int_0^t g_\omega(t,s,u_\omega(s,x(s,\omega)))ds\right)$$
$$\overset{\mathbb{P}.1}{\leq} \sup_{t\in[0,\gamma]} D(x(t,\omega), x_n(t,\omega)) + \sup_{t\in[0,\gamma]} D\left(x_n(t,\omega), x_0(\omega) \ominus (-1)\int_0^t g_\omega(t,s,u_\omega(s,x_{n-1}(s,\omega)))ds\right)$$
$$+ \sup_{t\in[0,\gamma]} D\left(x_0(\omega) \ominus (-1)\int_0^t g_\omega(t,s,u_\omega(s,x_{n-1}(s,\omega)))ds, x_0(\omega) \ominus (-1)\int_0^t g_\omega(t,s,u_\omega(s,x(s,\omega)))ds\right)$$
$$\overset{\mathbb{P}.1}{\leq} \sup_{t\in[0,\gamma]} D(x(t,\omega), x_n(t,\omega)) + \sup_{t\in[0,\gamma]} D\left(x_n(t,\omega), x_0(\omega) \ominus (-1)\int_0^t g_\omega(t,s,u_\omega(s,x_{n-1}(s,\omega)))ds\right)$$
$$+ L_g(\omega)L_u(\omega) \sup_{t\in[0,\gamma]} D(x_{n-1}(t,\omega), x(t,\omega))$$

It is easy to see the first term of the right-hand side of the inequality uniformly converges to zero, whereas the second is equal to zero. One obtains

$$D\left(x(t,\omega), x_0(\omega) \ominus (-1)\int_0^t g_\omega(t,s,u_\omega(s,x(s,\omega)))ds\right) \overset{[0,\gamma],\,\mathbb{P}.1}{=} 0.$$

Hence the fuzzy process $x(t,\omega)$ is a solution to (3.2). □

In what follows, we will discuss the continuity of the solution to the random fuzzy integral equations of Urysohn- Volterra type with respect to the coefficients of the equations. Let us consider the equation

$$x(t,\omega) \overset{[0,\gamma],\,\mathbb{P}.1}{=} x_0(\omega) \ominus (-1)\int_0^t g_\omega(t,s,u_\omega(s,x(s,\omega)))ds \tag{3.4}$$

and a sequence of equation

$$x_n(t,\omega) \overset{[0,\gamma],\,\mathbb{P}.1}{=} x_{0,n}(\omega) \ominus (-1)\int_0^t g_\omega(t,s,u_\omega(s,x_n(s,\omega)))ds. \tag{3.5}$$

Theorem 3.3. Let $g : \Omega\times[0,b]\times[0,b]\times\mathscr{B}_\rho(x_0) \to E^d$ and $u : \Omega\times[0,b]\times\mathscr{B}_\rho(x_0) \to E^d$ be such as in Theorem 3.2. Let $x(t,\omega)$ denote the solution of (3.4) and $x_n(t,\omega)$ denote the solution of (3.5). Then

$$\sup_{t\in[0,\gamma)} D(x_n(t,\omega), x(t,\omega)) \overset{\mathbb{P}.1}{\leq} M\exp\{L_g(\omega)L_u(\omega)b\},$$

where $M \overset{[0,\gamma],\,\mathbb{P}.1}{=} D(x_{0,n}(\omega), x_0(\omega))$, $L_g(\omega) \overset{\mathbb{P}.1}{=} \sup_{s\in[0,\gamma]} L_g(s,\omega)$, $L_u(\omega) \overset{\mathbb{P}.1}{=} \sup_{s\in[0,\gamma]} L_u(s,\omega)$.

Proof. For every $t\in[0,\gamma]$, every $\omega\in\Omega$ we have

$$\begin{aligned}
&D(x_n(t,\omega), x(t,\omega))\\
&= D\left(x_0(\omega)\ominus(-1)\int_0^t g_\omega(t,s,u_\omega(s,x(s,\omega)))ds,\, x_{0,n}(\omega)\ominus(-1)\int_0^t g_\omega(t,s,u_\omega(s,x_n(s,\omega)))ds\right)\\
&\overset{[0,\gamma],\,\mathbb{P}.1}{\leq} D(x_{0,n}(\omega), x_0(\omega)) + D\left(\int_0^t g_\omega(t,s,u_\omega(s,x(s,\omega)))ds,\, \int_0^t g_\omega(t,s,u_\omega(s,x_n(s,\omega)))ds\right)\\
&\overset{[0,\gamma],\,\mathbb{P}.1}{\leq} D(x_{0,n}(\omega), x_0(\omega)) + \int_0^t D(g_\omega(t,s,u_\omega(s,x(s,\omega))), g_\omega(t,s,u_\omega(s,x_n(s,\omega))))ds\\
&\overset{[0,\gamma],\,\mathbb{P}.1}{\leq} D(x_{0,n}(\omega), x_0(\omega)) + L_g(\omega)L_u(\omega)\int_0^t D(x_n(s,\omega), x(s,\omega))ds
\end{aligned}$$

Due to Gronwall's lemma in [13] we obtain

$$D(x_n(t,\omega), x(t,\omega)) \overset{[0,\gamma],\,\mathbb{P}.1}{\leq} M\exp\{L_g(\omega)L_u(\omega)t\},$$

where $M \overset{[0,\gamma],\,\mathbb{P}.1}{=} D(x_{0,n}(\omega), x_0(\omega))$, $L_g(\omega) \overset{\mathbb{P}.1}{=} \sup_{s\in[0,\gamma]} L_g(s,\omega)$, $L_u(\omega) \overset{\mathbb{P}.1}{=} \sup_{s\in[0,\gamma]} L_u(s,\omega)$.

□

Corollary 3.4. Under the assumptions of Theorem 3.3. Let $x(t,\omega)$ and $x_n(t,\omega)$ denotes the solution of (3.5). Then

$$\sup_{t\in[0,\gamma]} D(x_n(t,\omega), x(t,\omega)) \overset{\mathbb{P}.1}{\leq} 0.$$

4. Example

In the section, we shall consider the special case of the random fuzzy integral equations of Urysohn- Volterra type. If $u_\omega(t, x(t,\omega)) = x(t,\omega)$ then the problem (3.2) can be written as

$$x(t,\omega) \overset{[0,b],\,\mathbb{P}.1}{=} x_0(\omega) \ominus (-1)\int_0^t g_\omega(t,s,x(t,\omega))ds. \tag{4.1}$$

Let us denote the α-cuts ($\alpha \in [0,1]$) of x as $[x(t,\omega)]^\alpha = [x_l^\alpha(t,\omega), x_r^\alpha(t,\omega)]$ and $[x_0(\omega)]^\alpha = [f_l^\alpha(t,\omega), f_r^\alpha(t,\omega)]$, where $x_l^\alpha(t,\omega)$, $x_r^\alpha(t,\omega)$ $f_l^\alpha(t,\omega)$ and $f_r^\alpha(t,\omega)$ are some real-valued stochastic process. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Consider the following random fuzzy integral equations

$$x(t,\omega) \overset{[0,b],\,\mathbb{P}.1}{=} x_0(\omega) \ominus (-1)\int_0^t k(t,s,\omega)x(s,\omega)ds. \tag{4.2}$$

Using the α-cuts of x, f we obtain

$$\begin{cases} x_l^\alpha(t,\omega) \overset{[0,b],\,\mathbb{P}.1}{=} f_l^\alpha(t,\omega) + \int_0^t \underline{k(t,s,\omega)x^\alpha(s,\omega)}ds, \\ x_r^\alpha(t,\omega) \overset{[0,b],\,\mathbb{P}.1}{=} f_r^\alpha(t,\omega) + \int_0^t \overline{k(t,s,\omega)x^\alpha(s,\omega)}ds \\ x_l^\alpha(0,\omega) \overset{\mathbb{P}.1}{=} f_l^\alpha(0,\omega), \\ x_r^\alpha(0,\omega) \overset{\mathbb{P}.1}{=} f_r^\alpha(0,\omega). \end{cases} \tag{4.3}$$

or

$$\begin{cases} x_r^\alpha(t,\omega) \overset{[0,b],\,\mathbb{P}.1}{=} f_l^\alpha(t,\omega) + \int_0^t \underline{k(t,s,\omega)x^\alpha(s,\omega)}ds, \\ x_l^\alpha(t,\omega) \overset{[0,b],\,\mathbb{P}.1}{=} f_r^\alpha(t,\omega) + \int_0^t \overline{k(t,s,\omega)x^\alpha(s,\omega)}ds, \\ x_l^\alpha(0,\omega) \overset{\mathbb{P}.1}{=} f_l^\alpha(0,\omega), \\ x_r^\alpha(0,\omega) \overset{\mathbb{P}.1}{=} f_r^\alpha(0,\omega). \end{cases} \tag{4.4}$$

where

$$\underline{k(t,s,\omega)x^\alpha(s,\omega)} = \begin{cases} k(t,s,\omega)x_l^\alpha(s,\omega), & k(t,s,\omega) \overset{[0,b],\,\mathbb{P}.1}{\geq} 0, \\ k(t,s,\omega)x_r^\alpha(s,\omega), & k(t,s,\omega) \overset{[0,b],\,\mathbb{P}.1}{<} 0, \end{cases}$$

$$\overline{k(t,s,\omega)x^\alpha(s,\omega)} = \begin{cases} k(t,s,\omega)x_r^\alpha(s,\omega), & k(t,s,\omega) \overset{[0,b],\,\mathbb{P}.1}{\geq} 0, \\ k(t,s,\omega)x_l^\alpha(s,\omega), & k(t,s,\omega) \overset{[0,b],\,\mathbb{P}.1}{<} 0. \end{cases}$$

Example 4.1. Let $\Omega = (0,1)$, $\mathcal{F}$- Borel σ-field of subsets of Ω, $\mathbb{P}$-Lebesgue measure on $(\Omega, \mathcal{F})$. Let us consider the integral equation

$$x(t,\omega) \overset{[0,5],\,\mathbb{P}.1}{=} [-\omega, 0, \omega] - \int_0^t x(s,\omega)ds, \tag{4.5}$$

where $x_0(\omega) \overset{[0,5],\,\mathbb{P}.1}{=} [-\omega, 0, \omega]$, $k(t, s, \omega) \overset{[0,5],\,\mathbb{P}.1}{=} 1$.

To determine the initial condition, we substitute $t = 0$ into both sides of (4.5) to find $x(0, \omega) \overset{\mathbb{P}.1}{=} [-\omega, 0, \omega]$. Then the problem (4.5) is equivalent to the following system:

$$(4.6) \qquad \begin{cases} x_l^\alpha(t,\omega) \overset{[0,5],\,\mathbb{P}.1}{=} (\alpha-1)\omega - \int_0^t x_r^\alpha(s,\omega)ds, \\ x_r^\alpha(t,\omega) \overset{[0,5],\,\mathbb{P}.1}{=} (1-\alpha)\omega - \int_0^t x_l^\alpha(s,\omega)ds, \\ x_l^\alpha(0,\omega) \overset{\mathbb{P}.1}{=} (\alpha-1)\omega, \\ x_r^\alpha(0,\omega) \overset{\mathbb{P}.1}{=} (1-\alpha)\omega. \end{cases}$$

The exact solution $x(\cdot, \cdot) : [0, 5] \times \Omega \to E^1$ is given by

$$[x(t, \omega)]^\alpha = [(\alpha - 1)\omega e^{-t}, 0, (1 - \alpha)\omega e^{-t}].$$

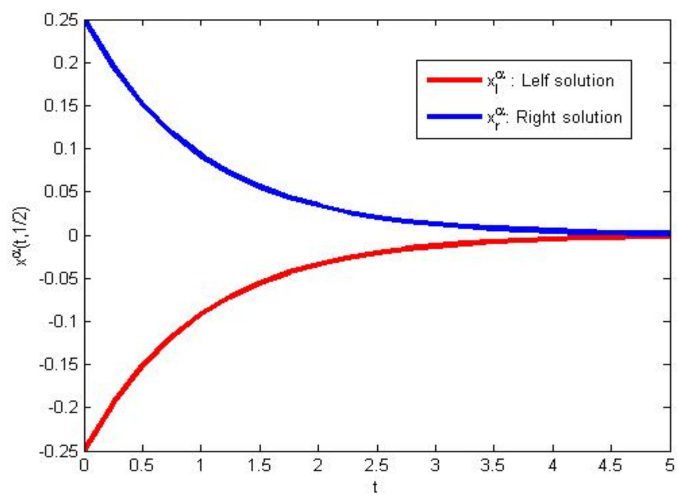

Figure 1. Graph 2D of the solution to (4.5)

From the graph of $[x(t, \omega)]^\alpha$ is drawn Fig 1. We see that diameter of valued of solution $[x(\cdot, \omega)]^\alpha$, ω is fixed, is decreasing with $\mathbb{P}.1$.

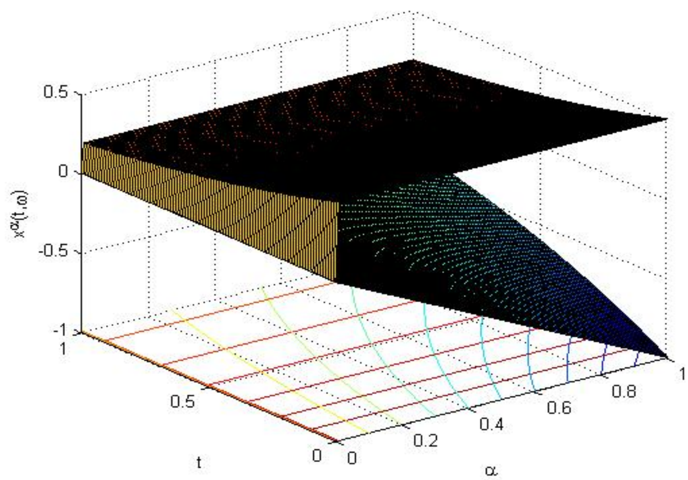

FIGURE 2. Graph 3D of the solution to (4.5)

Example 4.2. Let $\Omega = (1,3)$, $\mathcal{F}$- Borel σ-field of subsets of Ω, $\mathbb{P}$-Lebesgue measure on $(\Omega, \mathcal{F})$. Let us consider the integral equation

$$x(t,\omega) \overset{[0,5],\,\mathbb{P}.1}{=} [1+\omega, 2+\omega, 3+\omega] - \int_0^t e^{-(1+\omega)} x(s,\omega) ds, \tag{4.7}$$

where $x_0(\omega) \overset{[0,5],\,\mathbb{P}.1}{=} 0$, $k(t,s,\omega) \overset{[0,5],\,\mathbb{P}.1}{=} 1$.

To determine the initial condition, we substitute $t = 0$ into both sides of (4.7) to find $x(0,\omega) \overset{\mathbb{P}.1}{=} [1+\omega, 2+\omega, 3+\omega]$. Then the problem (4.7) is equivalent to the following a system:

$$\begin{cases} x_l^\alpha(t,\omega) \overset{[0,5],\,\mathbb{P}.1}{=} (1+\omega-\alpha) - \int_0^t e^{-(1+\omega)} x_r^\alpha(s,\omega) ds, \\ x_r^\alpha(t,\omega) \overset{[0,5],\,\mathbb{P}.1}{=} (3+\omega-\alpha) - \int_0^t e^{-(1+\omega)} x_l^\alpha(s,\omega) ds, \\ x_l^\alpha(0,\omega) \overset{\mathbb{P}.1}{=} 1+\omega-\alpha, \\ x_r^\alpha(0,\omega) \overset{\mathbb{P}.1}{=} 3+\omega-\alpha. \end{cases} \tag{4.8}$$

The exact solution $x(\cdot,\cdot) : [0,5] \times \Omega \to E^1$ is given by

$$[x(t,\omega)]^\alpha = [(1+\omega-\alpha)e^{-(1+\omega)t}, (3+\omega-\alpha)e^{-(1+\omega)t}].$$

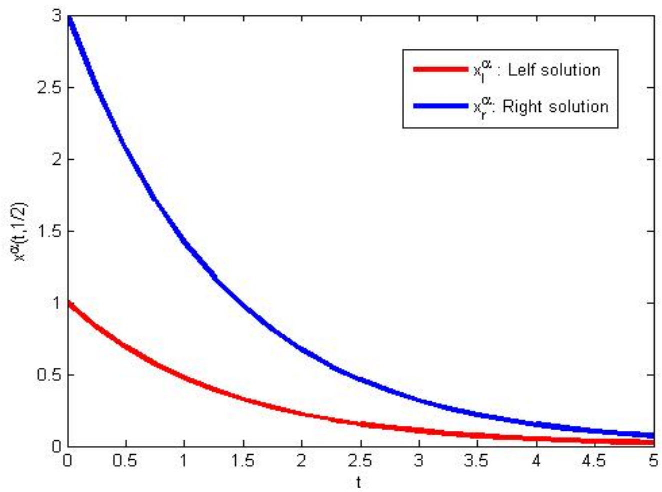

FIGURE 3. Graph 2D of the solution to (4.7)

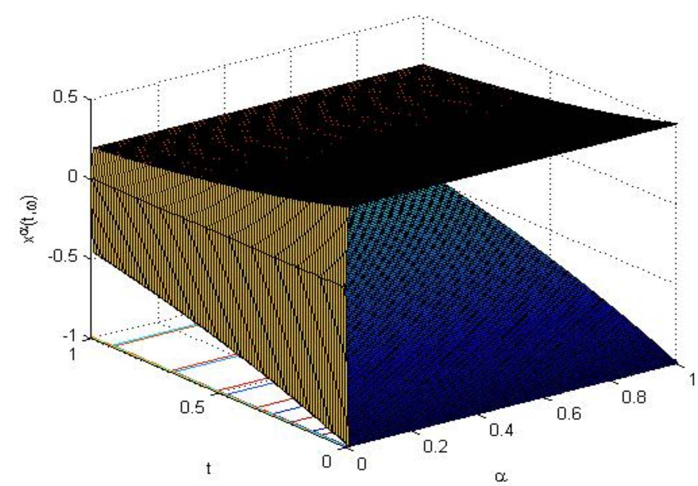

FIGURE 4. Graph 3D of the solution to (4.7)

From the graph of $[x(t,\omega)]^\alpha$ is drawn Fig 3. We see that diameter of valued of solution $[x(\cdot,\omega)]^\alpha$, ω is fixed, is decreasing with $\mathbb{P}.1$.

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References

[1] A. Bencsik, B. Bede, J. Tar and J. Fodor, Fuzzy differential equations in modeling hydraulic differential servo cylinders, in: Third Romanian-Hungarian Joint Symposium on Applied Computational Intelligence, SACI, Timisoara, Romania 2006.

[2] A. Esfahani, O. S. Fard and T. A. Bidgoli, On the existence and uniqueness of solutions to fuzzy boundary value problems, Ann. Fuzzy Math. Inform. 7 (2014) 14–25.

[3] B. Bede, I. J. Rudas and A. L. Bencsik, First order linear fuzzy differential equation sunder generalized differentiability, Inform. Sci. 177 (2007) 1648–1662.

[4] B. Bede and Sorin G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, Fuzzy Sets and Systems 151 (2005) 581–599.

[5] C. Lungan and V. Lupulescu, Random dynamical systems on time scales, Electron. J. Differential Equations 86 (2012) 01–14.

[6] V. Lakshmikantham and R. N. Mohapatra, Theory of Fuzzy Differential. Equations and Inclusions, Taylor & Francis, UK 2004.

[7] V. Lupulescu and U. Abbas, Fuzzy delay differential equations, Fuzzy Optim. Decis. Mak. 11 (2012) 99–111.

[8] V. Lupulescu, On a class of fuzzy functional differential equations, Fuzzy Sets and Systems 160 (2009) 1547–1562.

[9] J. Y. Park and J. U. Jeong, On random fuzzy functional differential equations, Fuzzy Sets and Systems 223 (2013) 89–99.

[10] J. Y. Park, S. Y. Lee and J. U. Jeong, The approximate solutions of fuzzy functional integral equations, Fuzzy Sets and Systems 110 (2000) 79–90.

[11] Juan J. Nieto and Rosana Rodriguez-Lopez, Bounded solutions for fuzzy differential and integral equations, Chaos Solitons Fractals 27 (2006) 1376–1386.

[12] L. C. Barros, R. C. Bassanezi and P. A. Tonelli, Fuzzy modelling in population dynamics, Ecol. Model. 128 (2000) 27–33.

[13] M. T. Malinowski, On random fuzzy differential equations, Fuzzy Sets and Systems 160 (2009) 3152–3165.

[14] M. T. Malinowski, Existence theorems for solutions to random fuzzy differential equations, Nonlinear Analysis: Theory, Methods & Applications 72 (2010) 1515–1532.

[15] M. T. Malinowski, Random fuzzy differential equations under generalized Lipschitz condition, Nonlinear Anal. Real World Appl. 13 (2012) 860–881.

[16] M. T. Malinowski, Peano type theorem for random fuzzy initial value problem, Discuss. Math. Differ. Incl. Control Optim. 31 (2011) 5–22.

[17] M. L. Puri and D. A. Ralescu, The concept of normality for fuzzy random variables , Ann. Probab. 13 (1985) 1373–1379.

[18] M. L. Puri and D. A. Ralescu , Fuzzy random variables, J. Math. Anal. Appl. 114 (1986) 409–422.

[19] M. Friedman, M. Ma and A. Kandel, Numerical solutions of fuzzy differential and integral equations, Fuzzy Sets and Systems 106 (1999) 35–48.

[20] M. A. Darwish, On the existence of a fuzzy integral equation of Urysohn-Volterra type, Discuss. Math. Differ. Incl. Control Optim. 28 (2008) 75–82.

[21] M. Oberguggenberger, S. Pittschmann, Differential equations with fuzzy parameters,Math. Modelling Syst. 5 (1999) 181–202.

[22] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems 24 (1987) 301–317.

[23] P. Diamond and P. Kloede, Metric spaces of fuzzy sets, World Scientific, UK 1998.

[24] Congxin Wu, Shiji Song, Haiyan Wang, On the basic solutions to the generalized fuzzy integral equation, Fuzzy Sets and Systems 95 (1998) 255–260.

[25] Weiyin Fei, Existence and uniqueness of solution for fuzzy random differential equations with non-Lipschitz coefficients, Inform. Sci. 177 (2007) 4329–4337

[26] Weiyin Fei, A generalization of Bihari's inequality and fuzzy random differential equations with non-Lipschitz coefficients, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 15 (2007) 425–439.

[27] Y. Feng, Fuzzy stochastic differential systems, Fuzzy Sets and Systems 115 (2000) 351–363.
[28] Y. K. Kim, Measurability for fuzzy valued functions, Fuzzy Sets and Systems 129 (2002) 105–109.

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