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Fuzzy n-fold integral and fuzzy n-fold Boolean ideals in BL-algebras

Akbar Paad

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Abstract. In this paper, we introduce concepts of fuzzy n-fold integral and fuzzy n-fold Boolean ideals in BL-algebras and we state and prove several property of fuzzy n-fold integral and fuzzy n-fold Boolean ideals. Using a level subset of a fuzzy set in a BL-algebra, we give characterization of fuzzy n-fold integral and fuzzy n-fold Boolean ideals. Also, we prove that the homomorphic image and preimage of fuzzy n-fold integral and fuzzy n-fold Boolean ideals are fuzzy n-fold integral and fuzzy n-fold Boolean ideals, respectively. Finally, we study relationship among fuzzy n-fold integral, fuzzy n-fold Boolean and fuzzy prime ideals in BL-algebras.

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Corresponding Author: Akbar Paad (akbar.paad@gmail.com)

1. Introduction

BL-algebras are the algebraic structure for Hájek basic logic [6] in order to investigate many valued logic by algebraic means. His motivations for introducing BL-algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Lukasiewicz Logic, Gödel Logic and Product Logic. This Basic Logic (BL for short) is proposed as "the most general"many-valued logic with truth values in [0,1] and BL-algebras are the corresponding Lindenbaum-Tarski algebras. The second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on [0, 1]. Most familiar example of a BL-algebra is the unit interval [0,1] endowed with the structure induced by a continuous t-norm. In 1958, Chang [1] introduced the concept of an MV-algebra which is one of the most classes of BL-algebras. Turunen [13] introduced the notion of an implicative filter and a *Boolean* filter and proved that these notions are equivalent in BL-algebras. *Boolean* filters are an important class of filters, because the quotient BL-algebra induced by these filters are *Boolean* algebras. Heveshki and Eslami [5] introduced the notions of n-fold implicative filter and n-fold positive implicative filter and they prove some relations between these filters and construct quotient algebras via these filters in 2008. Also, Motamed and Borumand Saeid [10] introduced the notions of n-fold obstinate filter in 2011. Moreover, Lele [7, 8] studied the notion of fuzzy n-fold (positive) implicative filter and fuzzy n-fold obstinate filter in BL-algebras. In 2012, Borzooei and Paad [11], introduced the notions of n-fold integral filter and n-fold integral BL algebra. The notion of (fuzzy) ideal has been introduced in many algebraic structures such as lattices, rings, MV-algebras. Ideal theory is very effective tool for studying various algebraic and logical systems. In the theory of MV-algebras, as various algebraic structures, the notion of ideal is at the center, while in BL-algebras, the focus has been on deductive systems also filters. The study of BL-algebras has experienced a tremendous growth over resent years and the main focus has been on filters. In the meantime, several authors have claimed in recent works that the notion of ideals is missing in BL-algebras. Zhang et al. [15] studied the notion of fuzzy ideals in BL-algebras and in 2013, Lele [7], introduced the notion of Boolean ideals and analyzed the relationship between ideals and filters by using the set of complement elements.

Now, in this paper, we introduce concepts of fuzzy n-fold integral and fuzzy n-fold Boolean ideals in BL-algebras and we state and prove several property of fuzzy n-fold integral and fuzzy n-fold Boolean ideals. Using a level subset of a fuzzy set in a BL-algebra, we give characterization of fuzzy n-fold integral and fuzzy n-fold Boolean ideals. Also, we prove that the homomorphic image and preimage of fuzzy n-fold integral and fuzzy n-fold Boolean ideals are fuzzy n-fold integral and fuzzy n-fold Boolean ideals, respectively. Finally, we study relationship among fuzzy n-fold integral, fuzzy n-fold Boolean and fuzzy prime ideals in BL-algebras.

2. Preliminaries

In this section, we give some fundamental definitions and results. For more details, refer to the references.

Definition 2.1 ([6])**.** A BL-algebra is an algebra $(L, \vee, \wedge, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that $(BL1)$ $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,

$(BL2)$ $(L, \odot, 1)$ is a commutative monoid,

$(BL3)$ $z \leq x \to y$ if and only if $x \odot z \leq y$, for all $x, y, z \in L$,

$(BL4)$ $x \wedge y = x \odot (x \to y)$,

$(BL5)$ $(x \to y) \vee (y \to x) = 1$.

We denote $x^n = \overbrace{x \odot ... \odot x}^{n-times}$, if $n > 0$ and $x^0 = 1$. Also, we denote $\overbrace{(x \to (...(x \to (x \to}^{n-times}$ $y)))...)$ by $x^n \to y$, for all $x, y \in L$. A BL-algebra L is called an MV-algebra, if $(x^-)^- = x$, for all $x \in L$, where $x^- = x \to 0$.

Proposition 2.2 ([2, 3])**.** *In any BL-algebra the following hold :*

$(BL6)$ $x \leq y$ *if and only if* $x \to y = 1$,

$(BL7)$ $x \odot y \leq x, y$ *and* $x^{n+1} \leq x^n$, $\forall n \in \mathbb{N}$,

$(BL8)$ $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$,

$(BL9)$ $0 \leq x$ *and* $x \odot x^- = 0$,

$(BL10)$ $1 \to x = x$ *and* $x \to 1 = 1$,
$(BL11)$ $(x \wedge y)^- = x^- \vee y^-$ *and* $(x \vee y)^- = x^- \wedge y^-$
for all $x, y, z \in L$.

Definition 2.3 ([5, 11]). Let L be a BL-algebra and F be a non-empty subset of L. Then

(i) F is called a filter of L, if $x \odot y \in F$, for any $x, y \in F$ and if $x \in F$ and $x \leq y$ then $y \in F$, for all $x, y \in L$.

(ii) A filter F is called a Boolean filter, if $x \vee x^- \in F$, for all $x \in L$.

(iii) A filter F is called an n-fold integral filter, if for all $x, y \in L$,

$$(x^n \odot y^n)^- \in F \text{ implies } (x^n)^- \in F \text{ or } (y^n)^- \in F$$

(iv) F is called an n-fold positive implicative filter of L, if $1 \in F$ and for all $x, y, z \in L$,

$$x \to ((y^n \to z) \to y) \in F \text{ and } x \in F \text{ imply } y \in F$$

(v) L is called an n-fold integral BL-algebra, if for all $x, y \in L$,

$$(x^n \odot y^n) = 0 \text{ then } x^n = 0 \text{ or } y^n = 0$$

Definition 2.4 ([7, 12]). Let L be a BL-algebra and I be a non-empty subset of L. Then

(i) I is called an *ideal* of L, if $x \oslash y = x^- \to y \in I$, for any $x, y \in I$ and if $y \in I$ and $x \leq y$ then $x \in I$, for all $x, y \in L$.

(ii) A proper ideal I of L is called a prime ideal of L, if $x \wedge y \in I$ implies $x \in I$ or $y \in I$, for all $x, y \in L$.

(iii) An ideal I of L is called an n-fold integral ideal, if for all $x, y \in L$,

$$(x \odot y)^n \in I \text{ implies } x^n \in I \text{ or } y^n \in I$$

(iv) An ideal I of L is called an n-fold Boolean ideal, if $x^n \wedge (x^n)^- \in I$, for all $x \in L$.

Definition 2.5 ([6]). Let L_1 and L_2 be two BL-algebras. Then the map $f : L_1 \to L_2$ is called a BL-algebra homomorphism if and only if it satisfies the following conditions, for every $x, y \in L_1$:

(i) $f(0) = 0$,
(ii) $f(x \odot y) = f(x) \odot f(y)$,
(iii) $f(x \to y) = f(x) \to f(y)$.

If f is a bijective, then the homomorphism f is called BL-algebra isomorphism. In this case we write $L_1 \cong L_2$.

The following theorems are from [12] and we refer the reader to them, for more details.

Theorem 2.6. *Let F be an n-fold positive implicative filter of L. Then $N(F)$ is an n-fold Boolean ideal of L, where $N(F)$ is the set of complement elements (with respect to $F \subseteq L$) and is defined by*

$$N(F) = \{x \in L \mid x^- \in F\}$$

Theorem 2.7. *Let $I \subseteq J$, where I and J be two ideals of L and I be an n-fold integral(Boolean) ideal of L. Then J is an n-fold integral(Boolean) ideal, too.*

Theorem 2.8. *In any BL-algebra L, the following conditions are equivalent :*
(i) $\{0\}$ *is an n-fold integral ideal,*
(ii) *any ideal of L is an n-fold integral ideal,*
(iii) *L is an n-fold integral BL-algebra.*

Theorem 2.9. *Let I be an ideal of L. Then the following conditions are equivalent :*
(i) *I is an n-fold integral ideal of L,*
(ii) *I is a maximal and n-fold Boolean ideal of L,*
(iii) *I is a prime and n-fold Boolean ideal of L,*
(iv) *I is a proper ideal and for all $x \in L$, $x^n \in I$ or $(x^n)^- \in I$.*

In the following, we give some fuzzy algebraic results on BL-algebras that come form references [8, 9, 14].

Definition 2.10. Let L be a BL-algebra and $\mu : L \to [0,1]$ be a fuzzy set on L. Then
(i) μ is called a fuzzy ideal on L, if and only if $\mu(x) \leq \mu(0)$ and $\mu((x^- \to y^-)^-) \wedge \mu(x) \leq \mu(y)$, for all $x, y \in L$. It is easy to see that for any fuzzy ideal μ, $\mu(x^{--}) = \mu(x)$.
(ii) μ is called a fuzzy filter on L, if and only if $\mu(x) \leq \mu(1)$ and $\mu(x \to y) \wedge \mu(x) \leq \mu(y)$, for all $x, y \in L$.
(iii) A fuzzy ideal μ is called a fuzzy prime ideal on L, if $\mu(x \wedge y) = \mu(x) \vee \mu(y)$, for all $x, y \in L$.
(iv) A fuzzy filter μ is called a fuzzy n-fold Boolean filter on L, if $\mu(x^n \vee (x^n)^-) = \mu(1)$, for all $x \in L$.

Lemma 2.11. *Let L be a BL-algebra, μ be a fuzzy ideal and η fuzzy filter on L. Then the following properties hold :*
(i) *if $x \leq y$, then $\mu(y) \leq \mu(x)$ and $\eta(x) \leq \eta(y)$, for all $x, y \in L$.*
(ii) *$N(\mu)$ is a fuzzy filter on L and $N(\eta)$ is a fuzzy ideal on L.*

Theorem 2.12. *Let L be a BL-algebra, μ be a fuzzy set on L and $\mu_t = \{x \in L \mid \mu(x) \geq t\}$, for each $t \in [0,1]$. Then*
(i) *μ is a fuzzy filter on L if and only if for each $t \in [0,1]$, $\varnothing \neq \mu_t$ is a filter of L.*
(ii) *μ is a fuzzy ideal on L if and only if for each $t \in [0,1]$, $\varnothing \neq \mu_t$ is an ideal of L.*
(iii) *A fuzzy ideal μ on L is a fuzzy prime ideal if and only if for each $t \in [0,1]$, $\varnothing \neq \mu_t$ is a prime ideal of L.*
(iv) *A fuzzy filter μ on L is a fuzzy n-fold positive implicative filter if and only if for each $t \in [0,1]$, $\varnothing \neq \mu_t$ is an n-fold positive implicative filter of L.*
(v) *A fuzzy filter μ on L is a fuzzy n-fold Boolean filter if and only if μ is a fuzzy n-fold positive implicative filter on L.*

Note. From now on, in this paper we let L be a BL-algebra, unless otherwise is stated.

3. Fuzzy n-fold integral ideals in BL-algebras

Definition 3.1. Let μ be a fuzzy ideal on L. Then μ is called a fuzzy n-fold integral ideal, if for all $x, y \in L$, it satisfies:

$$\mu(x^n \odot y^n) = \mu(x^n) \vee \mu(y^n)$$

A fuzzy 1-fold integral ideal on L is called a fuzzy integral ideal.

The following example shows that fuzzy n-fold integral ideals are exist but, a fuzzy ideal of L may not be a fuzzy n-fold integral ideal.

Example 3.2 ([5]). Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Let $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and operations $\odot$ and $\rightarrow$ are defined as the following tables:

Table. 1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

Table. 2

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a BL-algebra. Now, let the fuzzy set μ on L is defined by

$$\mu(a) = \mu(b) = \mu(1) = t_1, \ \mu(0) = t_2, \quad (0 \leq t_1 < t_2 \leq 1)$$

It is easy to check that μ is a fuzzy ideal and it is a fuzzy 3-fold integral ideal. But, it is not a fuzzy 2-fold integral ideal. Because, $\mu(b^2 \odot b^2) = \mu(a \odot a) = \mu(0) = t_2$ and $\mu(b^2) = \mu(a) = t_1$. Hence, $\mu(b^2 \odot b^2) \neq \mu(b^2) \vee \mu(b^2)$.

Theorem 3.3. *Let μ be a fuzzy ideal on L. Then μ is a fuzzy n-fold integral ideal if and only if for each $t \in [0, 1]$, $\varnothing \neq \mu_t$ is an n-fold integral ideal of L.*

Proof. Let μ be a fuzzy n-fold integral ideal on L and $(x^n \odot y^n) \in \mu_t$, for $t \in [0, 1]$ and $x, y \in L$. Then $\mu(x^n \odot y^n) \geq t$. Since $\mu(x^n \odot y^n) = \mu(x^n) \vee \mu(y^n)$, then $\mu(x^n) \vee \mu(y^n) \geq t$. Now, by contrary if $x^n \notin \mu_t$ and $y^n \notin \mu_t$, then $\mu(x^n) < t$ and $\mu(y^n) < t$. Hence, $\mu(x^n) \vee \mu(y^n) < t$ and it is a contradiction. Therefore, $x^n \in \mu_t$ or $y^n \in \mu_t$ and so μ_t is an n-fold integral ideal of L.
Conversely, assume that for each $t \in [0, 1]$, $\varnothing \neq \mu_t$ is an n-fold integral ideal of L, then we prove that μ is a fuzzy n-fold integral ideal on L. Since by $(BL7)$, $x^n \odot y^n \leq x^n$, then by Lemma 2.11(i), $\mu(x^n), \mu(y^n) \leq \mu(x^n \odot y^n)$ and so $\mu(x^n) \vee \mu(y^n) \leq \mu(x^n \odot y^n)$. Now, we show, $\mu(x^n \odot y^n) \leq \mu(x^n) \vee \mu(y^n)$. In the other wise, there exist $a, b \in L$ such that $\mu(a^n \odot b^n) > \mu(a^n) \vee \mu(b^n)$. Let

$$t_0 = 1/2\{\mu(a^n \odot b^n) + \mu(a^n) \vee \mu(b^n)\}$$

Then we have

$$\mu(a^n) \vee \mu(b^n) < t_0 < \mu(a^n \odot b^n)$$

and so $(a^n \odot b^n) \in \mu_{t_0}$. Now, since μ_{t_0} is an n-fold integral ideal of L, then $a^n \in \mu_{t_0}$ or $b^n \in \mu_{t_0}$. Hence, $\mu(a^n) \geq t_0$ or $\mu(b^n) \geq t_0$ and so $\mu(a^n) \vee \mu(b^n) \geq t_0$, it is a contradiction. Therefore,

$$\mu((x^n \odot y^n)^-) \leq \mu((x^n)^-) \vee \mu((y^n)^-)$$

and so μ is a fuzzy n-fold integral ideal on L. $\square$

In the following theorem, we make a link between, fuzzy n-fold integral ideals and fuzzy $(n+1)$-fold integral ideals in BL-algebaras.

Theorem 3.4. *Let μ be a fuzzy n-fold integral ideal on L. Then μ is a fuzzy $(n+1)$-fold integral ideal on L.*

Proof. Let μ be a fuzzy n-fold integral ideal on L. Then it is easy to check that $\mu(x^{n+1} \odot y^{n+1}) \geq \mu(x^{n+1}) \vee \mu(y^{n+1})$, for all $x, y \in L$. Since by $(BL7)$, $x^{n+n} \odot y^{n+n} \leq x^{n+1} \odot y^{n+1}$, then by Lemma 2.11$(i)$, $\mu(x^{n+1} \odot y^{n+1}) \leq \mu(x^{n+n} \odot y^{n+n})$. Now, since

$$\begin{aligned}
\mu(x^{n+n} \odot y^{n+n}) &= \mu(x^{2n} \odot y^{2n}) \\
&= \mu((x^2)^n \odot (y^2)^n) \\
&= \mu((x^2)^n) \vee \mu((y^2)^n) \\
&= \mu(x^{n+n}) \vee \mu(y^{n+n}) \\
&= \mu(x^n \odot x^n) \vee \mu(y^n \odot y^n), \quad \text{Since } \mu \text{ is a fuzzy } n\text{-fold integral ideal} \\
&= \mu(x^n) \vee \mu(x^n) \vee \mu(y^n) \vee \mu(y^n) \\
&= \mu(x^n) \vee \mu(y^n) \\
&\leq \mu(x^{n+1}) \vee \mu(y^{n+1}), \text{ by } (BL7) \text{ and Lemma } 2.11(i)
\end{aligned}$$

Hence, $\mu(x^{n+1} \odot y^{n+1}) \leq \mu(x^{n+1}) \vee \mu(y^{n+1})$ and so $\mu(x^{n+1} \odot y^{n+1}) = \mu(x^{n+1}) \vee \mu(y^{n+1})$. Therefore, μ is a fuzzy $(n+1)$-fold integral ideal on L. □

By mathematical induction, we can prove that every fuzzy n-fold integral ideal is a fuzzy $(n+k)$-fold integral ideal, for any integer $k \geq 0$.

Note. Example 3.2, shows that the converse of Theorem 3.4, is not correct in general.

Theorem 3.5. *(Extension property for fuzzy n-fold integral ideals) Let μ and η be two fuzzy ideals on L such that $\mu \subseteq \eta$. If μ is a fuzzy n-fold integral ideal on L, then η is a fuzzy n-fold integral ideal on L.*

Proof. Let μ be a fuzzy n-fold integral ideal on L. Then by Theorem 3.3, $\varnothing \neq \mu_t$ is an n-fold integral ideal of L, for each $t \in [0,1]$ and since $\mu \subseteq \eta$, then $\mu(x) \leq \eta(x)$, for all $x \in L$. Now, if $x \in \mu_t$, then $\mu(x) \geq t$ and so $\eta(x) \geq t$. Hence, $x \in \eta_t$ and so $\mu_t \subseteq \eta_t$ for each $t \in [0,1]$. By Theorem 2.7, since μ_t is an n-fold integral ideal of L, then η_t is an n-fold integral ideal of L, for each $t \in [0,1]$. Hence, by Theorem 3.3, η is a fuzzy n-fold integral ideal on L. □

Theorem 3.6. *Let fuzzy set μ on L be defined by*

$$\mu(x) = \begin{cases} 0, & x \neq 0, \\ \alpha, & x = 0, \end{cases}$$

for fixed $\alpha \in (0,1]$. Then the following conditions are equivalent :

(i) L is an n-fold integral BL-algebra,

(ii) Any fuzzy ideal on L is a fuzzy n-fold integral ideal on L,

(iii) μ is a fuzzy n-fold integral ideal on L.

Proof. $(i) \Rightarrow (ii)$: Let L be an n-fold integral BL-algebra and η be a fuzzy ideal on L. Then by Theorem 2.12(ii), for each $t \in [0,1]$, $\varnothing \neq \eta_t$ is an ideal of L and so by

Theorem 2.8, η_t is an n-fold integral ideal of L, for each $t \in [0, 1]$. Therefore, by Theorem 3.3, η is a fuzzy n-fold integral ideal on L.

$(ii) \Rightarrow (iii)$: First, we will prove that μ is a fuzzy ideal on L. By definition of μ, for any $x \in L$, $\mu(x) \leq \mu(0)$. Now, let $x, y \in L$. We consider two following cases for y. If $y = 0$, then

$$\mu((x^- \to y^-)^-) \wedge \mu(x) \leq \alpha = \mu(0) = \mu(y)$$

If $y \neq 0$, then we consider two following cases for x. If $x = 0$, then by $(BL10)$,

$$\begin{array}{rcl} \mu((x^- \to y^-)^-) \wedge \mu(x) & = & \mu((0^- \to y^-)^-) \wedge \mu(0) \\ & = & \mu((1 \to y^-)^-) \wedge \mu(0) \\ & = & \mu(y^{--}) \wedge \mu(0) \\ & = & \mu(y) \wedge \mu(0), \quad \text{By Definition 2.10}(i) \\ & \leq & \mu(y) \end{array}$$

If $x \neq 0$, then

$$\mu((x^- \to y^-)^-) \wedge \mu(x) = \mu((x^- \to y^-)^-) \wedge 0 = 0 \leq \mu(y)$$

Hence, μ is a fuzzy ideal on L and so by (ii), it is a fuzzy n-fold integral ideal on L.
$(iii) \Rightarrow (i)$: Since μ is a fuzzy n-fold integral ideal, then by Theorem 3.3, $\mu_\alpha = \{x \in L \mid \mu(x) \geq \alpha\} = \{0\}$, is an n-fold integral ideal of L. Now, let $x^n \odot y^n = 0$, for $x, y \in L$. Then $x^n \odot y^n \in \{0\}$ and so $x^n \in \{0\}$ or $y^n \in \{0\}$. Hence, $x^n = 0$ or $y^n = 0$. Therefore, L is an n-fold integral BL-algebra. □

Corollary 3.7. *Let fuzzy set μ on L be defined by*

$$\mu(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0, \end{cases}$$

Then the following are equivalent :
(i) L is an integral BL-algebra,
(ii) Any fuzzy ideal on L is a fuzzy integral ideal on L,
(iii) μ is a fuzzy integral ideal on L.

Proof. Let $n = 1$ in Theorem 3.6. Then the proof is clear. □

Theorem 3.8. *Let μ be a fuzzy ideal and ν a fuzzy filter on L. Then*
(i) μ is a fuzzy n-fold integral ideal on L if and only if $N(\mu)$ is a fuzzy n-fold integral filter on L.
(ii) ν is a fuzzy n-fold integral filter on L if and only if $N(\nu)$ is a fuzzy n-fold integral ideal on L.

Proof. (i) Let μ be a fuzzy n-fold integral ideal on L. Then by Lemma 2.11(ii), $N(\mu)$ is a fuzzy filter on L and for all $x, y \in L$,

$$\begin{aligned} N(\mu)((x^n \odot y^n)^-) &= \mu((x^n \odot y^n)^{--}) \\ &= \mu(x^n \odot y^n), \quad \text{By Definition 2.10}(i) \\ &= \mu(x^n) \vee \mu(y^n) \\ &= \mu((x^n)^{--}) \vee \mu((y^n)^{--}), \quad \text{By Definition 2.10}(i) \\ &= N(\mu)((x^n)^-) \vee N(\mu)((y^n)^-) \end{aligned}$$

Therefore, $N(\mu)$ is a fuzzy n-fold integral filter on L. Conversely, let $N(\mu)$ be a fuzzy n-fold integral filter on L. Then for all $x, y \in L$,

$$\begin{aligned} \mu(x^n \odot y^n) &= \mu((x^n \odot y^n)^{--}), \quad \text{By Definition 2.10}(i) \\ &= N(\mu)((x^n \odot y^n)^-) \\ &= N(\mu)((x^n)^-) \vee N(\mu)((y^n)^-) \\ &= \mu((x^n)^{--}) \vee \mu((y^n)^{--}) \\ &= \mu(x^n) \vee \mu(y^n), \quad \text{By Definition 2.10}(i) \end{aligned}$$

Therefore, μ is a fuzzy n-fold integral ideal on L.
(ii) Let ν be a fuzzy n-fold integral filter on L. Then by Lemma 2.11(ii), $N(\nu)$ is a fuzzy ideal on L and for all $x, y \in L$,

$$\begin{aligned} N(\nu)(x^n \odot y^n) &= \nu((x^n \odot y^n)^-) \\ &= \nu((x^n)^-) \vee \nu((y^n)^-) \\ &= N(\nu)(x^n) \vee N(\nu)(y^n) \end{aligned}$$

Therefore, $N(\nu)$ is a fuzzy n-fold integral ideal on L. Conversely, let $N(\nu)$ be a fuzzy n-fold integral ideal on L. Then for all $x, y \in L$,

$$\begin{aligned} \nu((x^n \odot y^n)^-) &= N(\nu)(x^n \odot y^n) \\ &= N(\nu)(x^n) \vee N(\nu)(y^n) \\ &= \nu((x^n)^-) \vee \nu((y^n)^-) \end{aligned}$$

Therefore, ν is a fuzzy n-fold integral filter on L. □

Definition 3.9 ([6]). Let L_1 and L_2 be two BL-algebras, μ a fuzzy subset of L_1, η a fuzzy subset of L_2 and $f : L_1 \to L_2$ a BL-homomorphism. The image of μ under f denoted by $f(\mu)$ is a fuzzy set of L_2 defined by :

$f(\mu)(y) = \sup_{x \in f^{-1}(y)} \mu(x)$ if $f^{-1}(y) \neq \varnothing$ and $f(\mu)(y) = 0$ if $f^{-1}(y) = \varnothing$, for all $y \in L_2$.
The preimage of η under f denoted by $f^{-1}(\eta)$ is a fuzzy set of L_1 defined by: $f^{-1}(\eta)(x) = \eta(f(x))$, for all $x \in L_1$.

Lemma 3.10. *Let $f : L_1 \to L_2$ be a BL-homomorphism, μ a fuzzy ideal on L_2 and η a fuzzy ideal on L_1. Then*
(i) *$f^{-1}(\mu)$ is a fuzzy ideal on L_1,*
(ii) *If f is a BL-isomorphism, then $f(\eta)$ is a fuzzy ideal on L_2.*

Proof. (i) Let μ be a fuzzy ideal on L_2. Since f is a BL-homomorphism and $0 \leq x$, then $f(0) \leq f(x)$. Hence, by Lemma 2.11(i), $\mu(f(x)) \leq \mu(f(0))$ and so $f^{-1}(\mu)(x) \leq f^{-1}(\mu)(0)$. Moreover, for all $x, y \in L_1$,

$$\begin{aligned} f^{-1}(\mu)((x^- \to y^-)^-) \wedge f^{-1}(\mu)(x) &= \mu(f((x^- \to y^-)^-)) \wedge \mu(f(x)) \\ &= \mu((f(x)^- \to f(y)^-)^-)) \wedge \mu(f(x)), \quad \text{By Definition 2.5} \\ &\leq \mu(f(y)) \\ &= f^{-1}(\mu)(y) \end{aligned}$$

Therefore, $f^{-1}(\mu)$ is a fuzzy ideal on L_1.
(ii) Let η be a fuzzy ideal on L_1 and f a BL-isomorphism. Then $\eta(x) \leq \eta(0)$, for all $x \in L_1$. Hence, for all $t \in f^{-1}(x)$, $\eta(t) \leq \eta(0)$ and so $f(\eta)(x) = \sup\limits_{t \in f^{-1}(x)} \eta(t) \leq \eta(0)$. Now, since $f(0) = 0$ and f is a BL-isomorphism, then $f(\eta)(0) = \sup\limits_{t \in f^{-1}(0)} \eta(t) = \eta(0)$ and so for all $x \in L_1$, $f(\eta)(x) \leq f(\eta)(0)$. Now, suppose that $y_1, y_2 \in L$, then since f is a BL-isomorphism, then there exisit $x_1, x_2 \in L_1$, such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

$$f(\eta)((y_1^- \to y_2^-)^-) = \sup\{\eta(z) | z \in f^{-1}((y_1^- \to y_2^-)^-)\}$$

Now, if $z \in f^{-1}((y_1^- \to y_2^-)^-)$, then by Definition 2.5,

$$f(z) = (y_1^- \to y_2^-)^- = (f(x_1)^- \to f(x_2)^-)^- = f((x_1^- \to x_2^-)^-)$$

Since f is a a BL-isomorphism, then $z = (x_1^- \to x_2^-)^-$. Hence,

$$f(\eta)((y_1^- \to y_2^-)^-) = \sup\{\eta(z) | z = (x_1^- \to {x_2}^-)^-\} = \eta(({x_1}^- \to {x_2}^-)^-)$$

By similar way, $f(\eta)(y_1) = \eta(x_1)$ and $f(\eta)(y_2) = \eta(x_2)$. Hence,

$$\begin{aligned} f(\eta)((y_1^- \to y_2^-)^-) \wedge f(\eta)(y_1) &= \eta((x_1^- \to x_2^-)^-) \wedge \eta(x_1) \\ &\leq \eta(x_2) \\ &= f(\eta)(y_2) \end{aligned}$$

Therefore, $f(\eta)$ is a fuzzy ideal on L_2. $\square$

Proposition 3.11. *Let $f : L_1 \to L_2$ be a BL-homomorphism and μ be a fuzzy n-fold integral ideal on L_2. Then $f^{-1}(\mu)$ is a fuzzy n-fold integral ideal on L_1.*

Proof. Let μ be a fuzzy n-fold integral ideal on L_2 and $x, y \in L_1$. Then by Lemma 3.10(i), $f^{-1}(\mu)$ is a fuzzy ideal on L_1. Now,

$$\begin{aligned} f^{-1}(\mu)(x^n \odot y^n) &= \mu(f(x^n \odot y^n)) \\ &= \mu((f(x)^n \odot f(y)^n)), \text{ By Definition 2.5} \\ &= \mu(f(x)^n)) \vee \mu(f(y)^n), \text{ Since } \mu \text{ is a fuzzy } n\text{-fold integral ideal on } L_2 \\ &= \mu(f(x^n)) \vee \mu(f(y^n)) \\ &= f^{-1}(\mu)(x^n) \vee f^{-1}(\mu)(y^n) \end{aligned}$$

Therefore, $f^{-1}(\mu)$ is a fuzzy n-fold integral ideal on L_1. $\square$

Proposition 3.12. *Let $f : L_1 \to L_2$ be a BL-isomorphism and μ be a fuzzy n-fold integral ideal on L_1. Then $f(\mu)$ is a fuzzy n-fold integral ideal on L_2.*

Proof. Let μ be a fuzzy n-fold integral ideal on L_1. Since f is a BL-isomorphism, then by Lemma 3.10(ii), $f(\mu)$ is a fuzzy ideal on L_2.

Now, we show that $f(\mu)(y_1^n \odot y_2^n) = f(\mu)(y_1^n) \vee f(\mu)(y_2^n)$, for all $y_1, y_2 \in L_2$. Since f is a BL-isomorphism then there exist $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Moreover, if $t \in f^{-1}(y_1^n)$, then $y_1^n = f(t) = f(x_1^n)$ and since, f is a BL-isomorphism, then $t = x_1^n$. Hence, $\sup\limits_{t \in f^{-1}((y_1)^n)} \mu(t) = \mu(x_1^n)$. By similar way, $\sup\limits_{t \in f^{-1}((y_2)^n)} \mu(t) = \mu(x_2^n)$ and $\sup\limits_{t \in f^{-1}(y_1^n \odot y_2^n)} \mu(t) = \mu(x_1^n \odot x_2^n)$. Therefore,

$$\begin{aligned} f(\mu)(y_1^n) \vee f(\mu)(y_2^n) &= \sup_{t \in f^{-1}(y_1^n)} \mu(t) \vee \sup_{t \in f^{-1}(y_2^n)} \mu(t) \\ &= \mu(x_1^n) \vee \mu(x_2^n), \text{ Since } \mu \text{ is a fuzzy } n\text{-fold integral ideal on } L_1 \\ &= \mu(x_1^n \odot x_2^n) \\ &= \sup_{t \in f^{-1}(y_1^n \odot y_2^n)} \mu(t) \\ &= f(\mu)(y_1^n \odot y_2^n) \end{aligned}$$

Therefore, $f(\mu)$ is a fuzzy n-fold integral ideal on L_2. □

Theorem 3.13. *Let μ be a fuzzy n-fold integral ideal on L. Then μ is a fuzzy prime ideal on L.*

Proof. Let μ be a fuzzy n-fold integral ideal on L. Then by Theorem 3.3, $\varnothing \neq \mu_t$ is an n-fold integral ideal of L, for each $t \in [0,1]$ and so by Theorem 2.9, $\varnothing \neq \mu_t$ is a prime ideal of L, for each $t \in [0,1]$. Hence, by Theorem 2.12(iii), μ is a fuzzy prime ideal on L. □

The following example shows that the converse of Theorem 3.13, is not correct in general.

Example 3.14 ([10])**.** Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Let $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and operations $\odot$ and $\to$ are defined as the following tables:

Table. 3

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Table. 4

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a BL-algebra. Now, let the fuzzy set μ on L is defined by

$$\mu(a) = \mu(b) = \mu(1) = 0.5, \;\; \mu(0) = 0.7$$

It is easy to check that μ is a fuzzy prime ideal. But, it is not a fuzzy integral ideal on L. Becuase, $\mu(a \odot a) = \mu(0) = 0.7$ and $\mu(a) \vee \mu(a) = 0.5$ and so $\mu(a \odot a) \neq \mu(a) \vee \mu(a)$.

4. Fuzzy n-fold Boolean ideals in BL-algebras

Definition 4.1. Let μ be a fuzzy ideal on L. Then μ is called a fuzzy n-fold Boolean ideal, if for all $x \in L$, it satisfies:

$$\mu(x^n \wedge (x^n)^-) = \mu(0)$$

A fuzzy 1-fold Boolean ideal on L is called a fuzzy Boolean ideal.

Example 4.2. Let L be BL-algebra in Example 3.14. Now, let the fuzzy set μ on L is defined by

$$\mu(1) = 0.5,\ \ \mu(a) = \mu(b) = \mu(0) = 0.7$$

Then μ is a fuzzy 2-fold Boolean ideal on L.

Theorem 4.3. *Let μ be a fuzzy ideal on L. Then μ is a fuzzy n-fold Boolean ideal if and only if for each $t \in [0,1]$, $\varnothing \neq \mu_t$ is an n-fold Boolean ideal of L.*

Proof. Let μ be a fuzzy n-fold Boolean ideal on L. Then by Theorem 2.12(ii), for each $t \in [0,1]$, $\varnothing \neq \mu_t$ is an ideal of L. Hence, $0 \in \mu_t$ and so $\mu(0) \geq t$. Since, $\mu(x^n \wedge (x^n)^-) = \mu(0)$, for all $x \in L$, then $\mu(x^n \wedge (x^n)^-) \geq t$. Therefore, for all $x \in L$, $x^n \wedge (x^n)^- \in \mu_t$ and so $\varnothing \neq \mu_t$ is an n-fold Boolean ideal of L, for each $t \in [0,1]$.
Conversely, let for each $t \in [0,1]$, $\varnothing \neq \mu_t$ is an n-fold Boolean ideal. Since, $\mu(0) \geq \mu(0)$, then $0 \in \mu(0)_{\mu(0)}$. Hence, $\mu(0)_{\mu(0)} \neq \varnothing$ and so $\mu(0)_{\mu(0)}$ is an n-fold Boolean ideal of L. Therefore, for all $x \in L$, $x^n \wedge (x^n)^- \in \mu(0)_{\mu(0)}$ and so $\mu(x^n \wedge (x^n)^-) \geq \mu(0)$, for all $x \in L$. Since μ is a fuzzy ideal, then $\mu(x^n \wedge (x^n)^-) \leq \mu(0)$, for all $x \in L$. Thus, $\mu(x^n \wedge (x^n)^-) = \mu(0)$, for all $x \in L$ and so μ is a fuzzy n-fold Boolean ideal on L. □

Example 4.4 ([4])**.** Let $L = [0,1]$, $x \odot y = \min\{x, y\}$ and

$$x \to y = \begin{cases} 1, & if\ x \leq y, \\ y, & if\ y < x, \end{cases}$$

Then $([0,1], \vee, \wedge, \odot, \to, 0, 1)$ is a BL-algebra. Now, let the fuzzy set μ on L is defined by

$$\mu(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0, \end{cases}$$

It is easy to check that μ is a fuzzy ideal on infinite BL-algebra L. Therefore, by Theorem 2.12(ii), for each $t \in [0,1]$, $\varnothing \neq \mu_t$ is an ideal of L. Moreover, since $x \wedge x^- = 0$, for all $x \in L$ and for each $t \in [0,1]$, $x \wedge x^- = 0 \in \mu_t$, for all $x \in L$, then for each $t \in [0,1]$, $\varnothing \neq \mu_t$ is a Boolean ideal of L. Hence, by Theorem 4.3, μ is a fuzzy Boolean ideal on infinite BL-algebra L.

Theorem 4.5. *(Extension property for fuzzy n-fold Boolean ideals) Let μ and η are two fuzzy ideals on L such that $\mu \subseteq \eta$. If μ is a fuzzy n-fold Boolean ideal, then η is a fuzzy n-fold Boolean ideal, too.*

Proof. Let μ be a fuzzy n-fold Boolean ideal on L. Then by Theorem 4.3, $\varnothing \neq \mu_t$ is an n-fold Boolean ideal, for each $t \in [0,1]$ and since $\mu \subseteq \eta$, then for each $t \in [0,1]$, $\mu_t \subseteq \eta_t$. By Theorem 2.7, since μ_t is an n-fold Boolean ideal of L, then η_t is an

n-fold Boolean ideal of L, for each $t \in [0,1]$. Hence, by Theorem 4.3, η is a fuzzy n-fold Boolean ideal on L. □

Note. Example 3.2, shows that the converse of Theorem 4.5, is not correct in general. Since $\mu(b^2 \wedge (b^2)^-) = \mu(a \wedge b) = \mu(a) \neq \mu(0)$ and $\mu(b^3 \wedge (b^3)^-) = \mu(0)$. Hence, μ is a fuzzy 3-fold Boolean ideal, but it is not a fuzzy 2-fold Boolean ideal on L.

Theorem 4.6. *Let μ be a fuzzy ideal and ν a fuzzy filter on L. Then*
(i) If ν is a fuzzy n-fold Boolean filter on L, Then $N(\nu)$ is a fuzzy n-fold Boolean ideal on L.
(ii) μ is a fuzzy n-fold Boolean ideal on L if and only if $N(\mu)$ is a fuzzy n-fold Boolean filter on L.

Proof. (i) Let ν be a fuzzy n-fold Boolean filter on L. Then by Theorem 2.12(v), ν is a fuzzy n-fold positive implicative filter on L and so by Theorem 2.12(iv), for each $t \in [0,1]$, $\varnothing \neq \nu_t$ is an n-fold positive implicative filter of L. Moreover, for each $t \in [0,1]$

$$\begin{aligned}(N(\nu))_t &= \{x \in L \mid (N(\nu))(x) \geq t\}\\ &= \{x \in L \mid \nu(x^-) \geq t\}\\ &= \{x \in L \mid x^- \in \nu_t\}\\ &= N(\nu_t)\end{aligned}$$

Hence, by Theorem 2.6, $\varnothing \neq (N(\nu))_t = N(\nu_t)$ is a an n-fold Boolean ideal of L. Therefore, by Theorem 4.3, $N(\nu)$ is a fuzzy n-fold Boolean ideal on L.

(ii) Let μ be a fuzzy Boolean ideal on L. Then by Lemma 2.11(ii), $N(\mu)$ is a fuzzy filter on L and for all $x \in L$,

$$\begin{aligned}N(\mu)(x^n \vee (x^n)^-) &= \mu((x^n \vee (x^n)^-)^-), \text{ By } (BL11)\\ &= \mu((x^n)^- \wedge ((x^n)^-)^-)\\ &= \mu(0)\\ &= N(\mu)(1)\end{aligned}$$

By $(BL7)$, $x^n \vee (x^n)^- \leq x \vee (x^n)^-$ and by Lemma 2.11(i), $N(\mu)(x^n \vee (x^n)^-) \leq N(\mu)(x \vee (x^n)^-)$. Hence, $N(\mu)(x \vee (x^n)^-) = N(\mu)(1)$ and so $N(\mu)$ is a fuzzy n-fold Boolean filter on L. Conversely, since $N(N(\mu))(x) = N(\mu)(x^-) = \mu(x^{--}) = \mu(x)$, then $N(N(\mu)) = \mu$. Now, since $N(\mu)$ is a fuzzy n-fold Boolean filter on L, then by (i), $N(N(\mu)) = \mu$ is a fuzzy n-fold Boolean ideal on L.

□

The following example shows that the converse of Theorem 4.6(i), is not correct in general.

Example 4.7. [?] Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Let $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and operations $\odot$ and $\rightarrow$ are defined as the following tables:

Table. 5

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

Table. 6

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a BL-algebra. Now, let the fuzzy set μ on L is defined by

$$\mu(0) = 0.4,\ \mu(a) = \mu(b) = 0.6,\ \mu(1) = 0.8$$

It is easy to check that μ is a fuzzy filter, but it is not a fuzzy 2-fold Boolean filter. Because,

$$\mu(a^2 \vee (a^2)^-) = \mu(a \vee a^-) = \mu(a \vee 0) = \mu(a) = 0.6 \neq 0.8 = \mu(1)$$

Now, fuzzy set $N(\mu)$ which is equal to

$$N(\mu)(1) = N(\mu)(a) = N(\mu)(b) = \mu(0) = 0.4,\ N(\mu)(0) = \mu(1) = 0.8$$

is a fuzzy ideal on L and since

$$N(\mu)(a^2 \wedge (a^2)^-) = N(\mu)(b^2 \wedge (b^2)^-) = N(\mu)(a \wedge 0) = N(\mu)(0) = 0.8$$

Therefore, $N(\mu)$ is a fuzzy 2-fold Boolean ideal on L.

Lemma 4.8. *Let I be an ideal of L and $\alpha \in I$. Then $\alpha^- \in N(I)$.*

Proof. Let I be an ideal of L and $\alpha \in I$. By $(BL9)$, $\alpha^- \odot \alpha^{--} = 0$, since I is an ideal of L, then $\alpha^{--} \in I$. Therefore, $\alpha^- \in N(I)$. □

Theorem 4.9. *Let μ be a fuzzy ideal on L. Then the following conditions are equivalent :*

(i) μ is a fuzzy n-fold integral ideal on L,

(ii) μ is a fuzzy prime and fuzzy n-fold Boolean ideal on L,

(iii) μ is a non-constant fuzzy ideal and for all $x \in L$, $\mu(x^n) = \mu(0)$ or $\mu((x^n)^-) = \mu(0)$.

Proof. $(i) \Rightarrow (ii)$: Let μ be a fuzzy n-fold integral ideal on L. Then by Theorem 3.13, μ is a fuzzy prime ideal and by Theorem 3.3, for each $t \in [0, 1]$, $\varnothing \neq \mu_t$, is an n-fold integral ideal of L. Hence, by Theorem 2.9, for each $t \in [0, 1]$, $\varnothing \neq \mu_t$, is an n-fold Boolean ideal of L and so by Theorem 4.3, μ is a fuzzy n-fold Boolean ideal on L.

$(ii) \Rightarrow (iii)$: Let μ be a fuzzy prime and fuzzy n-fold Boolean ideal on L. Then μ is a non-constant fuzzy ideal and $\mu(x^n \wedge (x^n)^-) = \mu(0)$, for all $x \in L$. Since μ is a fuzzy prime ideal on L, then

$$\mu(0) = \mu(x^n \wedge (x^n)^-) = \mu(x^n) \vee \mu((x^n)^-)$$

Hence, $\mu(x^n) = \mu(0)$ or $\mu((x^n)^-) = \mu(0)$.

$(iii) \Rightarrow (i)$: Let μ be a non-constant fuzzy ideal on L and for all $x \in L$, $\mu(x^n) = \mu(0)$ or $\mu((x^n)^-) = \mu(0)$. Then by $(BL7)$, $x^n \odot y^n \leq x^n, y^n$ and so by Lemma 2.11(i),

$\mu(x^n), \mu(y^n) \leq \mu(x^n \odot y^n)$. Hence, $\mu(x^n) \vee \mu(y^n) \leq \mu(x^n \odot y^n)$. Now, we prove that $\mu(x^n \odot y^n) \leq \mu(x^n) \vee \mu(y^n)$, for all $x, y \in L$. In the other wise, if ther exist $a, b \in L$, such that $\mu(a^n) \vee \mu(b^n) < \mu(a^n \odot b^n)$. Now, let $t_0 = 1/2\{\mu(a^n) \vee \mu(b^n) + \mu(a^n \odot b^n)\}$. Then

$$\mu(a^n) \vee \mu(b^n) < t_0 < \mu(a^n \odot b^n)$$

And so $a^n \odot b^n \in \mu_{t_0}$ and since $\mu(a^n) < t_0$ and $\mu(b^n) < t_0$, then $a^n \notin \mu_{t_0}$ and $b^n \notin \mu_{t_0}$. Since $t_0 < \mu(a^n \odot b^n) \leq \mu(0)$, then $\mu(a^n) < \mu(0)$ and $\mu(b^n) < \mu(0)$. Hence, $\mu((a^n)^-) = \mu(0)$ and $\mu((b^n)^-) = \mu(0) \geq t_0$ and so $(a^n)^-, (b^n)^- \in \mu_{t_0}$. Therefore, $(a^n), (b^n) \in N(\mu_{t_0})$ and since $N(\mu_{t_0})$ is a filter of L, then $(a^n) \odot (b^n) \in N(\mu_{t_0})$. Now, since $a^n \odot b^n \in \mu_{t_0}$ and μ_{t_0} is an ideal of L, the by Lemma 4.8, $(a^n \odot b^n)^- \in N(\mu_{t_0})$ and so by $(BL9)$, $0 = (a^n \odot b^n) \odot (a^n \odot b^n)^- \in N(\mu_{t_0})$. Hence, $1 \in \mu_{t_0}$ and so $\mu_{t_0} = L$. Therefore, $(a^n), (b^n) \in \mu_{t_0}$, which is a contradiction. Thus, $\mu(x^n \odot y^n) \leq \mu(x^n) \vee \mu(y^n)$, for all $x, y \in L$ and so μ is a fuzzy n-fold integral ideal on L. □

Proposition 4.10. *Let $f : L_1 \to L_2$ be a BL-homomorphism and μ be a fuzzy n-fold Boolean ideal on L_2. Then $f^{-1}(\mu)$ is a fuzzy n-fold Boolean ideal on L_1.*

Proof. Let μ be a fuzzy n-fold Boolean ideal on L_2 and $x, y \in L_1$. Then by Lemma 3.10(i), $f^{-1}(\mu)$ is a fuzzy ideal on L_1. Now,

$$\begin{aligned}
f^{-1}(\mu)(x^n \wedge (x^n)^-) &= \mu(f((x^n \wedge (x^n)^-)) \\
&= \mu(f(x^n \odot (x^n \to (x^n)^-)), \text{ By } (BL4) \\
&= \mu(f(x)^n \odot (f(x)^n \to (f(x)^n)^-), \text{ By Definition 2.5} \\
&= \mu(f(x^n) \wedge (f(x)^n)^-) \\
&= \mu(0_{L_2}), \text{ By Definition 3.9} \\
&= \mu(f(0_{L_1})) \\
&= f^{-1}(\mu)(0)
\end{aligned}$$

Therefore, $f^{-1}(\mu)$ is a fuzzy n-fold integral ideal on L_1. □

Proposition 4.11. *Let $f : L_1 \to L_2$ be a BL-isomorphism and μ be a fuzzy n-fold Boolean ideal on L_1. Then $f(\mu)$ is a fuzzy n-fold Boolean ideal on L_2.*

Proof. Let μ be a fuzzy n-fold Boolean ideal on L_1. Since f is a BL-isomorphism, then by Lemma 3.10(ii), $f(\mu)$ is a fuzzy ideal on L_2. Let $y \in L_2$. Then by Definition 3.9,

$$f(\mu)((y^n) \wedge (y^n)^-) = \sup_{t \in f^{-1}((y^n) \wedge (y^n)^-)} \mu(t)$$

Now, since f is a BL-isomorphism and $y \in L_2$, then there exist $x \in L_1$, such that $f(x) = y$ and so by Definition 2.5, $f((x^n) \wedge (x^n)^-) = (y^n) \wedge (y^n)^-$.

Moreover, if $t \in f^{-1}((y^n) \wedge (y^n)^-)$, then $f(t) = (y^n) \wedge (y^n)^- = f((x^n) \wedge (x^n)^-)$ and since f is a BL-isomorphism, then $t = (x^n) \wedge (x^n)^-$. Hence,

$$f(\mu)((y^n) \wedge (y^n)^-) = \sup_{t \in f^{-1}((y^n) \wedge (y^n)^-)} \mu(t) = \mu((x^n) \wedge (x^n)^-)$$

Since μ is a fuzzy n-fold Boolean ideal on L_1, then $\mu((x^n) \wedge (x^n)^-) = \mu(0_{L_1})$ and so $f(\mu)((y^n) \wedge (y^n)^-) = \mu(0_{L_1})$. Now, since f is a BL-isomorphism, by similar way,

$f(\mu)(0_{L_2}) = \sup_{t\in f^{-1}(0_{L_2})} \mu(t) = \mu(0_{L_1})$. Hence,

$$f(\mu)((y^n) \wedge (y^n)^-) = f(\mu)(0_{L_2})$$

Therefore, $f(\mu)$ is a fuzzy n-fold Boolean ideal on L_2. □

5. Conclusion

The results of this paper will be devoted to study fuzzy n-fold Boolean and fuzzy n-fold integral ideals on BL-algebras. Using a level subset of a fuzzy set in a BL-algebra, we studied characterization of fuzzy n-fold integral and fuzzy n-fold Boolean ideals. Also, we proved that the homomorphic image and preimage of fuzzy n-fold integral and fuzzy n-fold Boolean ideals are fuzzy n-fold integral and fuzzy n-fold Boolean ideals, respectively. Finally, we studied relationship among fuzzy n-fold integral, fuzzy n-fold Boolean and fuzzy prime ideals in BL-algebras.

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Akbar Paad (akbar.paad@gmail.com; a.paad@ub.ac.ir)
Department of Mathematics, University of Bojnord, Postal Code 9453155111, Bojnord, Iran