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Decomposition of an intuitionistic fuzzy matrix using implication operators

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Abstract. In this paper, we decompose a rectangular intuitionistic fuzzy matrix into a product of idempotent and a rectangular intuitionistic fuzzy matrix of the same dimension using implication operators, study some properties of decomposition and we have obtained equivalent condition for an intuitionistic fuzzy matrix to be reflexive and transitive.

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1. Introduction

After the introduction of the Intuitionistc fuzzy set theory by Atanassov [5], the fuzzy matrix theory has been extended to IFM theory using max-min composition by Im [11] et.al., and developed by several authors [15, 16, 24, 25, 26, 27]. Yager et.al[28] represented an Intuitionistic Fuzzy element as Intuitionistic Fuzzy values and for any two Intuitionistic Fuzzy values $\langle a, a' \rangle, \langle b, b' \rangle$, the addition and multiplication have been defined as, $\langle a, a' \rangle + \langle b, b' \rangle = \langle max\{a, b\}, min\{a', b'\} \rangle$ and $\langle a, a' \rangle \langle b, b' \rangle = \langle min\{a, b\}, max\{a', b'\} \rangle$. The set of all IFMs of order $m \times n$ is denoted by $\mathscr{F}_{mn}$ and $\mathscr{F}_n$ means square IFMs of order n. For $A \in \mathscr{F}_{mn}, B \in \mathscr{F}_{np}$ the Intuitionistic fuzzy matrix multiplication is defined as

1.$AB = (\bigvee_k (a_{ik} \wedge b_{kj}), \bigwedge_k (a'_{ik} \vee b'_{kj}))$ by Lee and Jeong [13],

2. $AB = (max\{min\{a_{ik}, b_{kj}\}\}, min\{max\{a'_{ik}, b'_{kj}\}\})$ by Liu and Huang[14] and

3. $AB = (\langle \sum_{k=1}^{n} a_{ik} b_{kj}, \prod_{k=1}^{n} (a'_{ik} + b'_{kj}) \rangle)$ by Sriram and Murugadas[24].

All the above three definitions have the same meaning. Pal et al.[20] studied and developed IFM in 2002. Shyamal and Pal[23] obtained the distance between IFM. Bhowmik and Pal [6, 7] studied about circulant IFM and generalized IFMs. Khan

and Pal [12] discussed about intuitionistic fuzzy tautological matrix. Adak et. al [1, 2, 3, 4] discussed some properties of generalized intuitionistic nilpotent fuzzy matrices over distributive lattice, applied generalized IFM in multi-criteria decision making and studied some properties of intuitionistic fuzzy block matrix. Mondal and Pal [17, 18] studied similarity relation, invertibility, eigen values of IFM and inclined IFM. Further Pradhan and Pal [21, 22] discussed some results on g-inverse of IFM and Atanassov's IFMs. The idempotent IFM, transitive IFM and nilpotent IFM play a vital role in the study of similarity measures in IFM theory.
Lee and Jeong[13] decomposed a transitive IFM into a sum of nilpotent IFM and symmetric IFM. Sriram and Murugadas[26] decomposed an IFM as a sum of $\langle\alpha,\alpha'\rangle$-cut IFMs. Hiroshi Hashimoto[8, 9, 10] used $\leftarrow, \leftharpoondown$ operators in Fuzzy matrices and obtained some interesting results. Sriram and Murugadas[24] applied this $\leftarrow$ operator to IFM and studied about g-inverse and sub-inverse of IFMs. Normally for A, B either in FM or in IFM $A \leftarrow B \geq A$, is true if we apply $\leftarrow$ component wise, which is used to find superiority between A and B. Similarly $A \leftharpoondown B \leq A$, is true if we apply $\leftharpoondown$ component wise, which is used to find inferiority between A and B.
Murugadas and Lalitha [19] obtained some relations between the operators $\leftarrow$ and $\leftharpoondown$. In this paper we decompose an IFM into a product of idempotent IFM and rectangular IFM.

2. Preliminaries

Definition 2.1 ([5])**.** An Intuitionistic Fuzzy Set(IFS) A in E (universal set) is defined as an object of the following form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in E\}$, where the functions: $\mu_A(x) : E \to [0,1]$ and $\nu_A(x) : E \to [0,1]$ define the membership and non-membership functions of the element $x \in E$ respectively and for every $x \in E : 0 \leq \mu_{A(x)} + \nu_A(x) \leq 1$.

Definition 2.2 ([24])**.** For $\langle x, x'\rangle, \langle y, y'\rangle \in IFS$, define

$$\langle x, x'\rangle \leftarrow \langle y, y'\rangle = \begin{cases} \langle 1, 0\rangle & \text{if } \langle x, x'\rangle \geq \langle y, y'\rangle \\ \langle x, x'\rangle & \text{if } \langle x, x'\rangle < \langle y, y'\rangle \end{cases}$$

and

$$\langle x, x'\rangle \leftharpoondown \langle y, y'\rangle = \begin{cases} \langle x, x'\rangle & \text{if } \langle x, x'\rangle > \langle y, y'\rangle \\ \langle 0, 1\rangle & \text{if } \langle x, x'\rangle \leq \langle y, y'\rangle. \end{cases}$$

Here $\langle x, x'\rangle \geq \langle y, y'\rangle$ means $x \geq y$ and $x' \leq y'$ and $\langle x, x'\rangle > \langle y, y'\rangle$ means either $x > y$ or $x' < y$.

Definition 2.3 ([26])**.** Let $X = \{x_1, x_2, ...x_m\}$ be a set of alternatives and $Y = \{y_1, y_2, ...y_n\}$ be the attribute set of each element of X. An Intuitionistic Fuzzy Matrix (IFM) is defined by $A = (\langle (x_i, y_j), \mu_A(x_i, y_j), \nu_A(x_i, y_j)\rangle)$ for $i = 1, 2...m$ and $j = 1, 2, ...n$, where $\mu_A : X \times Y \to [0,1]$ and $\nu_A : X \times Y \to [0,1]$ satisfy the condition $0 \leq \mu_A(x_i, y_j) + \nu_A(x_i, y_j) \leq 1$. For simplicity we denote an intuitionistic fuzzy matrix (IFM) as a matrix of pairs $A = (\langle a_{ij}, a'_{ij}\rangle)$ of non negative real numbers satisfying $a_{ij} + a'_{ij} \leq 1$ for all i, j.
For any two elements $A = (\langle a_{ij}, a'_{ij}\rangle), B = (\langle b_{ij}, b'_{ij}\rangle) \in \mathscr{F}_{mn}$ and $C \in \mathscr{F}_{np}$, define

1. $A \oplus B = (\langle max\{a_{ij}, b_{ij}\}, min\{a'_{ij}, b'_{ij}\}\rangle)$, (component wise addition).

2. $A \odot B = (\langle min\{a_{ij}, b_{ij}\}, max\{a'_{ij}b'_{ij}\}\rangle) = A \odot B$, (component wise multiplication) for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

3. $J = (\langle 1, 0\rangle)$ the Universal matrix(matrix in which all entries are $\langle 1, 0\rangle$).

4. $I = (\langle \delta_{ij}, \delta'_{ij}\rangle)$ (Identity Matrix) where $\langle \delta_{ij}, \delta'_{ij}\rangle = \begin{cases} \langle 1, 0\rangle & \text{if } i = j \\ \langle 0, 1\rangle & \text{if } i \neq j \end{cases}$.

The Zero matrix O is the matrix in which all the entries are $\langle 0, 1\rangle$.

5. $A \geq B$ if $a_{ij} \geq b_{ij}$ and $a'_{ij} \leq b'_{ij}$ for all i, j and $A > B$ if $a_{ij} > b_{ij}$ or $a'_{ij} < b'_{ij}$ for atleast one i, j, in which case A and B are comparable.

6.$\overline{A} = (\langle a'_{ij}, a_{ij}\rangle)$, (complement of A).

7. $A^T = (\langle a_{ji}, a'_{ji}\rangle)$, (transpose of A).

8. $AC = (\langle max_{k=1}^{n} min\{a_{ik}, c_{kj}\}, min_{k=1}^{n} max\{a'_{ik}, c'_{kj}\}\rangle) = (\langle \sum_{k=1}^{n} a_{ik}c_{kj}, \prod_{k=1}^{n}(a'_{ik} + c'_{kj})\rangle)$.

9. $A \leftarrow C = (min_{k=1}^{n}(\langle a_{ik}, a'_{ik}\rangle \leftarrow \langle c_{kj}, c'_{kj}\rangle)) = (\bigwedge_{k=1}^{n}(\langle a_{ik}, a'_{ik}\rangle \leftarrow \langle c_{kj}, c'_{kj}\rangle))$.

10. $A \leftharpoondown B = (\langle a_{ij}, a'_{ij}\rangle \leftharpoondown \langle b_{ij}, b'_{ij}\rangle)$, (component wise).

11. For $R \in \mathscr{F}_n, \Delta R = R \leftharpoondown R^T$, (component wise).

Definition 2.4 ([27]). Let $R \in \mathscr{F}_n$, then

1. If $R \geq I_n$, then R is reflexive.
2. If $R^2 \leq R$, then R is transitive.
3. If $R^2 \geq R$, then R is compact.
4. If R is reflexive and transitive, then R is idempotent.
5. In R, if all the diagonal entries are $\langle 0, 1\rangle$, then R is irreflexive.
6. R is symmetric if and only if $\langle r_{ij}, r'_{ij}\rangle = \langle r_{ji}, r'_{ji}\rangle$, for all i, j.

7.R is antisymmetric if and only if $R \odot R^T \leq I_n$, where I_n is the unit matrix with all entries in the main diagonal as $\langle 1, 0\rangle$ and remaining entries with $\langle 0, 1\rangle$. $R \odot R^T \leq I_n$ means $\langle r_{ij}, r'_{ij}\rangle\langle r_{ji}, r'_{ji}\rangle = \langle 0, 1\rangle$ for all $i \neq j$ and $\langle r_{ii}, r'_{ii}\rangle \leq \langle 1, 0\rangle$ for all i. So if $\langle r_{ij}, r'_{ij}\rangle = \langle 1, 0\rangle$, then $\langle r_{ji}, r'_{ji}\rangle = \langle 0, 1\rangle$.

8. If R is irreflexive and transitive, then R is nilpotent.

Lemma 2.5 ([24]). *If $A = (\langle a_{ij}, a'_{ij}\rangle)$ is an $m \times n$ IFM, then $A \leftarrow A^T$ is reflexive and transitive.*

3. Decomposition of an IFM using $\leftarrow$ and $\leftharpoondown$ operators

Throughout this section matrix means IFM.
Let A_i be the i^{th} row of A, if $A_i \geq A_j$, then $\langle g_{ij}, g'_{ij}\rangle = \langle 1, 0\rangle$, where $\langle g_{ij}, g'_{ij}\rangle$ is the $(i, j)^{th}$ entry of $G = A \leftarrow A^T$. Hence the matrix G represents inclusion among the rows of A.

Proposition 3.1. *Let $G = (\langle g_{ij}, g'_{ij}\rangle) \in \mathscr{F}_m$. Then the following conditions are equivalent.*
1. G is reflexive and transitive.
2.$G \leftarrow G^T = G$.

Proof. $1 \Rightarrow 2$
Suppose $\bigwedge_{k=1}^{n}(\langle g_{ik}, g'_{ik}\rangle \leftarrow \langle g_{jk}, g'_{jk}\rangle) = \langle c, c'\rangle > \langle 0, 1\rangle$.
Setting $k = j$ we have $\langle g_{ij}, g'_{ij}\rangle > \langle c, c'\rangle$.
Next we show that $G \leq G \leftarrow G^T$.

Suppose that $\langle g_{ij}, g'_{ij}\rangle = \langle c, c'\rangle > \langle 0, 1\rangle$
If $\langle g_{il}, g'_{il}\rangle < \langle c, c'\rangle$ and $\langle g_{il}, g'_{il}\rangle < \langle g_{jl}, g'_{jl}\rangle$ for some l, then
$\langle g_{il}, g'_{il}\rangle \geq \langle g_{ij}, g'_{ij}\rangle\langle g_{jl}, g'_{jl}\rangle = \langle c, c'\rangle\langle g_{jl}, g'_{jl}\rangle \geq \langle c, c'\rangle\langle g_{il}, g'_{il}\rangle = \langle g_{il}, g'_{il}\rangle$. So that $\langle g_{il}, g'_{il}\rangle = \langle g_{jl}, g'_{jl}\rangle$, which is a contradiction.
Hence $\bigwedge_{k=1}^{n}(\langle g_{ik}, g'_{ik}\rangle \leftarrow \langle g_{jk}, g'_{jk}\rangle) \geq \langle c, c'\rangle$, that is $G \leq G \leftarrow G^T$.
$2 \Rightarrow 1$ is true from Lemma 2.5 □

Lemma 3.2. *If* $A = (\langle a_{ij}, a'_{ij}\rangle) \in \mathscr{F}_{mn}$, *then* $(A \leftarrow A^T)A = A$.

Proof. Let $B = (\langle b_{ij}, b'_{ij}\rangle) = (A \leftarrow A^T)A$.
That is $\langle b_{ij}, b'_{ij}\rangle = \sum_{k=1}^{m}\bigwedge_{l=1}^{n}(\langle a_{il}, a'_{il}\rangle \leftarrow \langle a_{kl}, a'_{kl}\rangle)\langle a_{kj}, a'_{kj}\rangle$
Since $A \leftarrow A^T$ is reflexive , $A \leq B$. Next we prove that $A \geq B$
Suppose that $\langle b_{ij}, b'_{ij}\rangle > \langle a_{ij}, a'_{ij}\rangle$, then
$\bigwedge_{l=1}^{n}(\langle a_{il}, a'_{il}\rangle \leftarrow \langle a_{hl}, a'_{hl}\rangle) > \langle a_{ij}, a'_{ij}\rangle$, $\langle a_{hj}, a'_{hj}\rangle > \langle a_{ij}, a'_{ij}\rangle$ for some h. For $l = j$ we have $\langle a_{ij}, a'_{ij}\rangle \leftarrow \langle a_{hj}, a'_{hj}\rangle > \langle a_{ij}, a'_{ij}\rangle$, so that $\langle a_{ij}, a'_{ij}\rangle > \langle a'_{hj} a'_{hj}\rangle$, which is a contradiction. Hence $\langle b_{ij}, b'_{ij}\rangle \leq \langle a_{ij}, a'_{ij}\rangle$ for all i, j. Thus $A \geq B$. □

Remark 3.3. From Proposition 3.1 and Lemma 3.2 it is evident that $(A^T \leftarrow A)^T$ is also reflexive and transitive and $A(A^T \leftarrow A)^T = A$.

Lemma 3.4. *Let* $A = (\langle a_{ij}, a'_{ij}\rangle) \in \mathscr{F}_{mn}$, *and* $T = (\langle t_{ij}, t'_{ij}\rangle) \in \mathscr{F}_{m}$, *is transitive, then* $TA = T(A \leftharpoondown NA)$, *where* $N = (\langle n_{ij}, n'_{ij}\rangle) \in \mathscr{F}_m$ *is nilpotent such that* $N \leq T$.

Proof. Let $B = (\langle b_{ij}, b'_{ij}\rangle) = TA$ and $C = (\langle c_{ij}, c'_{ij}\rangle) = T(A \leftharpoondown NA)$.
This shows $\langle b_{ij}, b'_{ij}\rangle = \sum_{k=1}^{n}\langle t_{ik}, t'_{ik}\rangle\langle a_{kj}, a'_{kj}\rangle$ and
$\langle c_{ij}, c'_{ij}\rangle = \sum_{k=1}^{m}\langle t_{ik}, t'_{ik}\rangle(\langle a_{kj}, a'_{kj}\rangle \leftharpoondown \sum_{l=1}^{n}\langle n_{kl}, n'_{kl}\rangle\langle a_{lj}, a'_{lj}\rangle)$.
Clearly $C \leq B$, we claim that $C \geq B$.
Suppose not, let $\langle b_{ij}, b'_{ij}\rangle = \langle b, b'\rangle > \langle 0, 1\rangle$ and $\langle c_{ij}, c'_{ij}\rangle < \langle b, b'\rangle$.
This gives $\langle t_{il(0)}, t'_{il(0)}\rangle \geq \langle b, b'\rangle, \langle a_{l(0)j}, a'_{l(0)j}\rangle \geq \langle b, b'\rangle$ for some $k = l(0)$.
Since $\langle b_{ij}, b'_{ij}\rangle > \langle c'_{ij}, c'_{ij}\rangle$, we have
$\sum_{l=1}^{m}\langle n_{l(0)l}, n'_{l(0)l}\rangle\langle a_{lj}, a'_{lj}\rangle \geq \langle a_{l(0)j}, a'_{l(0)j}\rangle \geq \langle b, b'\rangle$. Thus
$\langle n_{l(0)l(1)}, n'_{l(0)l(1)}\rangle \geq \langle b, b'\rangle, \langle a_{l(1)j}, a'_{l(1)j}\rangle \geq \langle b, b'\rangle, \langle t_{l(0)l(1)}, t'_{l(0)l(1)}\rangle \geq \langle b, b'\rangle$ for some $l(1)$.
Therefore, $\langle t_{il(1)}, t'_{il(1)}\rangle \geq \langle b, b'\rangle, \langle a_{i(1)j}, a'_{i(1)j}\rangle \geq \langle b, b'\rangle, \langle n^{(1)}_{l(0)l(1)}, n'^{(1)}_{l(0)l(1)}\rangle \geq \langle b, b'\rangle$.
Again since $\langle b_{ij}, b'_{ij}\rangle > \langle c'_{ij}, c'_{ij}\rangle$, we have
$\sum_{l=1}^{m}\langle n_{l(0)l}, n'_{l(1)l}\rangle\langle a_{lj}, a'_{lj}\rangle \geq \langle a_{l(1)j}, a_{l(1)j}\rangle \geq \langle b, b'\rangle$.
Thus $\langle n_{l(1)l(2)}, n'_{l(1)l(2)}\rangle \geq \langle b, b'\rangle, \langle a_{l(2)j}, a'_{l(2)j}\rangle \geq \langle b, b'\rangle, \langle t_{l(1)l(2)}, t'_{l(1)l(2)}\rangle \geq \langle b, b'\rangle$ for some $l(2)$. Therefore,
$\langle t_{il(2)}, t'_{il(2)}\rangle \geq \langle b, b'\rangle, \langle a_{i(2)j}, a'_{i(2)j}\rangle \geq \langle b, b'\rangle, \langle n^{(2)}_{l(0)l(2)}, n'^{(2)}_{l(0)l(2)}\rangle \geq \langle b, b'\rangle$.
If we repeat the same procedure we get
$\langle t_{il(m)}, t'_{il(m)}\rangle \geq \langle b, b'\rangle, \langle a_{i(m)j}, a'_{i(m)j}\rangle \geq \langle b, b'\rangle, \langle n^{(m)}_{l(0)l(m)}, n'^{(m)}_{l(0)l(m)}\rangle \geq \langle b, b'\rangle$.
This contradicts the fact that N is nilpotent. Hence $B \leq C$.
Thus the Lemma holds.
In the dual fashion, we can prove the following Lemma. □

Lemma 3.5. *Let $A = (\langle a_{ij}, a'_{ij}\rangle) \in \mathscr{F}_{mn}$, and $R = (\langle r_{ij}, r'_{ij}\rangle) \in \mathscr{F}_n$, is transitive, then $AR = (A \leftharpoondown AP)R$, where $P = (\langle n_{ij}, n'_{ij}\rangle) \in \mathscr{F}_n$, is nilpotent such that $P \leq R$.*

Proof. $AR = (\langle \sum_{k=1}^{n} a_{ik} r_{kj}, \prod_{k=1}^{n}(a'_{ik} + r'_{kj}\rangle)),$
$A \leftharpoondown AP = (\langle a_{ij}, a'_{ij}\rangle \leftharpoondown \langle \sum_{k=1}^{n} a_{ik} p_{kj}, \prod_{k=1}^{n}(a'_{ik} + p'_{kj})\rangle)$
Let $AR = (\langle b_{ij}, b'_{ij}\rangle), A \leftharpoondown AP = (\langle t_{ij}, t'_{ij}\rangle),$
$(A \leftharpoondown AP)R = (\langle \sum_{k=1}^{n} t_{ik} r_{kj}, \prod_{k=1}^{n}(t'_{ik} + r'_{kj}\rangle)) = (\langle c_{ij}, c'_{ij}\rangle).$
Clearly, $(\langle c_{ij}, c'_{ij}\rangle) \leq (\langle b_{ij}, b'_{ij}\rangle)$.
Let us claim that $(\langle c_{ij}, c'_{ij}\rangle) \geq (\langle b_{ij}, b'_{ij}\rangle)$. Suppose $(\langle b_{ij}, b'_{ij}\rangle) \neq (\langle 0, 1\rangle) > (\langle c_{ij}, c'_{ij}\rangle)$.
Then $\langle a_{il}, a'_{lj}\rangle\langle r_{lj}, r'_{lj}\rangle > \langle t_{il}, t'_{il}\rangle\langle r_{lj}, r'_{lj}\rangle$ for some l.
$\Rightarrow \langle a_{il}, a'_{il}\rangle > \langle t_{il}, t'_{il}\rangle$
$\Rightarrow \langle a_{il}, a'_{il}\rangle > [\langle a_{il}, a'_{il}\rangle \leftharpoondown \langle\langle \sum_{k=1}^{n} a_{ik} p_{kl}, \prod_{k=1}^{n}(a'_{ik} + p'_{kl})\rangle]$
$\Rightarrow \langle a_{il}, a'_{il}\rangle > [\langle a_{il}, a'_{il}\rangle \leftharpoondown \langle a_{in} p_{nl}, a'_{in} + p'_{nl}\rangle]$ for some n.
$\Rightarrow \langle a_{il}, a'_{il}\rangle \leq \langle a_{in} p_{nl}, a'_{in} + p'_{nl}\rangle = \langle a_{in}, a'_{in}\rangle\langle p_{nl}, p'_{nl}\rangle$
$\Rightarrow \langle a_{in}, a'_{in}\rangle \geq \langle a_{il}, a'_{il}\rangle$ and $\langle p_{nl}, p'_{nl}\rangle \geq \langle a_{il}, a'_{il}\rangle$
$\Rightarrow \langle p_{nl}, p'_{nl}\rangle\langle p_{nl}, p'_{nl}\rangle = \langle p_{nl}, p'_{nl}\rangle^{(2)} \geq \langle a_{il}, a'_{il}\rangle$.
Proceeding like this we get $\langle p_{nl}, p'_{nl}\rangle^{(m)} \geq \langle a_{il}, a'_{il}\rangle$ that is $\langle 0, 1\rangle \geq \langle a_{il}, a'_{il}\rangle$ and hence $\langle a_{il}, a'_{lj}\rangle\langle r_{lj}, r'_{lj}\rangle \leq \langle 0, 1\rangle$ which shows $\langle b_{il}, b'_{il}\rangle \leq \langle 0, 1\rangle$, which is a contradiction to our assumption. Therefore $(\langle c_{ij}, c'_{ij}\rangle) \geq (\langle b_{ij}, b'_{ij}\rangle)$. □

Theorem 3.6. *If $A = (\langle a_{ij}, a'_{ij}\rangle) \in \mathscr{F}_{mn}$, then $A = (A \leftarrow A^T)(A \leftharpoondown NA)$, where $N = (\langle n_{ij}, n'_{ij}\rangle) \in \mathscr{F}_m$ is nilpotent such that $N \leq (A \leftarrow A^T)$.*

Proof. By Lemma 2.5 $(A \leftarrow A^T)$ is transitive, therefore by Lemma 3.4
$(A \leftarrow A^T)A = (A \leftarrow A^T)(A \leftharpoondown NA)$. Using Lemma 3.2
$A = (A \leftarrow A^T)(A \leftharpoondown NA)$.
Similarly we can obtain the following theorem. □

Theorem 3.7. *If $A = (\langle a_{ij}, a'_{ij}\rangle) \in \mathscr{F}_{mn}$, then*
$A = (A \leftharpoondown AN)(A^T \leftarrow A)^T$, where $N \in \mathscr{F}_n$ is nilpotent matrix such that $N \leq (A^T \leftarrow A)^T$.

Remark 3.8. Since an irreflexive and transitive matrix is nilpotent, in the Theorem 3.6 and Theorem 3.7 we can consider N as even irreflexive and transitive matrix.

Lemma 3.9. *Let $T = (\langle t_{ij}, t'_{ij}\rangle), U = (\langle u_{ij}, u'_{ij}\rangle) \in \mathscr{F}_m$ be transitive matrices. If $T \leq U$, then $T \leftharpoondown U^T$ is irreflexive and transitive.*

Proof. Let $V = (\langle v_{ij}, v'_{ij}\rangle) = T \leftharpoondown U^T$.
That is $\langle v_{ij}, v'_{ij}\rangle = \langle t_{ij}, t'_{ij}\rangle \leftharpoondown \langle u_{ji}, u'_{ji}\rangle$, then $\langle v_{ii}, v'_{ii}\rangle = \langle t_{ii}, t'_{ii}\rangle \leftharpoondown \langle u_{ii}, u'_{ii}\rangle = \langle 0, 1\rangle$, so that V is irreflexive. Suppose $\langle v_{ik}, v'_{ik}\rangle\langle v_{kj}, v'_{kj}\rangle = \langle c, c'\rangle > \langle 0, 1\rangle$. We have two cases

Case 1. $\langle t_{ik}, t'_{ik}\rangle = \langle c, c'\rangle, \langle t_{ik}, t'_{ik}\rangle > \langle u_{ki}, u'_{ki}\rangle, \langle t_{kj}, t'_{kj}\rangle \geq \langle c, c'\rangle$.

Case 2.$\langle t_{ik}, t'_{ik}\rangle \geq \langle c, c'\rangle, \langle t_{kj}, t'_{kj}\rangle = \langle c, c'\rangle, \langle t_{kj}, t'_{kj}\rangle > \langle u_{jk}, u'_{jk}\rangle$.

In case1,$\langle t_{ik}, t'_{ik}\rangle = \langle c, c'\rangle, \langle t_{kj}, t'_{kj}\rangle \geq \langle c, c'\rangle \Rightarrow \langle t_{ik}, t'_{ik}\rangle\langle t_{kj}, t'_{kj}\rangle = \langle t_{ij}, t'_{ij}\rangle \geq \langle c, c'\rangle$.
Suppose that $\langle u_{ji}, u'_{ji}\rangle \geq \langle c, c'\rangle$.
$\langle u_{ki}, u'_{ki}\rangle \geq \langle u_{kj}, u'_{kj}\rangle\langle u_{ji}, u'_{ji}\rangle \geq \langle c, c'\rangle$, which is a contradiction to the fact that $\langle t_{ik}, t'_{ik}\rangle = \langle c, c'\rangle, \langle t_{ik}, t'_{ik}\rangle > \langle u_{ki}, u'_{ki}\rangle$. In Case 2 also we have $\langle t_{ij}, t'_{ij}\rangle \geq \langle c, c'\rangle$,

$\langle u_{jk}, u'_{jk}\rangle \geq \langle u_{ji}, u'_{ji}\rangle\langle u_{ik}, u'_{ik}\rangle \geq \langle c, c'\rangle$ which is a contradiction.
Therefore $\langle u_{ji}, u'_{ji}\rangle < \langle c, c'\rangle$, so that $\langle v_{ij}, v'_{ij}\rangle \geq \langle c, c'\rangle$. Thus V is transitive. □

Using Theorem 3.6,Theorem 3.7 and Lemma 3.9, we obtain the following corollaries.

Corollary 3.10. *If $A \in \mathscr{F}_{mn}$,then$A = (A \leftarrow A^T)(A \leftharpoondown \Delta TA)$,where $T = A \leftarrow A^T$.*

Corollary 3.11. *If $A \in \mathscr{F}_{mn}$, then $A = (A \leftharpoondown A\Delta R)(A^T \leftarrow A)^T$, where $R = (A^T \leftarrow A)^T$.*

Proposition 3.12. *If $A \in \mathscr{F}_{mn}$, and $F \in \mathscr{F}_{nl}$, then $AF = (A \leftharpoondown AN)F$, where N is nilpotent and $N \leq F \leftarrow F^T$.*

Proof. By Lemma 3.5, $(A \leftharpoondown AN)(F \leftarrow F^T) = A(F \leftarrow F^T)$.
By Lemma 3.2, $(F \leftarrow F^T)F = F$, thus $(A \leftharpoondown AN)(F \leftarrow F^T)F = A(F \leftarrow F^T)F$. So that $(A \leftharpoondown AN)F = AF$. □

Proposition 3.13. *If $T \in \mathscr{F}_{mn}$, and $F \in \mathscr{F}_{nl}$,then $TF = T(F \leftharpoondown NF)$, where N is nilpotent and $N \leq (T^T \leftarrow T)^T$*

Proof. Using Lemma 3.4, we have for any transitive matrix T and a nilpotent matrix N,
$TF = T(F \leftharpoondown NF), N \leq T$, since $(T^T \leftarrow T)^T$ is transitive
$(T^T \leftarrow T)^T F = (T^T \leftarrow T)^T(F \leftharpoondown NF)$
$T(T^T \leftarrow T)^T F = T(T^T \leftarrow T)^T(F \leftharpoondown NF)$.
But from Remark 3.1, $T(T^T \leftarrow T)^T = T$, thus
$TF = T(F \leftharpoondown NF)$ with $N \leq (T^T \leftarrow T)^T$. □

Since irreflexive and transitive means nilpotent, in Proposition 3.12 and Proposition 3.13 we can replace N by ΔR. In the following Example we decompose an IFM into an idempotent IFM and a rectangular IFM.

Example 3.14. Let $A = \begin{bmatrix} \langle 0.1, 0.2\rangle & \langle 0.6, 0.3\rangle & \langle 0.5.0.5\rangle & \langle 0.1.0.2\rangle \\ \langle 0.1, 0.2\rangle & \langle 0.2, 0.4\rangle & \langle 0.2, 0.6\rangle & \langle 0.1, 0.2\rangle \\ \langle 0.2, 0.1\rangle & \langle 0.4, 0.2\rangle & \langle 0.6, 0.4\rangle & \langle 0.3, 0.1\rangle \end{bmatrix}$. Then

$R = A \leftarrow A^T = \begin{bmatrix} \langle 1, 0\rangle & \langle 1, 0\rangle & \langle 0.1, 0.2\rangle \\ \langle 0.2, 0.4\rangle & \langle 1, 0\rangle & \langle 0.1, 0.2\rangle \\ \langle 0.4, 0.2\rangle & \langle 1, 0\rangle & \langle 1, 0\rangle, \end{bmatrix}$

$\Delta R = R \leftharpoondown R^T = \begin{bmatrix} \langle 0, 1\rangle & \langle 1, 0\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.4, 0.2\rangle & \langle 1, 0\rangle & \langle 0, 1\rangle, \end{bmatrix}$

$\Delta RA = \begin{bmatrix} \langle 0.1, 0.2\rangle & \langle 0.2, 0.4\rangle & \langle 0.2, 0.6\rangle & \langle 0.1, 0.2\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.1, 0.2\rangle & \langle 0.4, 0.2\rangle & \langle 0.4, 0.2\rangle & \langle 0.1, 0.2\rangle, \end{bmatrix}$

$$A \leftharpoondown \Delta RA = \begin{bmatrix} \langle 0,1\rangle & \langle 0.6,0.3\rangle & \langle 0.5,0.5\rangle & \langle 0,1\rangle \\ \langle 0.1,0.2\rangle & \langle 0.2,0.4\rangle & \langle 0.2,0.6\rangle & \langle 0.1,0.2\rangle \\ \langle 0.2,0.1\rangle & \langle 0,1\rangle & \langle 0.6,0.4\rangle & \langle 0.3,0.1\rangle, \end{bmatrix}$$

$$R(A \leftharpoondown \Delta RA) = \begin{bmatrix} \langle 0.1,0.2\rangle & \langle 0.6,0.3\rangle & \langle 0.5.0.5\rangle & \langle 0.1.0.2\rangle \\ \langle 0.1,0.2\rangle & \langle 0.2,0.4\rangle & \langle 0.2,0.6\rangle & \langle 0.1,0.2\rangle \\ \langle 0.2,0.1\rangle & \langle 0.4,0.2\rangle & \langle 0.6,0.4\rangle & \langle 0.3,0.1\rangle. \end{bmatrix} = A$$

4. Conclusions

Decomposition of rectangular IFM may be useful for decomposition of intuitionistic fuzzy databases. Decomposition of IFM is closely related to reduction of intuitionistic fuzzy retrieval models. Can we decompose any IFM into a product of an idempotent IFM and a nilpotent IFM?. The question can be extended by replacing nilpotency by symmetry, transitive, compact etc.

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