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On λ-homeomorphism and λ-identification

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Abstract. Introducing certain basic notions of point-set topology such as that of homeomorphism, compactness, identification and retract in fuzzy setting with respect to a given fuzzy set λ, we obtain results interrelating these.

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1. Introduction

Since the introduction of fuzzy sets by Zadeh in 1965 [17] several mathematical structures have been introduced and studied replacing crisp sets by fuzzy sets [1, 5, 11, 12, 14]. In the sequel, fuzzy topology on a set got introduced by Chang in 1968 [4].

Various notions in topology have been transferred to fuzzy setting [9, 16] adopting conventional route. The core notions in topology such as continuity of a map, separation axioms, compactness and connectedness have been considered by several workers, including [2, 7, 8, 10], which remained a topic of interest for workers in this area for about three decades.

Because most of the definitions were based simply on the lines of those available in general topology, it appears that the notions in fuzzy topology like fuzzy compactness, fuzzy Hausdorffness and others could not be well-interconnected paving way to deeper investigations. Hoping for equivalent forms of the notions to provide a purposeful way, in [3], by choosing a fuzzy set λ on a set X, λ-Hausdorffness and λ-compactness, together with λ-continuity, have been introduced providing the following result:

'Let X be a λ-Hausdorff space, Y be a μ-compact space and $f : X \to Y$ be a map. Then f is λ-continuous if and only if the λ-graph of f is a fuzzy closed set of $X \times Y$',

which generalizes Theorem XI.2.7 of [6].

In the same spirit, in Section 4, we introduce λ-homeomorphism and obtain that a bijective λ-continuous map from a λ-compact space to an $f(\lambda)$-Hausdorff space is a λ-homeomorphism. It is relevant to mention that the definition of λ-compactness is slightly modified without affecting the results established in [3].

In Section 5, we introduce the notions of λ-identification and λ-retract. For a specific fuzzy set λ of a fuzzy space X, it is shown that a λ-retract is a λ-identification. Further, if λ is graph closed with respect to a retraction $r : X \to A,\ A \subseteq X$ and $X/K(r)$, where $K(r)$ denotes the kernel of r, has the λ-identification topology, then $X/K(r)$ is $q(\lambda)$-homeomorphic to A.

Several lemmas have been found which have come to use in establishing these results. Expecting their role beyond this paper, an entire section has been devoted to them.

The following section consists of necessary prerequisite.

2. Preliminaries

Throughout the paper, X, Y and Z are nonempty sets. The symbol $\mathbb{N}$ denotes the set of natural numbers and for $A \subseteq X$, χ_A denotes the characteristic map on A. By a fuzzy set of X, we mean a member of I^X, where I is the unit closed interval [0, 1] of the real line $\mathbb{R}$.

Definition 2.1 ([4]). A fuzzy topology on X is defined to be the family τX of fuzzy sets of X satisfying:

(i) $0, 1 \in \tau X$,

(ii) for a subfamily $\mathcal{F}$ of τX, $\vee\mathcal{F}$ belongs to τX, where

$$\vee\mathcal{F}(x) = sup\{\lambda(x) \mid \lambda \in \mathcal{F}\}, \quad x \in X,$$

(iii) for $\lambda_1, \lambda_2 \in \tau X$, $\lambda_1 \wedge \lambda_2 \in \tau X$, where

$$(\lambda_1 \wedge \lambda_2)(x) = min\{\lambda_1(x),\ \lambda_2(x)\}, \quad x \in X.$$

The pair $(X, \tau X)$ consisting of a nonempty set X and a fuzzy topology τX is said to be a fuzzy space. A member of τX is called a fuzzy open set of X and the complement of a fuzzy open set is called a fuzzy closed set of X.

Let $f : X \to Y$ be a map. Then the image and the pre-image of a fuzzy set are defined in the following way:

(a) If λ is a fuzzy set of X, then for $y \in Y$,

$$f(\lambda)(y) = \begin{cases} sup_{x' \in f^{-1}(y)}\{\lambda(x')\}, & y \in f(X), \\ 0, & \text{otherwise.} \end{cases}$$

(b) If μ is a fuzzy set of Y, then for $x \in X$,

$$f^{-1}(\mu)(x) = \mu(f(x)).$$

Let X and Y be fuzzy spaces and $f : X \to Y$ be a map. Then f is said be fuzzy continuous [4], if for every $\mu \in \tau Y$, $f^{-1}(\mu) \in \tau X$.

Definition 2.2 ([3]). Let X and Y be fuzzy spaces, $f : X \to Y$ be a map and λ be a fuzzy set of X. Then f is called λ-continuous if for each fuzzy closed set ν of Y, $\lambda \wedge f^{-1}(\nu)$ is a fuzzy closed set of X.

Let λ and μ be fuzzy sets of X and Y, respectively. Then $\lambda \times \mu$ is a fuzzy set of $X \times Y$ defined by

$$(\lambda \times \mu)(x, y) = min\{\lambda(x), \mu(y)\}, \quad (x, y) \in X \times Y.$$

If X and Y are fuzzy spaces, then the product fuzzy space is defined to be the space $(X \times Y, \tau X \times Y)$, where $\tau X \times Y$ is the fuzzy topology generated by the family $\{\lambda \times \mu : \lambda \in \tau X,\ \mu \in \tau Y\}$ [16].

By d_X, we denote the diagonal map from X to $X \times X$ sending $x \in X$ to the pair (x, x). If $f : X \to Y$ is a map, then the *graph of* f is the map $g_f : X \to X \times Y$ sending x in X to the pair $(x, f(x))$. For a fuzzy set λ of X, $d_X(\lambda)$ and $g_f(\lambda)$ are called the λ-*diagonal* of X and the λ-*graph* of f, respectively.

Definition 2.3 ([3]). Let λ be a fuzzy set of a fuzzy space X. Then X is said to be fuzzy λ-Hausdorff if the λ-diagonal $d_X(\lambda)$ of X is a fuzzy closed set of $X \times X$.

Definition 2.4 ([2]). A fuzzy set λ of X is said to be graph closed with respect to a mapping $f : X \to Y$, if for each $x \in X$,

$$sup_{x' \in f^{-1}(f(x))}\{\lambda(x')\} = \lambda(x).$$

Definition 2.5 ([2]). A fuzzy continuous map $f : X \to Y$ from a fuzzy space X to a fuzzy space Y is called a fuzzy perfect map if for any fuzzy space Z, the map $f \times I_Z : X \times Z \to Y \times Z$ is fuzzy closed.

Notions in fuzzy topology which are not described here can be found in [2, 3, 4, 15, 16, 17].

3. Lemmas

Lemma 3.1. *Let f be a map from X onto Y and λ be a fuzzy set of X. Then*

$$(f \times f)(d_X(\lambda)) \wedge (f(\lambda) \times f(\lambda)) = d_Y(f(\lambda)).$$

Proof. Let $(y_1, y_2) \in Y \times Y$. Then

$$\begin{aligned}\big((f \times f)(d_X(\lambda)) \wedge (f(\lambda) \times f(\lambda))\big)(y_1, y_2) \\ &= min\{sup_{(x_1,x_2)\in(f\times f)^{-1}(y_1,y_2)}\{d_X(\lambda)(x_1,x_2)\},\ f(\lambda)(y_1),\ f(\lambda)(y_2)\} \\ &= \begin{cases} sup_{x\in f^{-1}(y_1)}\{\lambda(x)\}, & \text{if } y_1 = y_2, \\ 0, & \text{otherwise,} \end{cases} \\ &= d_Y(f(\lambda))(y_1, y_2).\end{aligned}$$

□

Lemma 3.2. *Let $f : X \to Y$ be a map and λ be a fuzzy set of X. Then*

$$(f \times I_Y)^{-1}(d_Y(\mu)) \leq \lambda \times 1,$$

where $\mu = f(\lambda)$ and I_Y is the identity map on Y.

Proof. The result follows by noting that

$$(f \times I_Y)^{-1}(d_Y(\mu))(x, y) = \begin{cases} \lambda(x), & \text{if } y = f(x), \\ 0, & \text{otherwise.} \end{cases}$$

□

Lemma 3.3. *Let $f : X \to Y$ be a bijective map. Then*
(i) $p_Y((\alpha \times 1) \wedge g_f(\lambda)) = f(\alpha) \wedge \mu$,
(ii) $p_Y((\alpha \times 1) \wedge g_f(\lambda)) \leq p_Y(\lambda \times 1)$,
where α and λ are fuzzy sets of X and $\mu = f(\lambda)$.

Proof. For $y \in Y$, the following establishes (i).

$$\begin{aligned}p_Y((\alpha \times 1) \wedge g_f(\lambda))(y) &= sup_{x\in X}\{((\alpha \times 1) \wedge g_f(\lambda))(x, y)\} \\ &= \alpha(f^{-1}(y)) \wedge \lambda(f^{-1}(y)) \\ &= (f(\alpha) \wedge \mu)(y).\end{aligned}$$

Further, since $p_Y(\lambda \times 1)(y) = sup_{x\in X}\{\lambda(x)\}$, we have (ii). □

We also use the following lemma established in [3].

Lemma 3.4. *Let $f : X \to Y$ and μ be a fuzzy set of Y. Then*

$$g_f(f^{-1}(\mu)) = (f \times I_Y)^{-1}(d_Y(\mu)),$$

where I_Y is the identity map on Y.

4. λ-homeomorphism

In this section, we introduce the notion of a λ-closed map and modify the notion of λ-compactness introduced in [3] motivated by [6] and [13], accordingly. Besides, we introduce the notion of a λ-homeomorphism and obtain that λ-Hausdorffness is a λ-invariant (preserved under a λ-homeomorphism), while λ-compactness is a 1-invariant. Further, it is shown that λ-Hausdorffness is also preserved under a fuzzy perfect mapping. At the end of this section, we put a well-known result in point-set topology, according to which a bijective map from a compact space to a Hausdorff space is a homeomorphism, in fuzzy setting.

Definition 4.1. Let X and Y be fuzzy spaces, $f : X \to Y$ be a map and λ be a fuzzy set of X. Then f is said to be λ-closed if for a fuzzy closed set μ of X, $f(\lambda) \wedge f(\mu)$ is a fuzzy closed set of Y.

Definition 4.2. Let X and Y be fuzzy spaces, $f : X \to Y$ be a bijective map and λ be a fuzzy set of X. Then f is called a λ-homeomorphism if

(i) f is λ-continuous;
(ii) f^{-1} is $f(\lambda)$-continuous, or equivalently, f is λ-closed.

Theorem 4.3. *Let X be a λ-Hausdorff space, Y be a fuzzy space and $f : X \to Y$ be an onto map such that $f \times f$ is $(\lambda \times \lambda)$-closed. Then Y is $f(\lambda)$-Hausdorff.*

Proof. Follows from Lemma 3.1. □

Corollary 4.4. *The property of λ-Hausdorffness is a λ-invariant.*

Theorem 4.5. *Let f be a fuzzy perfect map from a λ-Hausdorff space X onto a fuzzy space Y. Then Y is $f(\lambda)$-Hausdorff.*

Proof. Since the product of two fuzzy perfect maps is a fuzzy perfect map [2], $f \times f$ is fuzzy perfect and hence fuzzy closed. Also, since $\lambda \times \lambda$ is a fuzzy closed set in $X \times X$, $f \times f$ is $(\lambda \times \lambda)$-closed. Thus, Theorem 4.3 provides the result. □

Using the following characterization of a Hausdorff compact space:

'A Hausdorff space is compact if and only if the projection p_Z from $X \times Z$ to Z is closed for every Hausdorff space Z ([6, 13])',

a λ-compact space has been defined in [3]. Taking the projection p_Z to be $(\lambda \times 1)$-closed, for a fuzzy set λ of X, we modify the notion of λ-compactness as follows:

Definition 4.6. Let λ be a fuzzy set of a fuzzy topological space X. Then X is said to be λ-compact if X is fuzzy λ-Hausdorff and the projection p_Z from $X \times Z$ to Z is $(\lambda \times 1)$-closed, for each Hausdorff space Z.

For a set X and $x_0 \in X$, define $\lambda : X \to I$ to be $\chi_{\{x_0\}}$ and $\tau X = \{0, 1, \lambda'\}$. It has been obtained in [3] that for a Hausdorff space Z, the projection p_Z is fuzzy closed. Since a fuzzy closed map is μ-closed, where μ is a fuzzy closed set, $(X, \tau X)$ is a λ-compact space.

Theorem 4.7. *Let f be a fuzzy continuous map from a λ-compact space X onto an $f(\lambda)$-Hausdorff space Y, where λ is a fuzzy set of X. Then Y is $f(\lambda)$-compact.*

Proof. Let Z be a Hausdorff space and p_Z^X and p_Z^Y be the projection maps from $X \times Z$ to Z and $Y \times Z$ to Z, respectively. Then, since p_Z^X is a $(\lambda \times 1)$-closed map, for a fuzzy closed set μ of $Y \times Z$, $p_Z^X((f \times I_Z)^{-1}(\mu)) \wedge p_Z^X(\lambda \times 1)$, where I_Z is the identity map on Z, is a fuzzy closed set in Z. For $(y, z) \in Y \times Z$, $p_Z^Y(y, z) = p_Z^X \circ (f \times I_Z)^{-1}(y, z)$. Also, for $z \in Z$,

$$p_Z^X(\lambda \times 1)(z) = \sup_{x \in X}\{\lambda(x)\} = p_Z^Y(f(\lambda) \times 1)(z).$$

Thus

$$p_Z^X((f \times I_Z)^{-1}(\mu)) \wedge p_Z^X(\lambda \times 1) = p_Z^Y(\mu) \wedge p_Z^Y(f(\lambda) \times 1).$$

Hence, p_Z^Y is $(f(\lambda) \times 1)$-closed. □

The following example shows that Theorem 4.7 need not be true when the fuzzy continuous map is replaced by a λ-continuous map or a λ-homeomorphism, where $\lambda \neq 1$.

Example 4.8. Let $x_0 \in I$. Then, for $\lambda = \chi_{\{x_0\}}$, the family $\{0, \lambda', 1\}$ is a fuzzy topology on I. Write I_1, for the fuzzy space I with this fuzzy topology. It is shown in [3] that I_1 is a λ-compact space.

Let $x_1 \neq x_0$ be in I and $0 < s < 1$. Set $\alpha = s \cdot \chi_{\{x_1\}}$. Then the family $\{0, \lambda', \alpha', \lambda' \wedge \alpha', 1\}$ describes another fuzzy topology on I. Denote I with this fuzzy topology by I_2.

It is straightforward to check that the identity map $f : I_1 \to I_2$ is λ-continuous. In fact, it is a λ-homeomorphism.

Now, we show that I_2 is not $f(\lambda)$-compact. Let Z be the fuzzy space obtained by considering I with the usual topology. Then $F = \alpha \times 1$ is a fuzzy closed set of $I_2 \times Z$. Since $p_Z(f(\lambda) \times 1) = 1$ and $p_Z(F)(z) = s$, for each $z \in Z$, $p_Z(F) \wedge p_Z(f(\lambda) \times 1)$ is not a fuzzy closed set of Z.

Theorem 4.9. *Let f be a λ-continuous bijective map from a λ-compact space X to a μ-Hausdorff space Y, where $\mu = f(\lambda)$. Then f is a λ-homeomorphism.*

Proof. It is sufficient to show that f is λ-closed. Note that lemmas 3.4 and 3.2 give that $g_f(\lambda)$ is a fuzzy closed set of Y. Let α be a fuzzy closed set of X. Then $(\alpha \times 1) \wedge g_f(\lambda)$ is a fuzzy closed set of $X \times Y$. Since X is λ-compact, p_Y is a $(\lambda \times 1)$-closed map, and hence the result follows from Lemma 3.3. □

5. λ-IDENTIFICATION

In this section, for a fuzzy set λ on a fuzzy space X, we introduce and study the λ-identification map.

Let $f : X \to Y$ be an onto map, where X is a fuzzy space and Y is a set, and λ be a fuzzy closed set of X. Then calling a fuzzy set μ of Y to be fuzzy closed if $f^{-1}(\mu) \wedge \lambda$ is fuzzy closed in X, we obtain a fuzzy topology on Y which is indeed the largest fuzzy topology on Y making f a λ-continuous map. We call this topology the λ-identification topology on Y.

It may be noted that for ordinary topological spaces and $\lambda = 1$, the λ-identification topology becomes the identification topology as described in [6].

Definition 5.1. Let X and Y be fuzzy spaces and f be a λ-continuous map from X onto Y, where λ is a fuzzy set of X. Then f is said to be a λ-identification map if Y has the λ-identification topology.

Let $(X, \tau X)$ be a fuzzy topological space and $A \subseteq X$. Then A is a fuzzy space with respect to the subspace topology given by

$$\tau A = \{\mu \wedge \chi_A \mid \mu \in \tau X\}.$$

Definition 5.2. Let X be a fuzzy space, $A \subseteq X$ and $r : X \to A$ be a λ-continuous map, where λ is a fuzzy set of X. Then r is called a λ-retraction if for every $x \in A$, $r(x) = x$.

Theorem 5.3. *A λ-retraction $r : X \to A$, where λ is a fuzzy closed set of X such that $\lambda \wedge \chi_A = \chi_A$, is a λ-identification map.*

Proof. Let μ be a fuzzy set of A such that $r^{-1}(\mu)\wedge\lambda$ is closed in X. Then $r^{-1}(\mu)\wedge\lambda\wedge\chi_A$ is a fuzzy closed set in A. The result follows by noting that $r^{-1}(\mu) \wedge \chi_A = \mu$. □

The following example shows that the condition $\lambda\wedge\chi_A = \chi_A$ in the above theorem cannot be omitted.

Example 5.4. For $n \in \mathbb{N} \cup \{0\}$, define $f_n : \mathbb{R} \to I$ by

$$f_n(z) = 0 \vee z^n \wedge 1, \quad z \in \mathbb{R}.$$

Consider the fuzzy topology on $\mathbb{R}$ having f_n, for $n = 0$ and all odd natural numbers, together with the constant map 0, as its members.

Let $r : \mathbb{R} \to I$ be the map taking $(-\infty, 0)$ to 0, $(1, \infty)$ to 1 and [0, 1] identically to [0, 1].

Consider $\lambda = 1 - f_3$. To show that r is λ-continuous, we obtain that $r^{-1}(\nu) \wedge \lambda$ is a fuzzy closed set of $\mathbb{R}$, where ν is a fuzzy closed set of I. Because r^{-1} preserves arbitrary meets, it is sufficient to consider $\nu = 1 - (f_n \wedge \chi_I)$, where n is an odd natural number. Since $r^{-1}(1-(f_n\wedge\chi_I))\wedge(1-f_3) = 1-f_k$, where $k = \max\{n, 3\}$, $r^{-1}(\nu) \wedge \lambda$ is a fuzzy closed set of $\mathbb{R}$.

In fact, r is a λ-retraction. However, the fuzzy subspace topology on I is not the λ-identification topology on I because for $\mu = (1 - f_4)|_I$, which is not a fuzzy closed set of I, $r^{-1}(\mu) \wedge \lambda$ is fuzzy closed in $\mathbb{R}$. Consequently, r is not a λ-identification.

Let $f : X \to Y$ be an onto map. Define an equivalence relation, denoted by $K(f)$, in X by $x\ K(f)\ x'$ if $f(x) = f(x')$, for $x,\ x' \in X$. Equip the quotient set $X/K(f)$ with the identification topology induced by the map $q : X \to X/K(f)$ sending $x \in X$ to its equivalence class $[x]$. Then we have the following:

Theorem 5.5. *Let X be a fuzzy space, $A \subseteq X$ and $r : X \to A$ be a λ-retraction, where λ is a fuzzy set of X which is graph closed with respect to the quotient map $q : X \to X/K(r)$. Then $X/K(r)$ is $q(\lambda)$-homeomorphic to A.*

Proof. Define $h : X/K(r) \to A$ by

$$h([x]) = r(x), \quad [x] \in X/K(r).$$

Then

(i) h is a bijective map.

(ii) Let μ be a fuzzy closed set of A. Then

$$h^{-1}(\mu) \wedge q(\lambda) = q(r^{-1}(\mu)) \wedge q(\lambda).$$

Since for $x \in X$,

$$\begin{aligned} q^{-1}(q(r^{-1}(\mu)) \wedge q(\lambda))(x) &= (q(r^{-1}(\mu)) \wedge q(\lambda))([x]) \\ &= min\{\mu(r(x)), sup_{x' \in q^{-1}(q(x))}\{\lambda(x')\}\} \\ &= \min\{\mu(r(x)), \lambda(x)\} \\ &= (r^{-1}(\mu) \wedge \lambda)(x), \end{aligned}$$

$h^{-1}(\mu) \wedge q(\lambda)$ is fuzzy closed in $X/K(r)$.

(iii) Let ν be a fuzzy closed set of $X/K(r)$. Then for $x \in A$,

$$\begin{aligned} (h(\nu) \wedge h(q(\lambda)))(x) &= min\{\nu([x]), sup_{x' \in q^{-1}(q(x))}\{\lambda(x')\}\} \\ &= (q^{-1}(\nu) \wedge \lambda \wedge \chi_A)(x). \end{aligned}$$

Since q is λ-continuous, we have that $h(\nu) \wedge h(q(\lambda))$ is closed in A. These show that h is a $q(\lambda)$-homeomorphism. □

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