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Cubic finite state machine

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Abstract. In this paper we present the concepts of cubic finite state machine, cubic finite subsystem and cubic finite submachine. The ideas of cubic immediate successor, cubic successor and strongly cubic connected are also introduced, and we investigate some of their algebraic properties.

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1. Introduction

The notion of a fuzzy set was introduced by Zadeh [14]. The theory of fuzzy automata was introduced by Wee [13]. Algebraic techniques to study fuzzy automata were introduced by Malik et al. in [7, 8, 9]. For a review of algebraic techniques used in crisp automata see [2]. Fuzzy finite automata has many applications, for example , they are useful for a knowledge based system designer since a knowledge based system should solve a problem from fuzzy knowledge and should also provide the user with reasons for arriving at certain conclusions. A design tool is more valuable if there exist guidelines to assist the designer to come with the best possible design. One of the major criteria for a best design is that it be minimal, for this see [1, 10, 12].

In [3], using the notion of soft sets, Hussain and Shabir introduced the concepts of soft finite state machines($SFSM$), soft successor, soft immediate successor, soft subsystems, soft submachines, weakly soft connected SFSM, strongly soft connected SFSM and studied related properties. They gave relations between strongly soft connected and soft submachine and also provided a characterization of general direct product of two soft finite state machines.

On the other hand cubic sets was introduced by Jun et al. in [5] and investigate several properties. In this paper we defined finite state machine on the basis of cubic set and investigate some of their algebraic properties.

Arrangement of this paper is as the following. In section 2, some basic notions related to finite state machines, fuzzy set and cubic sets are given. Notion of cubic finite machine is introduced in section 3. In this section concept of cubic successor and cubic immediate successor is introduced and some of its properties are discussed. This Section is devoted for the study of cubic sub-systems, cubic sub-machines and strongly connected submachines.

2. Preliminaries

In this section we review some basic definitions of finite state machine, fuzzy set and cubic set.

Definition 2.1. A six tuple $M = (Q, X, Y, f, g, s)$ is called a finite state machine if Q, X, and Y are non-empty sets, $f : Q \times X \to Q$, $g : Q \times X \longrightarrow Y$, and $s \in Q$.

The members of Q are called states. The members of X and Y are called input and output symbols respectively. The functions f and g are called the state transition and output functions, respectively. The state s is called the initial state.

A finite state automaton is a finite state machine such that the set of output symbols Y is $\{0, 1\}$. Further algebraic properties can be found in [4].

The concept of fuzzy automata was given by Wee in 1967 [13]. Later different approaches for the fuzzification of finite state machine presented in detail in [6, 11].

Now we recall the basic definitions of fuzzy set and cubic set.

Let X be a non empty fixed set. A fuzzy set in a set X is defined to be a function $f : X \longrightarrow [0, 1]$. If f and g are fuzzy subsets of X, then

(i) $f \subseteq g$ if and only if $f(x) \leq g(x)\ \ \forall\, x \in X$

(ii) $f = g$ if and only if $f \subseteq g$ and $g \subseteq f$

(iii) The complement of f, denoted by f^c, is defined as $f^c(x) = 1 - f(x)\ \ \forall x \in X$.

More generally if $\{f_i : i \in I\}$ is a family of fuzzy subsets of X, then by the union(join $\vee$) and intersection(meet $\wedge$) of this family we mean a fuzzy subsets

(iv) $(\vee_{i \in I} f_i)(x) = \sup\{f_i(x) : i \in I\}$

(v) $(\wedge_{i \in I} f_i)(x) = \inf\{f_i(x) : i \in I\}$

respectively for all $x \in X$.

By an interval number $\tilde{a} = [a^-, a^+]$ we mean a closed subinterval of $[0, 1]$ where $0 \leq a^- \leq a^+ \leq 1$. let us denote $D[0, 1]$, the set of all closed sub intervals of $[0, 1]$, and define refined minimum(briefly $r\min$) of two elements in $D[0, 1]$. We also define the symbols $\preceq$, $\succeq$ in case of two elements in $D[0, 1]$. Consider two interval number $\tilde{a}_1 = \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 = \left[a_2^-, a_2^+\right]$. Then

$$r\min\{\tilde{a}_1, \tilde{a}_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right],$$

$$\tilde{a}_1 \preceq \tilde{a}_2 \text{ if and only if } a_1^- \leq a_2^- \text{ and } a_1^+ \leq a_2^+,$$

and similarly we have $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. More generally, let $\widetilde{a_i} \in D[0, 1]$ where $i \in I$. We define

$$r\inf_{i \in I} \widetilde{a_i} = \left[\inf_{i \in I} a_i^-, \inf_{i \in I} a_i^+\right] \text{ and } r\sup_{i \in I} \widetilde{a_i} = \left[\sup_{i \in I} a_i^-, \sup_{i \in I} a_i^+\right]$$

For any $\tilde{a} \in D[0, 1]$, its complement denoted by $\tilde{a}^c$ and defined as

$$\tilde{a}^c = \left[1 - a^+, 1 - a^-\right]$$

An interval valued fuzzy set(briefly, an IVF set) in a set X is defined to be a function $A : X \longrightarrow D[0,1]$, where $D[0,1]$ is the set of all closed sub intervals of $[0,1]$. For every IVF sets A and B in X, we define

$$\begin{aligned} A &\subseteq B \text{ if and only if } A(x) \preceq B(x) \text{ for all } x \in X \text{ and} \\ A &= B \text{ if and only if } A(x) = B(x) \text{ for all } x \in X \end{aligned}$$

More generally if $\{A_i : i \in I\}$ is a family of IVF fuzzy subsets of X, then by the union $G = \cup_{i\in I} A_i$ and intersection $H = \cap_{i\in I} A_i$of this family are defined as follows:

$$G(x) = (\cup_{i\in I} A_i)(x) = r\sup_{i\in I} A_i(x)$$

and

$$F(x) = (\cap_{i\in I} A_i)(x) = r\inf A_i(x)$$

for all $x \in X$, respectively.

Definition 2.2 ([5]). Let X be a non empty set. By a cubic set in X we mean a structure

$$\mathcal{A} = \{\langle x, A(x), f(x)\rangle : x \in X\}$$

in which A is IVF set in X and f is a fuzzy set in X.

A cubic set $\mathcal{A} = \{\langle x, A(x), f(x)\rangle : x \in X\}$ is simply denoted by $\mathcal{A} = \langle A, f\rangle$.

Definition 2.3. Let $\mathcal{A} = \langle A, f\rangle$ and $\mathcal{B} = \langle B, g\rangle$ be cubic sets in X. Then we define
(*i*) $\mathcal{A} = \mathcal{B}$ if and only if $A = B$ and $f = g$.
(*ii*) $\mathcal{A} \subseteq \mathcal{B}$ if and only if $A \subseteq B$ and $f \geq g$.
More generally $\mathcal{A}_i = \{\langle x, A_i(x), f_i(x)\rangle : x \in X\}$ where $i \in I$, we define
(*iii*) $\cup_{i\in I}\mathcal{A}_i = \{\langle x, (\cup_{i\in I} A_i)(x), (\wedge_{i\in I} f_i)(x)\rangle : x \in X\}$.
(*iv*) $\cap_{i\in I}\mathcal{A}_i = \{\langle x, (\cap_{i\in I} A_i)(x), (\vee_{i\in I} f_i)(x)\rangle : x \in X\}$.

3. Cubic finite state machine

Definition 3.1. A cubic finite state machine($CFSM$) is a triple $M = (Q, X, \mathcal{A})$, where Q and X are finite non-empty set of states and input respectively, and $\mathcal{A} = \langle A, f\rangle$ is a cubic set in $Q \times X \times Q$.

Let X^* denote the set of all words of element of X of finite length, λ denote the empty word in X^* and $|x|$ denote the length of x for every $x \in X^*$.

Definition 3.2. Let $M = (Q, X, \mathcal{A})$ be a $CFSM$. Define cubic fuzzy set $\mathcal{A}^* = \langle A^*, f^*\rangle$ in $Q \times X^* \times Q$ by

$$\begin{aligned} A^*(q,\lambda,q_1) &= \begin{cases} [1,1] & \text{if } q = q_1 \\ [0,0] & \text{otherwise.} \end{cases} \quad \text{and} \quad f^*(q,\lambda,q_1) = \begin{cases} 0 \text{ if } q = q_1 \\ 1 \text{ otherwise.} \end{cases} \\ A^*(q,a,q_1) &= A(q,a,q_1) \qquad \text{and} \qquad f^*(q,a,q_1) = f(q,a,q_1) \end{aligned}$$

and

$$\begin{aligned} A^*(q,xa,q_1) &= r\sup_{s\in Q}\{r\inf\{A^*(q,x,s), A(s,a,q_1)\}\} \quad \text{and} \\ f^*(q,xa,q_1) &= \wedge_{s\in Q}\{f^*(q,x,s) \vee f(s,a,q_1)\} \end{aligned}$$

for all $q_1, q \in Q, x \in X^*$ and $a \in X$.

Lemma 3.3. *Let $M = (Q, X, \mathcal{A})$ be a $CFSM$. Then*

$$\begin{aligned} A^*(q, xy, q_1) &= r\sup_{s\in Q}\{r\inf\{A^*(q,x,s), A^*(s,y,q_1)\}\} \text{ and} \\ f^*(q,xy,q_1) &= \wedge_{s\in Q}\{f^*(q,x,s)\vee f^*(s,y,q_1)\} \end{aligned}$$

for all $q_1, q \in Q$ and $x, y \in X^*$.

Proof. Let $q_1, q \in Q$ and $x, y \in X^*$. We prove the result by induction on $|y| = n$ if $n = 0$, then $y = \lambda$ and so $xy = x\lambda = x$, hence

$$\begin{aligned} r\sup_{s\in Q}\{r\inf\{A^*(q,x,s), A^*(s,y,q_1)\}\} &= r\sup_{s\in Q}\{r\inf\{A^*(q,x,s), A^*(s,\lambda,q_1)\}\} \\ &= A^*(q,x,q_1) \end{aligned}$$

and

$$\begin{aligned} \wedge_{s\in Q}\{f^*(q,x,s)\vee f^*(s,y,q_1)\} &= \wedge_{s\in Q}\{f^*(q,x,s)\vee f^*(s,\lambda,q_1)\} \\ &= f^*(q,x,q_1). \end{aligned}$$

Thus the result is true for $n = 0$.

Suppose that the result is valid for all $u \in X^*$ such that $|u| = n-1$, $n > 0$. Let $y = ua$ where $u \in X^*$, $a \in X$ and $|u| = n-1$. Then

$$\begin{aligned} f^*(q,xy,q_1) &= f^*(q,xua,q_1) = r\sup_{s\in Q}\{r\inf\{A^*(q,xu,s), A(s,a,q_1)\}\} \\ &= r\sup_{s\in Q}\left\{r\inf\left\{\begin{array}{c} r\sup_{t\in Q}\{r\inf\{A^*(q,x,t), A^*(t,u,s)\}\}, \\ A(s,a,q_1) \end{array}\right\}\right\} \\ &= r\sup_{t,s\in Q}\{r\inf\{A^*(q,x,t), A^*(t,u,s), A(s,a,q_1)\}\} \\ &= r\sup_{t\in Q}\left\{r\inf\left\{\begin{array}{c} A^*(q,x,t), \\ r\sup_{s\in Q}\{r\inf\{A^*(t,u,s), A(s,a,q_1)\}\} \end{array}\right\}\right\} \\ &= r\sup_{t\in Q}\{r\inf\{A^*(q,x,t), A^*(t,ua,q_1)\}\} \\ &= r\sup_{t\in Q}\{r\inf\{A^*(q,x,t), A^*(t,y,q_1)\}\} \end{aligned}$$

and

$$\begin{aligned} f^*(q,xy,q_1) &= f^*(q,xua,q_1) = \wedge_{s\in Q}\{f^*(q,xu,s)\vee f(s,a,q_1)\} \\ &= \wedge_{s\in Q}\{\wedge_{t\in Q}\{f^*(q,x,t)\vee f^*(t,u,s)\}\vee f(s,a,q_1)\} \\ &= \wedge_{s,t\in Q}\{f^*(q,x,t)\vee f^*(t,u,s)\vee f(s,a,q_1)\} \\ &= \wedge_{t\in Q}\{f^*(q,x,t)\vee(\wedge_{s\in Q}\{f^*(t,u,s)\vee f(s,a,q_1)\})\} \\ &= \wedge_{t\in Q}\{f^*(q,x,t)\vee f^*(t,ua,q_1)\} \\ &= \wedge_{t\in Q}\{f^*(q,x,t)\vee f^*(t,y,q_1)\}. \end{aligned}$$

Hence the result is valid for $|y| = n$. This completes the proof. □

Definition 3.4. Let $M = (Q, X, \mathcal{A})$ be a $CFSM$ and let $q_1, q \in Q$. Then q_1 is called a cubic immediate successor of q if there exists $a \in X$ such that $A(q, a, q_1) \neq [0\ 0]$

and $f(q, a, q_1) < 1$. We denote by $CIS(q)$ the set of all cubic immediate successor of q.

We say that q_1 is a cubic successor of q if there exists $x \in X^*$ such that $A^*(q, a, q_1) \neq [0\ 0]$ and $f^*(q, a, q_1) < 1$. We denote by $CS(q)$ the set of all cubic successor of q. For any subset T of Q the set of all cubic successor of T denoted by $CS(T)$ is defined to be the set

$$CS(T) = \cup \{CS(t) : t \in T\}.$$

Proposition 3.5. *Let $M = (Q, X, \mathcal{A})$ be a CFSM. For any $q_1, q, s \in Q$, the following hold:*

(i) $q \in CS(q)$.

(ii) If $q_1 \in CS(q)$ and $s \in CS(q_1)$, then $s \in CS(q)$.

Proof. (i) It is obvious. Since

$$A^*(q, \lambda, q) = [1\ 1] \neq [0\ 0] \text{ and } f^*(q, \lambda, q) = 0 < 1$$

so we have $q \in CS(q)$.

(ii) Let $q_1 \in CS(q)$ and $s \in CS(q_1)$. Then there exist $x, y \in X^*$ such that

$$A^*(q, x, q_1) \neq [0\ 0] \text{ and } f^*(q, x, q_1) < 1$$

and

$$A^*(q_1, y, s) \neq [0\ 0] \text{ and } f^*(q_1, y, s) < 1$$

Using Lemma 3.3, we have

$$\begin{aligned} A^*(q, xy, s) &= r\sup_{t \in Q} \{r\inf\{A^*(q, x, t), A^*(t, y, s)\}\} \\ &\geq r\inf\{A^*(q, x, q_1), A^*(q_1, y, s)\} \\ &\neq [0\ 0] \end{aligned}$$

and

$$\begin{aligned} f^*(q, xy, s) &= \wedge_{t \in Q} \{f^*(q, x, t) \vee f^*(t, y, s)\} \\ &\leq f^*(q, x, q_1) \vee f^*(q_1, y, s) < 1. \end{aligned}$$

Hence $s \in CS(q)$. □

Proposition 3.6. *Let $M = (Q, X, \mathcal{A})$ be a CFSM. For any subset A and B of Q, the following assertions hold.*

(i) If $A \subseteq B$, then $CS(A) \subseteq CS(B)$.

(ii) $A \subseteq CS(A)$.

(iii) $CS(CS(A)) = CS(A)$.

(iv) $CS(A \cup B) = CS(A) \cup CS(B)$.

(v) $CS(A \cap B) \subseteq CS(A) \cap CS(B)$.

Proof. The proofs of (i) and (ii) are straightforward.

(iii) Obviously $CS(A) \subseteq CS(CS(A))$. If $q \in CS(CS(A))$, then $q \in CS(q_1)$ for some $q_1 \in CS(A)$. From $q_1 \in CS(A)$, there exists $s \in A$ such that $q_1 \in CS(s)$. It follows from Proposition 3.5 that $q \in CS(s) \subseteq CS(A)$ so that $CS(CS(A)) \subseteq CS(A)$. Thus (iii) is valid.

(iv) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$,

$$CS\left(A\right) \subseteq CS\left(A \cup B\right) \text{ and } CS\left(B\right) \subseteq CS\left(A \cup B\right).$$

Thus

$$CS\left(A\right) \cup CS\left(B\right) \subseteq CS\left(A \cup B\right).$$

Conversely, let $q \in CS\left(A \cup B\right) = \cup \left\{CS\left(z\right) : \ z \in A \cup B\right\}$. Then $q \in CS\left(z\right)$ for some $z \in A \cup B$. Thus there exist some $x \in X^*$ such that $A^*\left(z, x, q\right) \neq [0\ 0]$ and $f^*\left(z, x, q\right) < 1$.
So $q \in CS\left(A\right)$ or $q \in CS\left(B\right)$ and thus $q \in CS\left(A\right) \cup CS\left(B\right)$. Hence $CS(A \cup B) \subseteq CS\left(A\right) \cup CS\left(B\right)$. There $CS(A \cup B) = CS\left(A\right) \cup CS\left(B\right)$.

(v) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$,

$$CS(A \cap B) \subseteq CS(A) \text{ and } CS(A \cap B) \subseteq CS(B).$$

Thus $CS(A \cap B) \subseteq CS(A) \cap CS(B)$. □

Definition 3.7. Let $M = (Q, X, \mathcal{A})$ be a $CFSM$. We say that M satisfies the cubic exchange property if for all $q_1, q \in Q$ and $T \subseteq Q$, whenever

$$q_1 \in CS\left(T \cup \{q\}\right) \text{ and } q_1 \notin CS\left(T\right) \text{ then } q \in CS\left(T \cup \{q_1\}\right).$$

Theorem 3.8. *Let $M = (Q, X, \mathcal{A})$ be a $CFSM$. Then the following assertions are equivalent.*

(i) *M satisfies the cubic exchange property.*

(ii) $(\forall\ q_1, q \in Q)$ $q_1 \in CS\left(q\right)$ *if and only if* $q \in CS\left(q_1\right)$.

Proof. Assume that M satisfies the cubic exchange property.

Let $q_1, q \in Q$ be such that $q_1 \in CS\left(q\right) = CS\left(\varnothing \cup \{q\}\right)$. Note that $q_1 \notin CS\left(\varnothing\right)$ and so $q \in CS\left(\varnothing \cup \{q_1\}\right) = CS\left(q_1\right)$.

Similarly if $q \in CS\left(q_1\right)$ then $q_1 \in CS\left(q\right)$.

Conversely suppose that (ii) is valid.

Let $q_1, q \in Q$ and $T \subseteq Q$. If $q_1 \in CS\left(T \cup \{q\}\right)$ and $q_1 \notin CS\left(T\right)$, then $q_1 \in CS\left(q\right)$. It follows from (ii) that $q \in CS\left(q_1\right) \subseteq CS\left(T \cup \{q_1\}\right)$. Hence M satisfies the cubic exchange property. □

Definition 3.9. Let $M = (Q, X, \mathcal{A})$ be a $CFSM$. Let $\widetilde{\mathcal{A}} = \left(\widetilde{A}, \widetilde{f}\right)$ be a cubic subset in Q . Then $\left(Q, X, \mathcal{A}, \widetilde{\mathcal{A}}\right)$ is called a cubic subsystem of M if $\forall q_1, q \in Q, \forall x \in X^*$,

$$\begin{aligned} \widetilde{A}\left(q\right) &\geq r\inf\left\{\widetilde{A}\left(q_1\right), A\left(q_1, x, q\right)\right\} \text{ and} \\ \widetilde{f}\left(q\right) &\leq \widetilde{f}\left(q_1\right) \vee f\left(q_1, x, q\right) \end{aligned}$$

If $\left(Q, X, \mathcal{A}, \widetilde{\mathcal{A}}\right)$ is a cubic subsystem of M, then we write $\widetilde{M}$ for $\left(Q, X, \mathcal{A}, \widetilde{\mathcal{A}}\right)$.

Example 3.10. Let $Q = \{x, y\}$, $X = \{a\}$. Let $\mathcal{A} = \langle A, f\rangle$ be a cubic subset in $Q \times X \times Q$ defined as

$$A\left(t_1, a, t_2\right) = [0.4, 0.8] \text{ and } f\left(t_1, a, t_2\right) = \frac{1}{2} \text{ for all } t_1, t_2 \in Q.$$

Then $M = (Q, X, \mathcal{A})$ is $CFSM$. Let $\widetilde{\mathcal{A}} = \left(\widetilde{A}, \widetilde{f}\right)$ be a cubic subset in Q defined as

$$\begin{aligned} \widetilde{A}(x) &= [0.5, 0.9], \ \widetilde{A}(y) = [0.4, 0.8] \text{ and} \\ \widetilde{f}(x) &= \frac{1}{8}, \ \widetilde{f}(y) = \frac{1}{3}. \end{aligned}$$

It can easily be verified $\left(Q, X, \mathcal{A}, \widetilde{\mathcal{A}}\right)$ is a cubic subsystem of M.

Theorem 3.11. *Let $M = (Q, X, \mathcal{A})$ be a $CFSM$. Let $\widetilde{\mathcal{A}} = \left(\widetilde{A}, \widetilde{f}\right)$ be a cubic subset in Q . Then $\widetilde{M}$ is a cubic subsystem of M if and only if*

$$\begin{aligned} \widetilde{A}(q) &\geq r\inf\left\{\widetilde{A}(q_1), A^*(q_1, x, q)\right\} \text{ and} \\ \widetilde{f}(q) &\leq \widetilde{f}(q_1) \vee f^*(q_1, x, q) \qquad \forall q_1, q \in Q, \text{and } x \in X^*. \end{aligned}$$

Proof. Suppose that $\widetilde{M}$ is a cubic subsystem of M. Let $q_1, q \in Q$ and $x \in X^*$. The proof is by induction on $|x| = n$.

If $n = 0$, then $x = \lambda$. Now if $q_1 = q$, then

$$\begin{aligned} r\inf\left\{A^*(q, \lambda, q), \widetilde{A}(q)\right\} &= \widetilde{A}(q) \quad \text{and} \\ \widetilde{f}(q) \vee f^*(q, \lambda, q) &= \widetilde{f}(q). \end{aligned}$$

If $q \neq q_1$, then

$$\begin{aligned} r\inf\left\{A^*(q_1, \lambda, q), \widetilde{A}(q_1)\right\} &= [0, 0] \subseteq \widetilde{A}(q) \quad \text{and} \\ f^*(q_1, \lambda, q) \vee \widetilde{f}(q_1) &= 1 \geq \widetilde{f}(q). \end{aligned}$$

Thus the result is true for $n = 0$.

Suppose that result is valid for all $y \in X^*$ with $|y| = n - 1$, $n > 0$.

Let $x = ya$ where $a \in X$. Then

$$\begin{aligned} r\inf\left\{\widetilde{A}(q_1), A^*(q_1, x, q)\right\} &= r\inf\left\{\widetilde{A}(q_1), A^*(q_1, ya, q)\right\} \\ &= r\inf\left\{\widetilde{A}(q_1), \left[r\sup_{s\in Q}\left\{r\inf\left\{A^*(q_1, y, s), A(s, a, q)\right\}\right\}\right]\right\} \\ &= r\sup_{s\in Q}\left[r\inf\left\{\widetilde{A}(q_1), A^*(q_1, y, s), A(s, a, q)\right\}\right] \\ &\subseteq r\sup_{s\in Q}\left[r\inf\left\{\widetilde{A}(s), A(s, a, q)\right\}\right] \\ &\subseteq \widetilde{A}(q). \end{aligned}$$

Thus $\widetilde{A}(q_1) \cap A^*(q_1, x, q) \subseteq \widetilde{A}(q)$ and

$$\begin{aligned} \widetilde{f}(q_1) \vee f^*(q_1, x, q) &= \widetilde{f}(q_1) \vee f^*(q_1, ya, q) \\ &= \widetilde{f}(q_1) \vee \left\{\wedge_{s\in Q}\left\{f^*(q_1, y, s) \vee f^*(s, a, q)\right\}\right\} \\ &= \wedge_{s\in Q}\left\{\widetilde{f}(q_1) \vee f^*(q_1, y, s) \vee f^*(s, a, q)\right\} \\ &\geq \wedge_{s\in Q}\left\{\widetilde{f}(s) \vee f^*(s, a, q)\right\} \geq \widetilde{f}(q). \end{aligned}$$

So $\widetilde{f}(q_1) \vee f^*(q_1, x, q) \geq \widetilde{f}(q)$.

The converse is trivial. □

Definition 3.12. Let $M = (Q, X, \mathcal{A})$ be a $CFSM$. Let $T \subseteq Q$. Let $\mathcal{A}_T = (A_T, f_T)$ be a cubic set in $T \times X \times T$ and let $\Im = (T, X, \mathcal{A}_T)$ be a $CFSM$. Then $\Im$ is called a cubic submachine of M if

(*i*) $\mathcal{A}\mid_{T\times X\times T} = \mathcal{A}_T$,

(*ii*) $CS(T) \subseteq T$.

We assume that $\phi = (\varnothing, X, \mathcal{A})$ is a cubic submachine of M. Obviously, if $\Im'$ is a cubic submachine of $\Im$, and $\Im$ is cubic submachine of M, then $\Im'$ is a cubic submachine of M. A cubic submachine $\Im = (T, X, \mathcal{A}_T)$ of a $CFSM$ $M = (Q, X, \mathcal{A})$ is said to be proper if $T \neq \varnothing$ and $T \neq Q$.

Definition 3.13. A $CFSM$ $M = (Q, X, \mathcal{A})$ is said to be strongly cubic connected if $q_1 \in CS(q)$ for every $q_1, q \in Q$.

Theorem 3.14. *Let $M = (Q, X, \mathcal{A})$ be a $CFSM$ and let $\Im_i = (T_i, X, \mathcal{A}_{T_i})$ $i \in I$ be a family of cubic submachines of M. Then we have*

(*i*) *$\cap_{i\in I}\Im_i = (\cap_{i\in I}T_i, X, \cap_{i\in I}\mathcal{A}_{T_i})$ is a cubic submachine of M. Where $\cap_{i\in I}\mathcal{A}_{T_i} = (r\inf_{i\in I} A_{T_i}, \vee_{i\in I} f_{T_i})$.*

(*ii*) *$\cup_{i\in I}\Im_i = (\cup_{i\in I}T_i, X, \mathcal{A}')$ is a cubic submachine of M where $\mathcal{A}' = (A', f')$ is given by*

$$A' = A\mid_{\cup_{i\in I}T\times X\times\cup_{i\in I}T} \text{ and } f' = f\mid_{\cup_{i\in I}T\times X\times\cup_{i\in I}T}.$$

Proof. Let $(q, x, q_1) \in \cap_{i\in I}T_i \times X \times \cap_{i\in I}T_i$. Then

$$\begin{aligned}\left(r\inf_{i\in I} A_{T_i}\right)(q, x, q_1) &= r\inf_{i\in I} A_{T_i}(q, x, q_1) = r\inf_{i\in I} A(q, x, q_1) \\ &= A(q, x, q_1)\end{aligned}$$

and

$$\begin{aligned}(\vee_{i\in I} f_{T_i})(q, x, q_1) &= \vee_{i\in I} f_{T_i}(q, x, q_1) = \vee_{i\in I} f(q, x, q_1) \\ &= f(q, x, q_1).\end{aligned}$$

Thus $\mathcal{A}\mid_{\cap_{i\in I}T_i\times X\times\cap_{i\in I}T_i} = \cap_{i\in I}\mathcal{A}_{T_i}$.

On the one hand

$$CS(\cap_{i\in I}T_i) \subseteq \cap_{i\in I}CS(T_i) \subseteq \cap_{i\in I}T_i.$$

So $\cap_{i\in I}\Im_i$ is a cubic submachine of M.

(*ii*) Since

$$\begin{aligned}CS(\cup_{i\in I}T_i) &= \cup_{i\in I}CS(T_i) \\ &\subseteq \cup_{i\in I}T_i.\end{aligned}$$

Hence $\cup_{i\in I}\Im_i$ is a cubic submachine of M. □

Theorem 3.15. *A $CFSM$ $M = (Q, X, \mathcal{A})$ is strongly cubic connected if and only if M has no proper cubic submachine.*

Proof. Suppose that $M = (Q, X, \mathcal{A})$ is strongly cubic connected. Let $\Im = (T, X, \mathcal{A}_T)$ be a cubic submachine of M such that $T \neq \varnothing$. Then there exists $q \in T$, if $q_1 \in Q$, then $q_1 \in CS\left(q\right)$, since M is strongly cubic connected. It follows that $q_1 \in CS\left(q\right) \subseteq CS\left(T\right)$ so that $T = Q$. Hence $M = \Im$. *i.e* M has no proper cubic submachine.

Conversely M has no proper cubic submachines. Let $q_1, q \in Q$ and let $\Im = (CS\left(q\right), X, \mathcal{A}')$ where $\mathcal{A}' = \mathcal{A}\mid_{CS(q)\times X\times CS(q)}$. Then $\Im$ is a cubic submachine of M and $CS\left(q\right) \neq \varnothing$ and so $CS\left(q\right) = Q$.

Thus $q_1 \in CS\left(q\right)$, and therefore M is strongly Cubic connected. □

4. Conclusion

In this paper we have defined the notion of cubic finite state machine, cubic subsystem and cubic submachine. A necessary and sufficient condition for a cubic subsystem is also obtained and apply the concept of strong cubic connectedness of the CFSM. We hope the research can be continued along this direction. In future, cyclic cubic subsystem, simple strong cubic subsystem and the products of cubic finite state machine can be studied.

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