Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 10, No. 6, (December 2015), pp. 979–994
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Connected domination in fuzzy graphs using strong arcs

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Received 16 April 2015; Revised 25 May 2015; Accepted 29 June 2015

Abstract. In this paper, connected domination in fuzzy graphs using strong arcs is introduced. The strong connected domination number of different classes of fuzzy graphs is obtained. An upper bound for the strong connected domination number of fuzzy graphs is obtained. Strong connected domination in fuzzy trees is studied. It is established that the set of fuzzy cut nodes of a fuzzy tree is a strong connected dominating set. It is proved that in a fuzzy tree each node of a strong connected dominating set is incident on a fuzzy bridge. Also the characteristic properties of the existence of strong connected dominating set for a fuzzy graph and its complement are established.

2010 AMS Classification: 05C69

Keywords: fuzzy graph, strong arcs, connectedness, fuzzy trees, fuzzy cycles.

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1. Introduction

Fuzzy graphs were introduced by Rosenfeld, who has described the fuzzy analogue of several graph theoretic concepts like paths, cycles, trees and connectedness [15]. Bhutani and Rosenfeld have introduced the concept of strong arcs [2].
The work on fuzzy graphs was also done by Mordeson, Pradip, Talebi, and Yeh [10, 14, 24, 25]. It was during 1850s, a study of dominating sets in graphs started purely as a problem in the game of chess. Chess enthusiasts in Europe considered the problem of determining the minimum number of queens that can be placed on a chess board so that all the squares are either attacked by a queen or occupied by a queen. The concept of domination in graphs was introduced by Ore and Berge in 1962, the domination number and independent domination number are introduced by Cockayne and Hedetniemi [6]. Connected domination in graphs was discussed by Sampathkumar and Walikar [16]. Somasundaram and Somasundaram discussed

domination in fuzzy graphs. They defined domination using effective edges in fuzzy graph [18, 19]. Nagoorgani and Chandrasekharan defined domination in fuzzy graphs using strong arcs [12]. Manjusha and Sunitha discussed some concepts in domination and total domination in fuzzy graphs using strong arcs [7, 8]. In this paper we have discussed connected domination in fuzzy graphs using strong arcs.

This paper is organized as follows. Section 2 contains preliminaries and in Section 3, strong connected domination in fuzzy graphs is introduced. (Definitions 3.1–3.4). It is proved that the strong connected domination number of every connected fuzzy graph will not exceed that of its maximum spanning tree(Corollary 3.11). In Section 4, the strong connected domination number of classes of fuzzy graphs is obtained(Proposition 4.1, 4.2, Theorem 4.3, 4.6). In Section 5, the minimal strong connected domination in fuzzy graphs is defined and discussed some of their properties(Definition 5.1, Theorem 5.4, 5.5, 5.6, 5.7). In Section 6, strong connected domination in fuzzy trees is studied. It is established that in a non trivial fuzzy tree except K_2, each node of a strong connected dominating set is the end node of a fuzzy bridge (Theorem 6.5) and no node of a strong connected dominating set is the end node of a $\beta-$strong arc(Proposition 6.6). Also it is established that in a non trivial fuzzy tree the set of fuzzy cut nodes is a strong connected dominating set (Theorem 6.7). It is found that the strong connected domination number of a fuzzy tree is the weight of the set of all fuzzy cut nodes of G(Theorem 6.11). Also it is obtained an upper bound for the strong connected domination number of a connected fuzzy graph(Theorem 6.12). Finally in Section 7, the characteristic properties for the existence of a strong connected dominating set of a fuzzy graph and its complement are established(Theorem 7.4, 7.7, 7.8, 7.9, Corollary 7.10).

2. Preliminaries

We summarize briefly some basic definitions in fuzzy graphs which are presented in this paper.

Definition 2.1 ([15])**.** A fuzzy graph is denoted by $G : (V, \sigma, \mu)$ where V is a node set, σ is a fuzzy subset of V and μ is a fuzzy relation on σ. i.e., $\mu(x, y) \leq \sigma(x) \bigwedge \sigma(y)$ for all $x, y \in V$. We call σ the fuzzy node set of G and μ the fuzzy arc set of G, respectively. We consider fuzzy graph G with no loops and assume that V is finite and nonempty, μ is reflexive (i.e.,$\mu(x, x) = \sigma(x)$, for all x) and symmetric (i.e., $\mu(x, y) = \mu(y, x)$, for all (x, y)). In all the examples σ is chosen suitably.

Also, we denote the underlying crisp graph by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \{u \in V : \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in V \times V : \mu(u, v) > 0\}$. Throughout we assume that $\sigma^* = V$.

Definition 2.2 ([11])**.** The fuzzy graph $H : (\tau, \nu)$ is said to be a partial fuzzy subgraph of $G : (\sigma, \mu)$ if $\nu \subseteq \mu$ and $\tau \subseteq \sigma$. In particular we call $H : (\tau, \nu)$, a fuzzy subgraph of $G : (\sigma, \mu)$ if $\tau(u) = \sigma(u)$ for all $u \in \tau^*$ and $\nu(u, v) = \mu(u, v)$ for all $(u, v) \in \nu^*$. A fuzzy subgraph $H : (\tau, \nu)$ spans the fuzzy graph $G : (V, \sigma, \mu)$ if $\tau = \sigma$. The fuzzy graph $H : (P, \tau, \nu)$ is called an induced fuzzy subgraph of $G : (V, \sigma, \mu)$ induced by P if $P \subseteq V$ and $\tau(u) = \sigma(u)$ for all $u \in P$ and $\nu(u, v) = \mu(u, v)$ for all

$u, v \in P$. We shall use the notation $< P >$ to denote the fuzzy subgraph induced by P. $G : (V, \sigma, \mu)$ is called trivial if $|\sigma^*| = 1$.

Definition 2.3 ([15]). In a fuzzy graph $G : (V, \sigma, \mu)$, a path P of length n is a sequence of distinct nodes $u_0, u_1, ..., u_n$ such that $\mu(u_{i-1}, u_i) > 0, i = 1, 2, ..., n$ and the degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n$ and $n \geq 3$ then P is called a cycle and P is called a fuzzy cycle, if it contains more than one weakest arc. The strength of a cycle is the strength of the weakest arc in it. The strength of connectedness between two nodes x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by $CONN_G(x, y)$.

Definition 2.4 ([11]). A fuzzy graph $G : (\sigma, \mu)$ is connected if for every x, y in $\sigma^*, CONN_G(x, y) > 0$.

Definition 2.5 ([4, 18]). An arc (u, v) of a fuzzy graph is called an effective arc (M-strong arc) if $\mu(u, v) = \sigma(u) \bigwedge \sigma(v)$. Then u and v are called effective neighbors. The set of all effective neighbors of u is called effective neighborhood of u and is denoted by $EN(u)$.

Definition 2.6 ([1]). A fuzzy graph G is said to be complete if $\mu(u, v) = \sigma(u) \bigwedge \sigma(v)$, for all $u, v \in \sigma^*$ and is denoted by K_σ.

Definition 2.7 ([18]). The order p and size q of a fuzzy graph $G : (\sigma, \mu)$ are defined to be $p = \sum_{x \in V} \sigma(x)$ and $q = \sum_{(x,y) \in V \times V} \mu(x, y)$.

Definition 2.8 ([18]). Let $G : (V, \sigma, \mu)$ be a fuzzy graph and $S \subseteq V$. Then the scalar cardinality of S is defined to be $\sum_{v \in S} \sigma(v)$ and it is denoted by $|S|$. Let p denotes the scalar cardinality of V, also called the order of G.

Definition 2.9 ([2]). An arc of a fuzzy graph is called strong if its weight is at least as great as the strength of connectedness of its end nodes when it is deleted.

Depending on $CONN_G(x, y)$ of an arc (x, y) in a fuzzy graph G, Sunil Mathew and Sunitha [20] defined three different types of arcs. Note that $CONN_{G-(x,y)}(x, y)$ is the the strength of connectedness between x and y in the fuzzy graph obtained from G by deleting the arc (x, y).

Definition 2.10 ([20]). An arc (x, y) in G is $\alpha-$ strong if

$$\mu(x, y) > CONN_{G-(x,y)}(x, y).$$

An arc (x, y) in G is $\beta-$ strong if $\mu(x, y) = CONN_{G-(x,y)}(x, y)$.
An arc (x, y) in G is $\delta-$ arc if $\mu(x, y) < CONN_{G-(x,y)}(x, y)$.
Thus an arc (x, y) is a strong arc if it is either $\alpha-$ strong or $\beta-$ strong. A path P is called strong path if P contains only strong arcs.

Definition 2.11 ([18]). A fuzzy graph G is said to be bipartite if the node set V can be partitioned into two non empty sets V_1 and V_2 such that $\mu(v_1, v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$. Further if $\mu(u, v) = \sigma(u) \bigwedge \sigma(v)$ for all $u \in V_1$ and $v \in V_2$ then G is called a complete bipartite graph and is denoted by K_{σ_1,σ_2}, where σ_1 and σ_2 are respectively the restrictions of σ to V_1 and V_2.

Definition 2.12 ([15]). A connected fuzzy graph $G = (V, \sigma, \mu)$ is called a fuzzy tree if it has a fuzzy spanning subgraph $F : (\sigma, \nu)$, which is a tree [spanning tree], where for all arcs (x, y) not in F there exists a path from x to y in F whose strength is more than $\mu(x, y)$. Note that here F is a tree which contains all nodes of G and hence is a spanning tree of G.

Definition 2.13 ([11]). A maximum spanning tree of a connected fuzzy graph $G : (V, \sigma, \mu)$ is a fuzzy spanning subgraph $T : (\sigma, \nu)$, such that T is a tree, and for which $\sum_{u \neq v} \nu(u, v)$ is maximum. A node which is not an endnode of T is called an internal node of T.

Definition 2.14 ([12]). A node u is said to be isolated if $\mu(u, v) = 0$ for all $v \neq u$.

3. Strong connected domination in fuzzy graphs

The concept of domination in graphs was introduced by Ore and Berge in 1962 ,the domination number and independent domination number are introduced by Cockayne and Hedetniemi [6]. Connected domination in graphs was discussed by Sampathkumar and Walikar [16]. For the terminology of domination and connected domination in crisp graphs we refer to [5, 16].

For a node v of a graph $G : (V, E)$, recall that a neighbor of v is a node adjacent to v in G. Also the neighborhood $N(v)$ of v is the set of neighbors of v. The closed neighborhood $N[v]$ is defined as $N[v] = N(v) \cup \{v\}$. A node v in a graph G is said to dominate itself and each of its neighbors, that is v dominates the nodes in $N[v]$. A set S of nodes of G is a dominating set of G if every node of $V(G) - S$ is adjacent to some node in S. A minimum dominating set in a graph G is a dominating set of minimum cardinality. The cardinality of a minimum dominating set is called the domination number of G and is denoted by $\gamma(G)$. A dominating set S is a connected dominating set if it induces a connected subgraph in G. Since a dominating set must contain at least one node from every component of G, it follows that a connected dominating set exist for a graph G if and only if G is connected. The minimum of the cardinalities of the connected dominating sets of G is termed as the connected domination number of G, and is denoted as $\gamma_c(G)$.

Nagoorgani and Chandrasekaran [12] introduced the concept of domination using strong arcs in fuzzy graphs. According to Nagoorgani a node v in a fuzzy graph G is said to strongly dominate itself and each of its strong neighbors, i.e., v strongly dominates the nodes in $N_s[v]$. A set D of nodes of G is a strong dominating set of G if every node of $V(G) - D$ is a strong neighbor of some node in D. They defined a minimum strong dominating set in a fuzzy graph G as a strong dominating set with minimum number of nodes [12]. These concepts motivated researchers to reformulate some of the concepts in domination more effectively.

Also in [13] Nagoorgani defined a minimum strong dominating set as a strong dominating set of minimum scalar cardinality and the scalar cardinality of a minimum strong dominating set is called the strong domination number of G.

Manjusha and Sunitha [9] defined strong domination number using membership values (weights) of arcs in fuzzy graphs as follows.

Definition 3.1 ([9])**.** The weight of a strong dominating set D is defined as $W(D) = \sum_{u \in D} \mu(u, v)$, where $\mu(u, v)$ is the minimum of the membership values (weights) of strong arcs incident on u. The strong domination number of a fuzzy graph G is defined as the minimum weight of strong dominating sets of G and it is denoted by $\gamma_s(G)$ or simply γ_s. A minimum strong dominating set in a fuzzy graph G is a strong dominating set of minimum weight.

Let $\gamma_s(\overline{G})$ or $\overline{\gamma_s}$ denote the strong domination number of the complement of a fuzzy graph G.

Now we define connected domination in fuzzy graphs using strong arcs as follows.

Definition 3.2. A strong dominating set D of a fuzzy graph $G : (V, \sigma, \mu)$ is a strong connected dominating set of G if the induced fuzzy subgraph $< D >$ is connected.

Remark 3.3. Note that a fuzzy graph $G : (V, \sigma, \mu)$ contains a strong connected dominating set if and only if G is connected.

Definition 3.4. The weight of a strong connected dominating set D is defined as $W(D) = \sum_{u \in D} \mu(u, v)$, where $\mu(u, v)$ is the minimum of the membership values(weights) of strong arcs incident on u. The strong connected domination number of a fuzzy graph G is defined as the minimum weight of strong connected dominating sets of G and it is denoted by $\gamma_{sc}(G)$ or simply γ_{sc}. A minimum strong connected dominating set in a fuzzy graph G is a strong connected dominating set of minimum weight.

Let $\gamma_{sc}(\overline{G})$ or $\overline{\gamma_{sc}}$ denote the strong connected domination number of the complement of a fuzzy graph G.

Example 3.5. Consider the fuzzy graph in Figure 1. In this fuzzy graph (a, c) and (e, d) are $\delta-$ arcs and all others are strong arcs. Hence $D = \{b, c, d, f\}$ is a minimum strong connected dominating set and strong connected domination number is $\gamma_{sc}(G) = 0.2 + 0.2 + 0.2 + 0.5 = 1.1$.

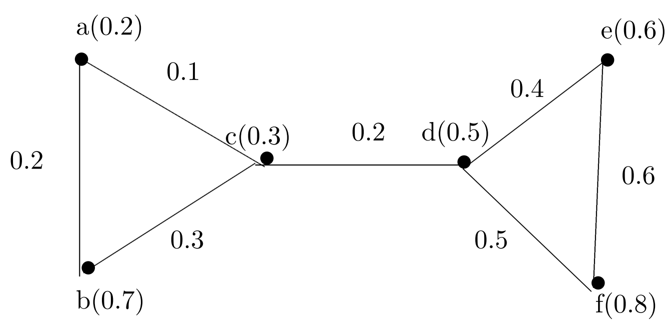

Figure 1: Illustration of strong connected domination

Proposition 3.6. *Any strong connected dominating set of a fuzzy graph* $G : (V, \sigma, \mu)$ *is a strong dominating set of* G.

Remark 3.7. The converse of proposition 3.6 need not be true as seen in the following example.

Example 3.8. Consider the fuzzy graph in Figure 2. In this fuzzy graph, $D = \{u, x\}$ is a strong dominating set, but not a strong connected dominating set, since the induced fuzzy subgraph $< D >$ is not connected.

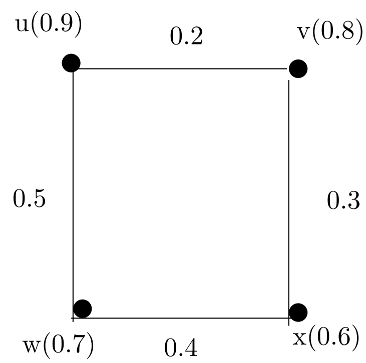

Figure 2: Example of a strong dominating set but not a strong connected dominating set

Since a strong connected dominating set is necessarily a strong dominating set, the following result is obvious.

Proposition 3.9. *For any connected fuzzy graph* $G : (V, \sigma, \mu)$ $\gamma_s(G) \leq \gamma_{sc}(G)$.

Theorem 3.10. *Let* $G : (V, \sigma, \mu)$ *be any connected fuzzy graph and* $H : (V, \nu, \tau)$ *be any maximum spanning tree of* G. *Then every strong dominating set of* H *is also a strong dominating set of* G *and consequently* $\gamma_s(G) \leq \gamma_s(H)$.

Proof. Let D be a strong dominating set of H. Since H is a maximum spanning tree of G we have $\sigma = \nu$. Hence the nodes in D strongly dominates all the nodes in $V \setminus D$. Hence D is a strong dominating set of G. Hence $\gamma_s(G) \leq \gamma_s(H)$. □

Corollary 3.11. *Let* $G : (V, \sigma, \mu)$ *be any connected fuzzy graph and* H *be any maximum spanning tree of* G. *Then every strong connected dominating set of* H *is also a strong connected dominating set of* G *and hence* $\gamma_{sc}(G) \leq \gamma_{sc}(H)$.

Remark 3.12. In general connected fuzzy graph if we take a spanning tree other than maximum spanning tree, then Theorem 3.10 and Corollary 3.11 need not be true as seen in the following example.

Example 3.13. Consider the fuzzy graph G and its spanning tree H as shown in Figure 3.

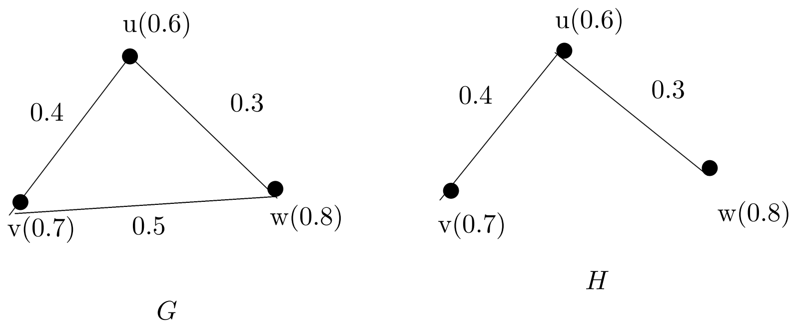

Figure 3: Example of a fuzzy graph G and its spanning tree H in which $\gamma_{sc}(H) < \gamma_{sc}(G)$

In G, $D = \{u\}$ is not a strong dominating set since (u, w) is a $\delta-$ arc and thus is not a strong connected dominating set of G. But it is a strong dominating and strong connected dominating set of H.
Consequently

$$\gamma_s(G) = 0.4, \gamma_{sc}(G) = 0.4$$
$$\gamma_s(H) = 0.3, \gamma_{sc}(H) = 0.3.$$

Hence

$$\gamma_s(H) < \gamma_s(G)$$
$$\gamma_{sc}(H) < \gamma_{sc}(G).$$

4. Strong connected domination number for classes of fuzzy graphs

In this section, we have determined the strong connected domination number of complete fuzzy graph, complete bipartite fuzzy graph, fuzzy cycles and join of a fuzzy graph with a complete fuzzy graph.

Proposition 4.1. *If $G : (V, \sigma, \mu)$ is a complete fuzzy graph, then*

$$\gamma_{sc}(G) = \wedge\{\mu(u, v) : u, v \in \sigma^*\}.$$

Proof. Since G is a complete fuzzy graph, all arcs are strong [21] and each node is adjacent to all other nodes. Hence $D = \{u\}$ is a strong connected dominating set for each $u \in \sigma^*$. Hence the result follows. □

Proposition 4.2.

$$\gamma_{sc}(K_{\sigma_1,\sigma_2}) = \begin{cases} \mu(u, v), & \text{if } |V_1| = 1 \text{ or } |V_2| = 1 \\ 2\mu(u, v), & \text{if } |V_1| \text{ and } |V_2| \geq 2 \end{cases}$$

where $\mu(u,v)$ is the weight of a weakest arc in K_{σ_1,σ_2}.

Proof. In K_{σ_1,σ_2}, all arcs are strong. Also each node in V_1 is adjacent with all nodes in V_2. Hence in K_{σ_1,σ_2}, the strong dominating sets are V_1, V_2 and any set containing at least 2 nodes, one in V_1 and other in V_2. Among this if V_1 or V_2 contains only one element say u, then $D = \{u\}$ is the minimum strong connected dominating set in G. Hence $\gamma_{sc}(K_{\sigma_1,\sigma_2}) = \mu(u,v)$ where $\mu(u,v)$ is the minimum of the weights of arcs incident on u. If both V_1 and V_2 contains more than one element then the set $\{u,v\}$ of nodes of any weakest arc (u,v) in K_{σ_1,σ_2} forms a strong connected dominating set.
Hence $\gamma_{sc}(K_{\sigma_1,\sigma_2}) = \mu(u,v) + \mu(u,v) = 2\mu(u,v)$. Hence the result. □

Theorem 4.3. *Let $G : (V,\sigma,\mu)$ be a fuzzy cycle where G^* is a cycle. Then, $\gamma_{sc}(G) = \bigwedge\{W(D) : D$ is a strong connected dominating set in G with $|D| \geq (n-2)\}$, where n is the number of nodes in G.*

Proof. In a fuzzy cycle every arc is strong. Also, the number of nodes in a strong connected dominating set of both G and G^* are same because each arc in both graphs are strong. In graph G^*, the strong connected domination number of G^* is obtained as $(n-2)$ [16]. Hence the minimum number of nodes in a strong connected dominating set of G is $(n-2)$. Hence the result follows. □

Definition 4.4 ([10, 11]). Union of two fuzzy graphs: Let $G_1 : (\sigma_1,\mu_1)$ and $G_2 : (\sigma_2,\mu_2)$ be two fuzzy graphs with $G_1^* : (V_1,E_1)$ and $G_2^* : (V_2,E_2)$ with $V_1 \cap V_2 = \phi$
and let $G^* = G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$ be the union of G_1^* and G_2^*.
Then the union of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G : (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ defined by

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u) & \text{if } u \in V_1 \setminus V_2 \\ \sigma_2(u) & \text{if } u \in V_2 \setminus V_1 \end{cases}$$

and

$$(\mu_1 \cup \mu_2)(u,v) = \begin{cases} \mu_1(u,v) & \text{if } (u,v) \in E_1 \setminus E_2 \\ \mu_2(u,v) & \text{if } (u,v) \in E_2 \setminus E_1 \end{cases}$$

Definition 4.5 ([10, 11]). Join of two fuzzy graphs: Consider the join $G^* = G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$ of graphs where E' is the set of all arcs joining the nodes of V_1 and V_2 where we assume that $V_1 \cap V_2 = \phi$. Then the join of two fuzzy graphs G_1 and G_2 is a fuzzy graph

$G = G_1 + G_2 : (\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ defined by

$$(\sigma_1 + \sigma_2)(u) = (\sigma_1 \cup \sigma_2)(u),\ u \in V_1 \cup V_2$$

and

$$(\mu_1 + \mu_2)(u, v) = \begin{cases} (\mu_1 \cup \mu_2)(u, v) & \text{if } (u, v) \in E_1 \cup E_2 \textit{ and} \\ \\ \sigma_1(u) \wedge \sigma_2(v) & \text{if } (u, v) \in E^{'} \end{cases}$$

Theorem 4.6. *For any fuzzy graph $G : (V, \sigma, \mu)$, $\gamma_{sc}(K_\sigma + G) = \mu(u, v)$ where $\mu(u, v)$ is the weight of a weakest arc incident on u for any node $u \in K_\sigma$.*

Proof. For any fuzzy graph G, any node in K_σ is adjacent to all other nodes in K_σ and G and note that all such arcs are strong arcs. Hence any singleton set $D = \{u\}$ for each node u in K_σ, is a strong connected dominating set of $K_\sigma + G$. Hence $\gamma_{sc}(K_\sigma + G) = \mu(u, v)$ where $\mu(u, v)$ is the weight of a weakest arc incident on u for any node $u \in K_\sigma$. □

5. Minimal strong connected domination in fuzzy graphs

In this section we have defined minimal strong connected dominating sets and discussed some properties.

Definition 5.1. A strong connected dominating set D of a connected fuzzy graph $G : (V, \sigma, \mu)$ is called a minimal strong connected dominating set if no proper subset of D is a strong connected dominating set of G.

Remark 5.2. Every minimum strong connected dominating set is minimal but not conversely.

Example 5.3. Consider the fuzzy graph in Figure 4.

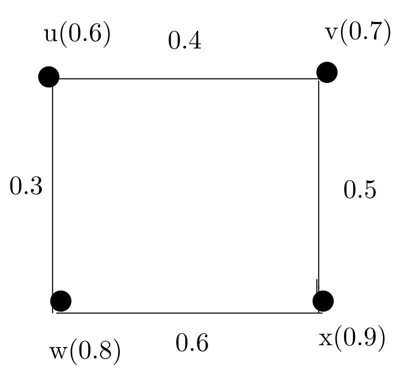

Figure 4: Illustration of minimal strong connected domination

In the fuzzy graph of Figure 4, $(u, v), (v, x), (x, w)$ are strong arcs and (u, w) is a $\delta-$ arc. $D = \{w, x\}$ is a minimal strong connected dominating set but not minimum strong connected dominating set since the set $\{u, v\}$ forms a minimum strong connected dominating set with $\gamma_{sc}(G) = 0.8$, but $W(D) = 1.1$.

Note that in a complete fuzzy graph the minimum and minimal strong connected dominated sets are same and any singleton set of nodes is the minimum strong connected dominating set. Hence the following theorems are obvious.

Theorem 5.4. *Every non trivial complete fuzzy graph G has a strong connected dominating set D whose complement $V \setminus D$ is also a strong connected dominating set.*

Theorem 5.5. *Let G be a complete fuzzy graph. If D is a minimal strong connected dominating set then $V \setminus D$ is a strong connected dominating set.*

Note that in a complete bipartite fuzzy graph the end nodes of any weakest arc forms a minimal strong connected dominating set. Hence the following theorems are obvious.

Theorem 5.6. *Every non trivial complete bipartite fuzzy graph G has a strong connected dominating set D of two elements whose complement $V \setminus D$ is also a strong connected dominating set.*

Theorem 5.7. *Let G be a complete bipartite fuzzy graph. If D is a minimal strong connected dominating set of two elements then $V \setminus D$ is a strong connected dominating set.*

Remark 5.8. Theorems 5.4 to 5.7 are not true in general connected fuzzy graphs as seen in the following example.

Example 5.9. Consider the fuzzy graph given in Figure 5.

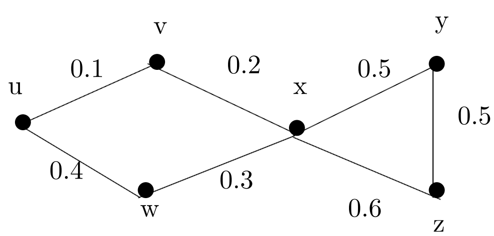

Figure 5: Example of a strong connected dominating set D

such that $V \setminus D$ is not a strong connected dominating set

In this fuzzy graph all node weights are taken as 1. $D = \{x, w\}$ is a minimal strong connected dominating set. But $V \setminus D = \{u, v, y, z\}$ is not a strong connected dominating set.

6. Strong connected domination in fuzzy trees

Note that in the definition of a fuzzy tree, F is the unique maximum spanning tree (MST) of G [23].

Definition 6.1 ([15]). An arc is called a fuzzy bridge of a fuzzy graph $G : (V, \sigma, \mu)$ if its removal reduces the strength of connectedness between some pair of nodes in G .

Definition 6.2 ([15]). A fuzzy cut node w is a node in G whose removal from G reduces the strength of connectedness between some pair of nodes other than w.

Definition 6.3 ([3]). A node z is called a fuzzy end node if it has exactly one strong neighbor in G. A non trivial fuzzy tree G contains at least two fuzzy end nodes and every node in G is either a fuzzy cut node or a fuzzy end node.

Definition 6.4 ([2, 23]). In a fuzzy tree G an arc is strong if and only if it is an arc of F where F is the associated unique maximum spanning tree of G.

Note that these strong arcs are α-strong and there are no β-strong arcs in a fuzzy tree [20]. Also note that in a fuzzy tree G an arc (x, y) is α-strong if and only if (x, y) is a fuzzy bridge of G [20].

Theorem 6.5. *In a non trivial fuzzy tree $G : (V, \sigma, \mu)$, each node of a strong connected dominating set is incident on an $\alpha-$strong arc (fuzzy bridge) of G.*

Proof. Let D be a strong connected dominating set of G. Let $u \in D$. Since D is a strong dominating set, there exists $v \in V \setminus D$ such that (u, v) is a strong arc. Then (u, v) is an arc of the unique MST F of G [2, 23]. Hence (u, v) is an $\alpha-$strong arc or a fuzzy bridge of G [15]. Since u is arbitrary, this is true for every node of the strong connected dominating set of G. This completes the proof. □

Proposition 6.6. *In a non trivial fuzzy tree $G : (V, \sigma, \mu)$, no node of a strong connected dominating set is an end node of a $\beta-$strong arc.*

Proof. Note that a fuzzy graph is a fuzzy tree if and only if it has no $\beta-$strong arcs [20]. Hence the proposition. □

Theorem 6.7. *In a non trivial fuzzy tree $G : (V, \sigma, \mu)$, except K_2, the set of all fuzzy cut nodes is a strong connected dominating set.*

Proof. Let D be the set of all fuzzy cut nodes of a non trivial fuzzy tree $G : (V, \sigma, \mu)$. Then D is a strong dominating set in G [7]. To prove that the induced fuzzy subgraph $< D >$ is connected. Note that the internal nodes of F are the fuzzy cutnodes of G [20]. Hence the fuzzy subgraph induced by the internal nodes of F is connected. Therefore $< D >$ is connected. Hence D is a strong connected dominating set of G. □

Proposition 6.8. *In a fuzzy tree $G : (V, \sigma, \mu)$, every node of G is strongly dominated by an end node of an $\alpha-$ strong arc.*

Proof. Note that in G all strong arcs are $\alpha-$ strong arcs. Hence the proposition. □

Remark 6.9. The set of all fuzzy end nodes need not be a strong connected dominating set in a non trivial fuzzy tree $G : (V, \sigma, \mu)$ except K_2.

Theorem 6.10. *In a fuzzy tree $G : (V, \sigma, \mu)$, each node of every strong connected dominating set is contained in the unique maximum spanning tree of G.*

Proof. Since G is a fuzzy tree, G has a unique maximum spanning tree F which contains all the nodes of G. In particular, F contains all nodes of every strong connected dominating set of G. This completes the proof. □

Theorem 6.11. *In a non trivial fuzzy tree $G : (V, \sigma, \mu)$ except K_2, $\gamma_{sc}(G) = W(S)$ where S is the set of all fuzzy cut nodes of G.*

Proof. Note that the set S of all fuzzy cut nodes of G is a strong connected dominating set of G (Theorem 6.7). Here we have to prove that S is a minimum strong connected dominating set. Suppose if possible S is not a minimum strong connected dominating set. Then there exists a strong connected dominating set $S^{'}$ such that $W(S^{'}) < W(S)$. Then $S^{'}$ has 4 choices.

1]. $S^{'}$ contains all fuzzy cut nodes and at least one fuzzy end node.

2]. At least one fuzzy cut node say w is not contained in $S^{'}$ and $S^{'}$ contains no fuzzy end node.

3]. $S^{'}$ is a combination of fuzzy cut nodes and fuzzy end nodes.

4]. $S^{'}$ contains only fuzzy end nodes.

In case 1 it is obvious that $W(S^{'}) > W(S)$.

In case 2 $< S^{'} >$ (the fuzzy sub graph induced by $S^{'}$) is not connected if w is an internal node of $< S >$ (the fuzzy sub graph induced by S) or $S^{'}$ is not a strong dominating set if w is an end node of the fuzzy subgraph $< S >$ for,
A fuzzy tree contains at least 2 fuzzy end nodes. If w is an end node of $< S >$ then one neighboring node of w is a fuzzy end node say u in G and w is the only strong neighbor of u in G. Therefore, if w is not contained in $< S^{'} >$ then u is not strongly dominated by any node in G. Hence $S^{'}$ is not a strong dominating set of G.

Case 3 has 3 possibilities.

a]. G has a unique maximum weighted arc adjacent to any fuzzy end node, then $W(S^{'}) > W(S)$ since weight of maximum arc is contributed to $W(S^{'})$ but not to $W(S)$

b]. The unique maximum weighted arc is adjacent to any fuzzy cut node then $W(S^{'}) \geq W(S)$

c]. G has more than one maximum weighted arc and one of these is adjacent to a fuzzy cut node and other is adjacent to a fuzzy end node then $W(S^{'}) > W(S)$.

In case 4 we can consider the cases a, b, c as in case 3, we get similar results.

Therefore in all the cases we get a contradiction. Hence the minimum strong connected dominating set of G is the set of all fuzzy cut nodes of G.

$$\text{Hence } \gamma_{sc}(G) = W(S).$$

□

Theorem 6.12. *Let $G : (V, \sigma, \mu)$ be a connected fuzzy graph and S be the set of all internal nodes of any maximum spanning tree of G. Then, $\gamma_{sc}(G) \leq W(S)$ and equality holds if G is a fuzzy tree.*

Proof. Every connected fuzzy graph has at least one maximum spanning tree T and $\gamma_{sc}(T) = W(S)$ [Theorem 6.11]. By Corollary 3.11, every strong connected dominating set of T is also a strong connected dominating set of G and hence $\gamma_{sc}(G) \leq \gamma_{sc}(T)$. Hence

$$\gamma_{sc}(G) \leq W(S)$$

and equality holds if G is a fuzzy tree by theorem 6.11. Hence the result. □

7. Strong connected domination in complement of fuzzy graphs

Sunitha and Vijayakumar [22] has defined the present notion of complement of a fuzzy graph. Sandeep and Sunitha have studied the connectivity concepts in a fuzzy graph and its complement [17].

Definition 7.1 ([22])**.** The complement of a fuzzy graph G, denoted by $\overline{G}$ or G^c is defined to be $\overline{G} = (V, \sigma, \overline{\mu})$ where $\overline{\mu}(x, y) = \sigma(x) \wedge \sigma(y) - \mu(x, y)$ for all $x, y \in V$.

Bhutani has defined the isomorphism between fuzzy graphs [1].

Definition 7.2 ([1])**.** Consider the fuzzy graphs $G_1 : (V_1, \sigma_1, \mu_1)$ and $G_2 : (V_2, \sigma_2, \mu_2)$ with $\sigma_1^* = V_1$ and $\sigma_2^* = V_2$. An isomorphism between two fuzzy graphs G_1 and G_2 is a bijective map $h : V_1 \to V_2$ that satisfies

$$\sigma_1(u) = \sigma_2(h(u)) \text{ for all } u \in V_1.$$
$$\mu_1(u, v) = \mu_2(h(u), h(v)) \text{ for all } u, v \in V_1 \text{ and we write } G_1 \approx G_2.$$

Definition 7.3 ([22])**.** A fuzzy graph G is self complementary if $G \approx \overline{G}$.

Theorem 7.4. *If G is a connected fuzzy graph with no $M-$strong arcs then G and G^c contains at least one strong connected dominating set.*

Proof. If G is a connected fuzzy graph with no $M-$strong arcs then G^c is also connected [17]. Hence both G and G^ccontains at least one strong connected dominating set. □

Remark 7.5. There are fuzzy graphs which contain $M-$strong arcs such that G and G^c contain strong connected dominating set [Example 7.6].

Example 7.6. Consider the fuzzy graph in Figure 6. Here (v, w) is the only $M - strong$ arc in G and G and G^c are connected. In G, $D = \{v\}$ is a strong connected dominating set and in G^c, $D = \{u\}$ is a strong connected dominating set.

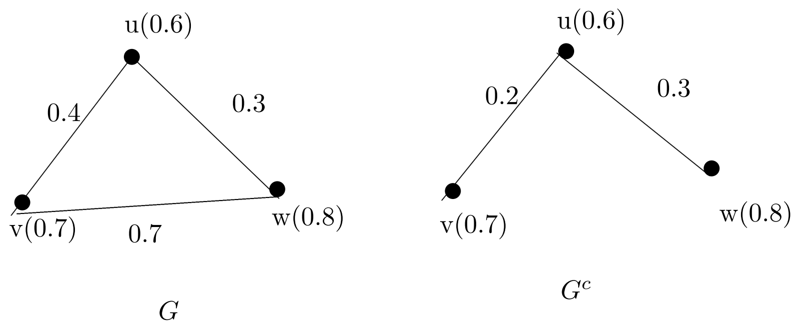

Figure 6: Illustraton of a strong connected dominating set in a fuzzy graph and its complement

Theorem 7.7. *Let $G : (V, \sigma, \mu)$ be a fuzzy graph. Then each of G and G^c contain at least one strong connected dominating set if and only if G contains at least one connected spanning fuzzy subgraph with no $M-$strong arcs.*

Proof. Note that for a fuzzy graph $G : (V, \sigma, \mu)$, G and G^c are connected if and only if G contains at least one connected spanning fuzzy subgraph with no $M-$strong arcs [17]. Thus it follows that for a fuzzy graph $G : (V, \sigma, \mu)$, G and G^c contains at least one strong connected dominating set if and only if G contains at least one connected spanning fuzzy subgraph with no $M-$strong arcs.

□

Theorem 7.8. *Let $G : (V, \sigma, \mu)$ be a fuzzy graph. Then each of G and G^c contain at least one strong connected dominating set if and only if G contains at least one fuzzy spanning tree having no $M-$strong arcs.*

Proof. Note that for a fuzzy graph $G : (V, \sigma, \mu)$, G and G^c are connected if and only if G contains at least one fuzzy spanning tree having no $M-$strong arcs [17]. Thus it follows that for a fuzzy graph $G : (V, \sigma, \mu)$, each of G and G^c contain at least one strong connected dominating set if and only if G contains at least one fuzzy spanning tree having no $M-$strong arcs. □

Theorem 7.9. *If $G : (V, \sigma, \mu)$ is a connected self complementary fuzzy graph ,then each of G and G^c contain at least one strong connected dominating set.*

Proof. Since G is self complementary G is isomorphic to G^c. Also G^c is connected since G is connected. Hence the result. □

Corollary 7.10. *If G is a connected fuzzy graph such that $\mu(u,v) = \frac{1}{2}(\sigma(u) \wedge \sigma(v))$ for all $u, v \in \sigma^*$ then each of G and G^c contain at least one strong connected dominating set.*

Proof. Since $\mu(u,v) = \frac{1}{2}(\sigma(u) \wedge \sigma(v))$ for all $u, v \in \sigma^*$, G is self complementary [22]. Hence the result follows. □

8. Conclusion

The concept of domination in graph is very rich both in theoretical developments and applications. More than thirty domination parameters have been investigated by different authors, and in this paper we have introduced the concept of strong connected domination and minimal strong connected domination in fuzzy graphs. The strong connected domination number of classes of fuzzy graphs is obtained. We have found some bounds for the strong connected domination number of fuzzy graphs and studied the same in fuzzy trees and fuzzy cycles. Also we have studied the concept of strong connected domination in complement of fuzzy graphs.

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