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On the class of weakly fuzzy countably compact spaces

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Abstract. In this paper, observing the drawback of the notion of fuzzy countably compact space introduced by Wong [16] that it is not a "good extension" of countably compact topological space, we have initiated a new notion of fuzzy countably compactness, namely weakly fuzzy countably compactness in fuzzy topological space which is a "good extension" of countably compactness of general topology. The concept of weakly fuzzy countably compact space generalized both of the notions of weakly fuzzy compact and fuzzy compact space, but is equivalent to weakly fuzzy compact space under some certain conditions. Several characterizations of this notion have been achieved. These besides, we have derived some conditions under which the product of weakly fuzzy countably compact spaces is weakly fuzzy countably compact.

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1. INTRODUCTION

Although the notion of compactness in general topology is an old concept, because of its wide applicability, even to day, a good number of mathematicians have focused their attention for the various generalizations of this notion. As the results countably compact, local compactness, paracompactness H-closedness [15], pseudocompactness etc. have been discovered.

Following the introduction of fuzzy sets by Zadeh [18], many research has been carried out in the areas of general theories as well as applications. The fuzzy topology was introduced by Chang [4]. Chang [4] defined fuzzy compact topological space in the usual manner. After Chang, several notions of fuzzy compactness were initiated, namely countably compactness [16], sequentially compactness [16] and semicompactness [16], weakly fuzzy compact [9] and fuzzy compact due to Lowen [9], strong fuzzy compactness [9] and ultra-fuzzy compactness [9], α-compact [6] and $\alpha^{\star}$-compact [6]. Lowen [9] showed that weakly fuzzy compact, strong fuzzy compactness, ultra-fuzzy compactness, fuzzy compactness due to Lowen, and α-compactness are good extensions of compactness, but fuzzy compactness due to Chang and $\alpha^{\star}$-compactness are not good extensions of compactness of general topology. Lowen [9] also established that the Tychonoff-like theorem on products are fulfilled by weakly fuzzy compact, strong fuzzy compactness and ultra-fuzzy compactness, fuzzy compactness due to Lowen and α-compactness, but not fuzzy compactness due to Chang. Some recent research works related to fuzzy compactness are found in the papers [2, 3, 7, 10, 11, 12, 13, 17].

In this paper, in section 3, we have first shown that fuzzy countably compactness introduced by Wong [16] is not a "good extension" of countably compactness of general topology and observing this drawback of the notion of fuzzy countably compactness of Wong [16], we have proposed a new definition of fuzzy countably compact space following the definition of weakly fuzzy compact space introduced by Lowen [8] which is pronounced as weakly fuzzy countably compact space. The classes of fuzzy countably compact spaces [16] and weakly fuzzy compact spaces [8] are properly contained in the class of weakly fuzzy countably compact spaces. In this section, we have established several characterizations of weakly fuzzy countably compact spaces. We have also shown that fuzzy continuous images of weakly fuzzy countably compact spaces are weakly fuzzy countably compact space. In section 4, we have derived some conditions under which the product of weakly fuzzy countably compact spaces is weakly fuzzy countably compact.

2. Preliminaries

Throughout this paper, spaces (X, σ) and (Y, δ) (or simply X and Y) represent non-empty fuzzy topological spaces due to Chang [4] and the symbols I, $\mathbb{N}$ and I^X have been used for the unit closed interval $[0, 1]$, the set of all natural numbers directed by usual order and the set of all functions with domain X and codomain I respectively. The support of a fuzzy set A is the set $\{x \in X : A(x) > 0\}$ and is denoted by $supp(A)$. A fuzzy set with only non-zero value $\lambda \in (0, 1]$ at only one element $x \in X$ is called a fuzzy point and is denoted by x_λ and the set of all fuzzy points of a fuzzy topological space is denoted by $Pt(X)$. For any two fuzzy sets A, B of X, $A \leq B$ if and only if $A(x) \leq B(x)$ for all $x \in X$. A fuzzy point x_λ is said to be in a fuzzy set A (denoted by $x_\lambda \in A$) if $x_\lambda \leq A$, that is, if $\lambda \leq A(x)$. The set of all fuzzy points of X contained in a fuzzy subset A of X is denoted by $Pt(A)$. The constant fuzzy set of X with value $\epsilon \in [0.1]$ is denoted by $\underline{\epsilon}$. A fuzzy set A is said to be quasi-coincident with B (written as $A\hat{q}B$) [14] if $A(x) + B(x) > 1$ for some $x \in X$. A fuzzy set A is said to be not quasi-coincident with B (written as $A\bar{q}B$) [14] if $A(x)+B(x) \leq 1$, for all $x \in X$. A fuzzy open set A of X is called fuzzy quasi-neighborhood of a fuzzy point x_λ if $x_\lambda \hat{q} A$. The set of all fuzzy quasi-neighborhood of a fuzzy point $x_\lambda \in Pt(X)$ is denoted by $\mathcal{Q}(X, x_\lambda)$. It is well-known that a function

$\psi : X \to Y$ is fuzzy continuous [4] if for every fuzzy point x_λ and every fuzzy quasi-neighborhood V of a fuzzy point $\psi(x_\lambda)$, there exists a fuzzy quasi-neighborhood U of a fuzzy point x_λ such that $\psi(U) \leq V$.

An ideal on a non-empty set $\mathbb{G}$ is defined as a non-empty family $\mathcal{I}$ of subsets of $\mathbb{G}$ satisfying (i) $\varnothing \in \mathcal{I}$, (ii) $A \in \mathcal{I}$ and $B \subset A \Rightarrow B \in \mathcal{I}$ and (iii) $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$. An ideal $\mathcal{I}$ on a set $\mathbb{G}$ is called non-trivial if $\mathbb{G} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ on $\mathbb{G}$ is called admissible [5] if $\mathbb{G} - \{n \in \mathbb{N} : n \geq \lambda\} \in \mathcal{I}$, for all $\lambda \in \mathbb{G}$. Throughout this paper, $\mathcal{I}$ stands for an admissible ideal on $\mathbb{G}$. For a net $\{F_n : n \in \mathbb{D}\}$, where $\mathbb{D}$ is a directed set of fuzzy sets in a fuzzy topological space X, the set $F\text{-}\limsup = \bigvee\{x_\lambda \in Pt(X)$: for each $n_0 \in \mathbb{N}$ and for each $U \in \mathcal{Q}(X, x_\lambda)$ there exists $n(\geq n_0) \in \mathbb{D}$ such that $U\hat{q}F_n\}$ is called fuzzy upper limit of the net. In other case, $F\text{-}\limsup = \underline{0}$. The notion of fuzzy upper limit was generalized by Afsan [1]. Let $\{A_n : n \in \mathbb{D}\}$, where $\mathbb{D}$ is a directed set be a net of fuzzy sets of a fuzzy topological space X. Then the fuzzy upper $\mathcal{I}$-limit [1] of $\{A_n : n \in \mathbb{D}\}$ is defined and denoted by $FIUL(A_n) = \vee\{x_\lambda \in Pt(X)$: for every fuzzy quasi-neighborhood $U \in \mathcal{Q}(X, x_\lambda)$, $\{n \in \mathbb{D} : A_n\hat{q}U\} \notin \mathcal{I}\}$.

A family $\Omega \subset I^X$ is said to be fuzzy locally finite if for every fuzzy point $x_\lambda \in Pt(X)$, there exists $U \in \mathcal{Q}(X, x_\lambda)$ such that U is quasi-coincident with finite number of members of Ω. We recall that a fuzzy point $x_\lambda \in Pt(X)$ is called fuzzy cluster point of a fuzzy sequence $\mathcal{S} = \{x^k_{\lambda_k} \in Pt(X) : k \in \mathbb{N}\}$ in X if for every $U \in \mathcal{Q}(X, x_\lambda)$ and every $k \in \mathbb{N}$, there exists an $m(\geq k) \in \mathbb{N}$ such that $U\hat{q}x^m_{\lambda_m}$. The set of all fuzzy cluster points of the sequence $\mathcal{S}$ is denoted by $\mathcal{A}(\mathcal{S})$. In fact, $x_\lambda \in \mathcal{A}(\mathcal{S})$ if there exists a subsequence of $\mathcal{T} = \{x_{\lambda^{k_i}_{k_i}} : i \in \mathbb{N}\}$ of $\mathcal{S}$ that converges to x_λ.

3. Weakly fuzzy countably compact space

We first recall the definition of fuzzy countably compact set due to Wong [16]. A fuzzy subset $\eta \in I^X$ is called countably compact if for each family $\{\mu_k \in \delta : k \in \mathbb{N}\}$ satisfying $\bigvee_{i=1}^{\infty} \mu_i \geq \eta$, there exist finite number of indices $k_1, k_2, ..., k_p \in \mathbb{N}$ such that $\bigvee_{i=1}^{p} \mu_{k_i} \geq \eta$. If $\eta = \underline{1}$ and η is a countably compact set, then X is called a fuzzy countably compact space.

The following example shows that fuzzy countably compactness of fuzzy topological space due to Wong [16] is not a "good extension" of countably compactness of general topological space.

Example 3.1. Let X be the closed unit interval $[0, 1]$ with usual topology τ. Then (X, τ) is compact and so countably compact, but not fuzzy countably compact because the countable open cover $\{\mu_n : n \in \mathbb{N}\}$, where $\mu_n(x) = 1 - \frac{1}{n}$ for $(X, \omega(\tau))$ has no finite subcover for $(X, \omega(\tau))$.

Now we propose a new definition of fuzzy countably compactness, namely weakly countably compactness following weakly compactness due to Lowen [8] that is a "good extension" of countably compactness of general topological space.

Definition 3.2. A fuzzy subset $\eta \in I^X$ is called weakly countably compact if for each family $\{\mu_k \in \delta : k \in \mathbb{N}\}$ satisfying $\bigvee_{i=1}^{\infty} \mu_i \geq \eta$ and for each $\epsilon > 0$, there exist finite number of indices $k_1, k_2, ..., k_p \in \mathbb{N}$ such that $(\bigvee_{i=1}^{p} \mu_{k_i})(x) \geq \eta(x) - \epsilon$ for all

$x \in X$. If $\eta = \underline{1}$ and η is a weakly countably compact set, then X is called a weakly fuzzy countably compact space.

Following Theorem shows that weakly countably compactness of fuzzy topological space is a "good extension" of countably compactness of general topological space. Since its proof is almost analogous to Theorem 4.1 in [8], we omit the proof.

Theorem 3.3. *The fuzzy topological space $(X, \omega(\tau))$ is weakly fuzzy countably compact if and only if (X, τ) is countably compact.*

Remark 3.4. (i) Every fuzzy countably compact space due to Wong [16] is weakly fuzzy countably compact.
(ii) Every weakly fuzzy compact space due to Lowen [8] is weakly fuzzy countably compact.

Following example shows that none of above statements are reversible.

Example 3.5. (i) Let $\mathcal{R}$ be the set of all real numbers. Clearly, for each $n \in \mathbb{N}$, $\delta_n = \{\underline{0}, A_n, \underline{1}\}$, where $A_n(x) = 1 - \frac{1}{n}$ is a fuzzy topology on $\mathcal{R}$ [16] and for each $n \in \mathbb{N}$, $(\mathcal{R}, \delta_n)$ is weakly fuzzy countably compact. Then clearly, the product fuzzy topology on $Y = \prod_{i=1}^{\infty} \mathcal{R}^i$ is $\delta = \{\underline{0}, \underline{1}\} \cup \{\pi_n^{-1}(A_n) : A_n \in \delta_n, n \in \mathbb{N}\}$. Then this space is weakly fuzzy countably compact. In fact, for the fuzzy open cover $\Omega = \{\pi_n^{-1}(A_n) : A_n \in \delta_n, n \in \mathbb{N}\}$ (or any of its subcover) for Y and for given $\epsilon > 0$, we can choose an integer $p > \epsilon^{-1}$ such that $\pi_p^{-1}(A_p) \in \Omega$ and $\pi_p^{-1}(A_p) \geq \overline{1 - \epsilon}$. But (Y, δ) is not fuzzy countably compact.

(ii) Let $(\beta\mathbb{N}, \tau)$ be the Stone-Cěck cementification of the the space of natural numbers with usual topology. Then its proper dense subspace (X, τ_X), where $X = \beta\mathbb{N} - \{x\}$ and $x \in \beta\mathbb{N} - \mathbb{N}$ is not compact, but countably compact. Then by Theorem 4.1 in [8] the fuzzy topological space $(X, \omega(\tau_X))$ is not weakly fuzzy compact space. Again, Theorem 3.3 ensures that $(X, \omega(\tau_X))$ is weakly fuzzy countably compact.

Definition 3.6. A fuzzy set $\eta \in I^X$ of X is said to have weakly fuzzy Lindelöf property if $\{\mu_\alpha \in \delta : \alpha \in \Delta\}$ satisfying $\bigvee_{\alpha \in \Delta} \mu_\alpha \geq \eta$ and for each $\epsilon > 0$, there exists countable subset $\Delta_0 \subset \Delta$ such that $(\bigvee_{\alpha \in \Delta_0} \mu_\alpha)(x) \geq \eta(x) - \epsilon$ for all $x \in X$. If $\eta = \underline{1}$ and η satisfies weakly fuzzy Lindelöf property, then X is called weakly fuzzy Lindelöf space.

Following two Theorems provide the conditions under which weakly fuzzy compactness and weakly fuzzy countably compactness are equivalent.

Theorem 3.7. *A fuzzy topological space is weakly fuzzy compact if and only if it is a weakly fuzzy countably compact space and weakly fuzzy Lindelöf space.*

Proof. The proof is straightforward. □

Theorem 3.8. *Let X be fuzzy second countable space. Then X is weakly fuzzy countably compact if and only if X is weakly fuzzy compact.*

Proof. It has already been seen that weakly fuzzy compact spaces are weakly fuzzy countably compact. So, we prove only the reverse implication. Let X be a weakly fuzzy countably compact space that is fuzzy second countable. Let $\mathcal{B}$ be a countable open cover of X. Consider any fuzzy open cover Σ of X. Then for each $V \in \Sigma$, there exists an $U \in \mathcal{B}$ such that $U \leq V$. Clearly $\{U \in \mathcal{B} : U \leq V, V \in \Sigma\}$ is a countable open cover of X. Now weakly fuzzy countably compactness of X ensures the existence of finite number of members $U_1, U_2, ..., U_k \in \mathcal{B}$ such that $\bigvee_{i=1}^{k} U_i \geq \underline{1-\epsilon}$. Consider $V_i \in \Sigma$ such that $U_i \leq V_i$ for each $i = 1, 2, ..., k$. Then $\bigvee_{i=1}^{k} V_i \geq \underline{1-\epsilon}$. □

Theorem 3.9. *For every fuzzy Hausdorff space X, the following conditions are equivalent: (i) The space X is weakly fuzzy countably compact.*
(ii) Every countable family $\{\mu_k : k \in \mathbb{N}\}$ of fuzzy closed subsets of X with the property that there exists $\epsilon > 0$ such that for any finite number of indices $k_1, k_2, ..., k_p \in \mathbb{N}$, $\bigwedge_{i=1}^{p} \mu_{k_i} \not\leq \underline{\epsilon}$ has non-empty intersection.
(iii) For every decreasing sequence $\{\mu_k : k \in \mathbb{N}\}$ of fuzzy closed subsets of X, with the property that there exists $\epsilon > 0$ such that for any $k \in \mathbb{N}$, $\mu_k \not\leq \underline{\epsilon}$, $\bigwedge\{\mu_k : k \in \mathbb{N}\} \neq \underline{0}$.

Proof. (i)⇔(ii): Let X be weakly fuzzy countably compact and $\{\mu_k : k \in \mathbb{N}\}$ be a countable family of fuzzy closed subsets of X with $\{\mu_k : k \in \mathbb{N}\} = \underline{0}$. Then $\{\mu'_k : k \in \mathbb{N}\}$ is a fuzzy open cover of X. Let $\epsilon > 0$ be an arbitrary. Then the weakly fuzzy countably compactness of X ensures the existence of finite number of indices $k_1, k_2, ..., k_p \in \mathbb{N}$ such that $\bigvee_{i=1}^{p} \mu'_{k_i} \geq \underline{1-\epsilon}$ and so $\bigwedge_{i=1}^{p} \mu_{k_i} \leq \underline{\epsilon}$.

Conversely, let X be not weakly fuzzy countably compact. Then there exists an $\epsilon > 0$ and a fuzzy open cover $\{\zeta_k : k \in \mathbb{N}\}$ of X such that for any finite number of indices $k_1, k_2, ..., k_p \in \mathbb{N}$, $\bigwedge_{i=1}^{p} \zeta_{k_i} \not\geq \underline{1-\epsilon}$. So $\{\zeta'_k : k \in \mathbb{N}\}$ is countable family of fuzzy closed subsets of X with the property that for any finite number of indices $k_1, k_2, ..., k_p \in \mathbb{N}$, $\bigwedge_{i=1}^{p} \zeta'_{k_i} \not\leq \underline{\epsilon}$. But $\bigwedge\{\zeta'_k : k \in \mathbb{N}\} = \underline{0}$.

(i)⇒(iii): Let X be weakly fuzzy countably compact. Let $\{\mu_k : k \in \mathbb{N}\}$ be a decreasing sequence of non-empty fuzzy closed subsets of X and $\epsilon > 0$ be arbitrary. Let $\bigwedge\{\mu_k : k \in \mathbb{N}\} = \underline{0}$. Then $\{\mu'_k : k \in \mathbb{N}\}$ is a fuzzy open cover of X. The weakly fuzzy countably compactness of X ensures the existence of a finite number of natural numbers $k_1 < k_2 <, ..., < k_p$ such that $\bigvee\{\mu'_{k_i} : i = 1, 2, ..., p\} \geq \underline{1-\epsilon}$, i.e. $\bigwedge\{\mu_{k_i} : i = 1, 2, ..., p\} \leq \underline{\epsilon}$. Since $\{\mu_k : k \in \mathbb{N}\}$ is a decreasing sequence $F_{k_p} \leq \underline{\epsilon}$.

(iii)⇒(ii): Let $\{\mu_k : k \in \mathbb{N}\}$ be a sequence of non-empty fuzzy closed subsets of X with the property that there exists $\epsilon > 0$ such that for any finite number of indices $k_1, k_2, ..., k_p \in \mathbb{N}$, $\bigwedge_{i=1}^{p} \mu_{k_i} \not\leq \underline{\epsilon}$. Consider the sequence $\{\zeta_k : k \in \mathbb{N}\}, \zeta_1 = \mu_1, \zeta_k = \mu_1 \wedge \mu_2 \wedge ... \wedge \mu_k$ for all $k = 2, 3, ...$. Then $\{\zeta_k : k \in \mathbb{N}\}$ is a decreasing sequence of non-empty fuzzy closed subsets of X, with the property that for any $k \in \mathbb{N}$, $\zeta_k \not\leq \underline{\epsilon}$ and so $\bigwedge\{\zeta_k : k \in \mathbb{N}\} \neq \underline{0}$. Hence $\bigwedge\{\mu_k : k \in \mathbb{N}\} \neq \underline{0}$. □

Theorem 3.10. *For every fuzzy topological space X, the following conditions are equivalent:*
(i) The space X is weakly fuzzy countably compact.
(ii) Every locally finite countable family $\{\mu_k : k \in \mathbb{N}\}$ of non-empty fuzzy subsets of X, with the property that there exists $\epsilon > 0$ such that for any $k \in \mathbb{N}$, $\mu_k \not\leq \underline{\epsilon}$, is finite.
(iii) Every locally finite sequence $\mathcal{S} = \{x^k_{\lambda_k} \in Pt(X) : k \in \mathbb{N}\}$ in X, with the property

that there exists $\epsilon > 0$ such that for any $k \in \mathbb{N}$, $x^k_{\lambda_k} \not\leq \underline{\epsilon}$, is finite.
(iv) For every fuzzy sequence $\mathcal{S} = \{x^k_{\lambda_k} \in Pt(X) : k \in \mathbb{N}\}$ in X, with the property that there exists $\epsilon > 0$ such that for any $k \in \mathbb{N}$, $x^k_{\lambda_k} \not\leq \underline{\epsilon}$, $\mathcal{A}(\mathcal{S}) \neq \underline{0}$.

Proof. (i)⇒(ii): Let X be a weakly fuzzy countably compact space. Let $\{\mu_k : k \in \mathbb{N}\}$ be a sequence of non-empty fuzzy subsets of X, with the property that there exists $\epsilon > 0$ such that for any $k \in \mathbb{N}$, $\mu_k \not\leq \underline{\epsilon}$ whose range is not finite. Consider the sequence $\{\zeta_k : k \in \mathbb{N}\}, \zeta_k = \bigvee_{i=k}^{\infty} cl(\mu_i)$. Then it is a decreasing sequence of non-empty fuzzy closed subsets of X, with the property that for any $k \in \mathbb{N}$, $\zeta_k \not\leq \underline{\epsilon}$ and so by Theorem 3.4, $\bigwedge\{\zeta_k : k \in \mathbb{N}\} \neq \underline{0}$. Let $x_\lambda \in \zeta_k$ for all $k \in \mathbb{N}$. Now $x_\lambda \in \zeta_k$ implies that $x_\lambda \in cl(\mu_{i_k})$ for some $i_k \geq k$. Thus $x_\lambda \in \bigwedge\{cl(\mu_{i_k}) : k \in \mathbb{N}\}$. So, for every $U \in \mathcal{Q}(X, x_\lambda)$, $U\hat{q}\mu_{i_k}$ for all $k \in \mathbb{N}$. So $\{\mu_k : k \in \mathbb{N}\}$ is not locally finite.

(ii)⇒(iii): The proof is straightforward.

(iii)⇒(iv): If the range of $\mathcal{S}$ is finite, then nothing to prove. So, let the range of $\mathcal{S}$ is not finite and $\mathcal{A}(\mathcal{S}) = \underline{0}$. Then for each $x_\lambda \in Pt(X)$, there exists an $U \in \mathcal{Q}(X, x_\lambda)$ and an $m \in \mathbb{N}$ such that $U\bar{q}x^k_{\lambda_k}$ for all $k \geq m$. So $\mathcal{S}$ is locally finite and so $\mathcal{S}$ is finite, which is a contradiction.

(iv)⇒(i): Let X be not weakly fuzzy countably compact. Then it has a countable fuzzy open cover $\Sigma = \{U_k : k \in \mathbb{N}\}$ and an $\epsilon > 0$ such that for any finite number of indices $k_1, k_2, ..., k_p \in \mathbb{N}$ such that $(\bigvee_{i=1}^{p} U_{k_i})(x) < 1 - \epsilon$ for some $x \in X$. Consider the sequence $\{x^k_{\lambda_k} : k \in \mathbb{N}\}, x^k_{\lambda_k} \in (\bigvee_{i=1}^{k} U_k)'$. Then for any $k \in \mathbb{N}$, $x^k_{\lambda_k} \not\leq \underline{\epsilon}$. Also, for any fuzzy point x_λ, there exists an $U_i \in \Sigma$ with the property that $\lambda_k \leq 1 - \max\{U_1(x^k), U_2(x^k), ..., U_i(x^k), ..., U_k(x^k)\} \leq 1 - U_i(x)$ for all $k(\geq i) \in \mathbb{N}$. Thus $\mathcal{A}(\mathcal{S}) = \underline{0}$. □

Corollary 3.11. *Let X be a fuzzy topological space and for every fuzzy sequence $\mathcal{S} = \{x^k_{\lambda_k} \in Pt(X) : k \in \mathbb{N}\}$ in X, $\mathcal{A}(\mathcal{S}) \neq \underline{0}$. Then X is weakly fuzzy countably compact.*

Corollary 3.12. *If every fuzzy sequence $\mathcal{S} = \{x^k_{\lambda_k} \in Pt(X) : k \in \mathbb{N}\}$ in a fuzzy topological space X has a convergent fuzzy subsequence, then X is weakly fuzzy countably compact.*

Corollary 3.13. *Let X be a first-countable fuzzy topological space. Then X is weakly fuzzy countably compact if and only if every fuzzy sequence $\mathcal{S} = \{x^k_{\lambda_k} \in Pt(X) : k \in \mathbb{N}\}$ in X, with the property that there exists $\epsilon > 0$ such that for any $k \in \mathbb{N}$, $x^k_{\lambda_k} \not\leq \underline{\epsilon}$ has a convergent fuzzy subsequence.*

Theorem 3.14. *Let for every sequence $\{F_n : n \in \mathbb{N}\}$ of fuzzy closed sets of a fuzzy topological space X,, for every ideal $\mathcal{I}$ on $\mathbb{N}$ with $FIUL(F_n) = \bar{0}$ and for every $\epsilon > 0$, $\{n \in \mathbb{N} : F_n \not\leq \bar{\epsilon}\} \in \mathcal{I}$. Then X is weakly fuzzy countably compact.*

Proof. Let the condition of the theorem holds. Let $\Sigma = \{U_n : n \in \mathbb{N}\}$ be a countable fuzzy open cover of X and $\epsilon > 0$. Consider the sequence $\{F_n : n \in \mathbb{N}\}$, $F_n = \bigwedge\{U'_i : i = 1, 2, ..., n\}$. Let $x_\lambda \in Pt(X)$. Since $\bigwedge\{F_n : n \in \mathbb{N}\} = \underline{0}$, there exists an $n_0 \in \mathbb{N}$ such that $x_\lambda \notin F_{n_0}$. Then $F_{n_0}(x) < \lambda$ and so $U = F'_{n_0} \in \mathcal{Q}(X, x_\lambda)$. Since

$\{F_n : n \in \mathbb{N}\}$ is monotonically decreasing, we have $F_n \bar{q} U$ for every natural number $n \geq n_0$. Since $\mathcal{I}$ is admissible, $N - \{n \in \mathbb{N} : n \geq n_0\} \in \mathcal{I}$ and so the inclusion $\{n \in \mathbb{N} : F_n \hat{q} U\} \subset N - \{n \in \mathbb{N} : n \geq n_0\}$ ensures that $\{n \in \mathbb{N} : F_n \hat{q} U\} \in \mathcal{I}$. Then $FIUL(F_n) = \bar{0}$. Then the hypothesis of the theorem, $\{n \in \mathbb{N} : F_n \not\leq \bar{\epsilon}\} \in \mathcal{I}$. Since $\mathcal{I}$ is non-trivial, there exists $n_0 \in \mathbb{N}$ such that $\bigwedge_{i=1}^{n_0} U_i' = F_{n_0} \leq \underline{\epsilon}$, i.e. $\bigvee_{i=1}^{n_0} U_i \geq \underline{1-\epsilon}$. Hence X is weakly fuzzy countably compact. □

Using the similar technique, following results can be established.

Corollary 3.15. *Let for every net $\{F_n : n \in \mathbb{D}\}$, where $\mathbb{D}$ is a directed set of fuzzy closed sets of a fuzzy topological space X, for every ideal $\mathcal{I}$ on $\mathbb{D}$ with $FIUL(F_n) = \bar{0}$ and for every $\epsilon > 0$, $\{n \in \mathbb{N} : F_n \not\leq \bar{\epsilon}\} \in \mathcal{I}$. Then X is weakly fuzzy compact.*

Since weakly fuzzy compact space X satisfies the conditions of corollary 3.15 in [1], we have the characterization of weak fuzzy compact spaces.

Corollary 3.16. *A fuzzy topological space X is weakly fuzzy compact if and only if for every net $\{F_n : n \in \mathbb{D}\}$, where $\mathbb{D}$ is a directed set of fuzzy closed sets of a fuzzy topological space X, for every ideal $\mathcal{I}$ on $\mathbb{D}$ with $FIUL(F_n) = \bar{0}$ and for every $\epsilon > 0$, $\{n \in \mathbb{N} : F_n \not\leq \bar{\epsilon}\} \in \mathcal{I}$. Then X is weakly fuzzy compact.*

Since by Theorem 3.8, weakly fuzzy countably compact fuzzy and weakly fuzzy compact are equivalent in presence of fuzzy topological property fuzzy second countability, we get the following result.

Corollary 3.17. *A fuzzy topological space X is weakly fuzzy compact if and only if for every net $\{F_n : n \in \mathbb{D}\}$, where $\mathbb{D}$ is a directed set of fuzzy closed sets of a fuzzy topological space X, for every ideal $\mathcal{I}$ on $\mathbb{D}$ with $FIUL(F_n) = \bar{0}$ and for every $\epsilon > 0$, $\{n \in \mathbb{N} : F_n \not\leq \bar{\epsilon}\} \in \mathcal{I}$. Then X is weakly fuzzy countably compact.*

Theorem 3.18. *Let for every sequence $\{F_n : n \in \mathbb{N}\}$ of fuzzy closed sets of a fuzzy topological space X with $F\text{-}\lim\sup(F_n) = \underline{0}$ and for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $F_n \leq \underline{\epsilon}$ for every $n \geq n_0$. Then X is weakly fuzzy countably compact.*

Proof. Let $\Sigma = \{U_n : n \in \mathbb{N}\}$ be a countable fuzzy open cover of X and $\epsilon > 0$. Consider the sequence $\{F_n : n \in \mathbb{N}\}$ as defined in Theorem 3.14. Then for each $x_\lambda \in Pt(X)$, there exists an $n_0 \in \mathbb{N}$ such that $F_n \bar{q} U$ for every natural number $n \geq n_0$. So $F\text{-}\lim\sup = \underline{0}$. Then the hypothesis of the theorem, there exists $n_0 \in \mathbb{N}$ such that $F_n \leq \underline{\epsilon}$ for every $n \geq n_0$. In particular, $\bigwedge_{i=1}^{n_0} U_i' = F_{n_0} \leq \underline{\epsilon}$, i.e. $\bigvee_{i=1}^{n_0} U_i \geq \underline{1-\epsilon}$. Hence X is weakly fuzzy countably compact. □

Since by Theorem 3.8, weakly fuzzy countably compact fuzzy second countable space is weakly fuzzy compact, we get the following result.

Corollary 3.19. *Let X be fuzzy second countable space. Then X is weakly fuzzy countably compact if and only if for every sequence $\{F_n : n \in \mathbb{N}\}$ of fuzzy closed sets of a fuzzy topological space X with $F\text{-}\lim\sup(F_n) = \underline{0}$ and for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $F_n \leq \underline{\epsilon}$ for every $n \geq n_0$.*

Theorem 3.20. *Fuzzy continuous images of weakly fuzzy countably compact sets are weakly fuzzy countably compact.*

Proof. Let $\psi : X \to Y$ be a fuzzy continuous function, η is a weakly fuzzy compact set of X and $\Sigma = \{\mu_i : i \in \mathbb{N}\}$ be a family of fuzzy open sets of Y such that $\bigvee_{i=1}^{\infty} \mu_i \geq \psi(\eta)$. Then $\bigvee_{i=1}^{\infty} \psi^{-1}(\mu_i) = \psi^{-1}(\bigvee_{i=1}^{\infty} \mu_i) \geq \psi^{-1}(\psi(\eta)) \geq \eta$. Now weakly fuzzy compactness of η ensures the existence of finite number of indices $k_1, k_2, ..., k_p \in \mathbb{N}$ such that $\bigvee_{i=1}^{p} \psi^{-1}(\mu_i) \geq \eta - \epsilon$. Then $\bigvee_{i=1}^{p} \mu_i \geq \psi(\psi^{-1}(\bigvee_{i=1}^{p} \mu_i)) = \psi(\bigvee_{i=1}^{p} \psi^{-1}(\mu_i)) \geq \psi(\eta) - \epsilon$. So $\psi(\eta)$ is weakly fuzzy countably compact. □

4. Product of Weakly countably compact spaces

In this section, we have derived some conditions under which the product of weakly fuzzy countably compact spaces is weakly fuzzy countably compact.

Theorem 4.1. *Let X be a weakly fuzzy compact space and Y be a weakly fuzzy countably compact space. Then $X \times Y$ is weakly fuzzy countably compact.*

Proof. Let $\Sigma = \{U_k : k \in \mathbb{N}\}$ be a countable fuzzy open cover of $X \times Y$. Then for any fuzzy point $(x, y)_\lambda \in Pt(X \times Y)$ there exist $U_{x_\lambda} \in \mathcal{Q}(X \times Y, (x, y)_\lambda) \cap \Sigma$, $U_{y_\lambda}^{x_\lambda} \in \mathcal{Q}(X, x_\lambda)$ and $V_{y_\lambda}^{x_\lambda} \in \mathcal{Q}(Y, y_\lambda)$ such that $U_{y_\lambda}^{x_\lambda} \times V_{y_\lambda}^{x_\lambda} \leq U_{x_\lambda}$. Let $\epsilon > 0$ be an arbitrary. For each fixed $y_\lambda \in Pt(Y)$, $\{U_{y_\lambda}^{x_{\frac{1}{k}}} : x_{\frac{1}{k}} \in Pt(X), k \in \mathbb{N}\}$ and so $\{U_{y_\lambda}^{x_\lambda} : x_\lambda \in Pt(X)\}$ is a fuzzy open cover of X. The weakly fuzzy compactness of X ensures the existence of finite number of fuzzy points $x_{\lambda_1}^1, x_{\lambda_2}^2, ..., x_{\lambda_p}^p \in Pt(X)$ such that $\bigvee_{i=1}^{p} U_{y_\lambda}^{x_{\lambda_i}^i} \geq \underline{1_X - \epsilon}$. Now for each $i = 1, 2, ..., p$, $V_{y_\lambda}^{x_{\lambda_i}^i} \in \mathcal{Q}(Y, y_\lambda)$ and so $V_{y_\lambda} = \bigwedge_{i=1}^{p} V_{y_\lambda}^{x_{\lambda_i}^i} \in \mathcal{Q}(Y, y_\lambda)$. Again $\{V_{y_{\frac{1}{k}}} : y_{\frac{1}{k}} \in Pt(Y), k \in \mathbb{N}\}$ and so $\{V_{y_\lambda} : y_\lambda \in Pt(Y)\}$ is a fuzzy open cover of weakly fuzzy countably compact space Y and so there exist finite number of fuzzy points $y_{\lambda_1'}^1, y_{\lambda_2'}^2, ..., y_{\lambda_t'}^t \in \mathcal{Q}(Y, y_\lambda)$ such that $\bigvee_{j=1}^{t} V_{y_{\lambda_i'}^i} \geq \underline{1_Y - \epsilon}$. Clearly, $\bigvee_{i=1}^{p} \bigvee_{j=1}^{t} (U_{y_\lambda}^{x_{\lambda_i}^i} \times V_{y_{\lambda_j'}^j}) \geq \underline{1_{X \times Y} - \epsilon}$. Thus the countable cover $\{U_{y_\lambda}^{x_\lambda} \times V_{y_\lambda}^{x_\lambda}\}$ has a finite subfamily $\{U_{y_\lambda}^{x_{\lambda_i}^i} \times V_{y_{\lambda_j'}^j} : i = 1, 2, ...p; j = 1, 2, ..., t\}$ such that $\bigvee_{i=1}^{p} \bigvee_{j=1}^{t} (U_{y_\lambda}^{x_{\lambda_i}^i} \times V_{y_{\lambda_j'}^j}) \geq \underline{1_{X \times Y} - \epsilon}$. Let $U_i^j \in \Sigma$ satisfies $U_{y_\lambda}^{x_{\lambda_i}^i} \times V_{y_{\lambda_j'}^j} \leq U_i^j$ for all $(i, j) \in \{1, 2, ..., p\} \times \{j = 1, 2, ..., t\}$. Then $\bigvee_{i=1}^{p} \bigvee_{j=1}^{t} U_i^j \geq \underline{1_{X \times Y} - \epsilon}$. So $X \times Y$ is weakly fuzzy countably compact. □

Theorem 4.2. *Let X be a fuzzy topological space with the property that for any sequence $\mathcal{S} = \{x_{\lambda_k}^k \in Pt(X) : k \in \mathbb{N}\}$ in X, $T = \bigvee\{x_{\lambda_k}^k \in Pt(X) : k \in \mathbb{N}\}$ is contained in a weakly fuzzy compact set of X and Y be a weakly fuzzy countably compact space. Then $X \times Y$ is weakly fuzzy countably compact.*

Proof. Let $\{(x^k, y^k)_{\lambda_k} \in Pt(X) : k \in \mathbb{N}\}$ be a sequence in $X \times Y$, with the property that there exists an $\epsilon > 0$ such that $(x^k, y^k)_{\lambda_k} \not\leq \underline{\epsilon}$ for all $k \in \mathbb{N}$. By the property enjoyed by X, we can find a weakly fuzzy compact set K of X such that $T = \bigvee\{x_{\lambda_k}^k \in Pt(X) : k \in \mathbb{N}\} \leq K$. Now by Theorem 4.1, $K \times Y$ is weakly fuzzy countably compact. Now $\{(x^k, y^k)_{\lambda_k} \in Pt(X) : k \in \mathbb{N}\}$ is a sequence in $K \times Y$ with $(x^k, y^k)_{\lambda_k} \not\leq \underline{\epsilon}$ for all $k \in \mathbb{N}$ and thus it has at least one fuzzy cluster point. So, $X \times Y$ is weakly fuzzy countably compact. □

Theorem 4.3. *Let X be a fuzzy topological space (in fact weakly fuzzy countably compact) with the property that every sequence $\{x^k_{\lambda_k} \in Pt(X) : k \in \mathbb{N}\}$ in X has a convergent subsequence and Y be a weakly fuzzy countably compact space. Then $X \times Y$ is weakly fuzzy countably compact.*

Proof. Let $\mathcal{S} = \{(x^k, y^k)_{\lambda_k} \in Pt(X) : k \in \mathbb{N}\}$ be a sequence in $X \times Y$, with the property that there exists an $\epsilon > 0$ such that $(x^k, y^k)_{\lambda_k} \not\leq \underline{\epsilon}$ for all $k \in \mathbb{N}$. By the property enjoyed by X, $\{y^k_{\lambda_k} \in Pt(X) : k \in \mathbb{N}\}$ has a convergent subsequence $\{x^{k_i}_{\lambda_{k_i}} : i \in \mathbb{N}\}$ converging to a fuzzy point $x_{\lambda_1} \in Pt(X)$. If the range of $\{y^{k_i}_{\lambda_{k_i}} : i \in \mathbb{N}\}$ is finite, there exists a constant subsequence $\{y^{k_{i_n}}_{\lambda_{k_{i_n}}} : n \in \mathbb{N}\}, y^{k_{i_n}}_{\lambda_{k_{i_n}}} = y_{\lambda_2}$ of this sequence. If $\{y^{k_i}_{\lambda_{k_i}} : i \in \mathbb{N}\}$ is infinite, the condition $y^{k_i}_{\lambda_{k_i}} \not\leq \underline{\epsilon}$ for each $i \in \mathbb{N}$ and Theorem 3.10, imply that there exists a fuzzy cluster point y_{λ_2} of $\{y^{k_i}_{\lambda_{k_i}} : i \in \mathbb{N}\}$. Now we shall show that $(x, y)_\lambda$, where $\lambda = \min\{\lambda_1, \lambda_2\}$ is fuzzy cluster point of the sequence $\mathcal{S}$. Let $U \in \mathcal{Q}(X \times Y, (x, y)_\lambda)$ and $k \in \mathbb{N}$. Then there exist $U_1 \in \mathcal{Q}(X, x_\lambda)$ and $U_2 \in \mathcal{Q}(Y, y_\lambda)$ such that $U_1 \times U_2 \leq U$ and positive integer k_s such that $k_s \geq k$. Since $\{x^{k_i}_{\lambda_{k_i}} : i \in \mathbb{N}\}$ converges to x_{λ_1}, there exists $k_t (\geq k_s) \in \mathbb{N}$ such that $U_1 \hat{q} x^{k_i}_{\lambda_{k_i}}$ for all $k_i \geq k_t$. Since y_{λ_2} is a fuzzy cluster point of $\{y^{k_i}_{\lambda_{k_i}} : i \in \mathbb{N}\}$, there exists $k_m \in \mathbb{N}$ such that $k_m \geq k_t$ and $U_2 \hat{q} y^{k_m}_{\lambda_{k_m}}$. Clearly, $(U_1 \times U_2) \hat{q} (x^{k_m}, y^{k_m})_{\lambda_{k_m}}$ and so $U \hat{q} (x^{k_m}, y^{k_m})_{\lambda_{k_m}}$. So, $\mathcal{S} \neq \underline{0}$ and thus by Theorem 3.10, $X \times Y$ is weakly fuzzy countably compact. □

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