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Fuzzy setting of residuated multilattices

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Abstract. In this paper, we introduce the notion of fuzzy filter of residuated multilattices and investigated related properties. We establish the relation between fuzzy filter, fuzzy homomorphism and fuzzy congruence in the framework of multilattices. Finally, we prove that the quotient of a residuated multilattice by a fuzzy filter is a residuated multilattice.

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1. Introduction and Preliminaries

Several fuzzifications of formal concept analysis have been proposed to deal with uncertain information with application in different fields such as network calculus, algebraic structures used in soft constraint satisfaction problems, or when considering the fuzzy extensions of crisp formalisms, for instance, description logic or formal concept analysis. In this paper we will focus on fuzzy filter of residuated multilattices. Firstly we recall the existing definitions of m-filter and filter and introduce some new concepts such as fuzzy m-filter, fuzzy filter of residuated multilattice. We give a new characterization of fuzzy m-filter and fuzzy filter in term of α-cut set.

Residuation has a prominent role in the algebraic study of logical systems. We recall preliminaries definitions and results.

Definition 1.1 ([2])**.** $\mathcal{A} = (A, \leq \odot, \rightarrow, T,)$ is said to be partially ordered commutative residuated integral monoid, briefly a pocrim, if the following properties hold:

(1) $(A, \odot, T)$ is a commutative monoid with neutral element T.
(2) $(A, \leq)$ is a poset with a top element T.
(3) The operations $\odot$ and $\rightarrow$ satisfy the adjointness condition, $a \odot c \leq b$ if and only if $c \leq a \rightarrow b$, for all $a, b, c \in A$.

We now recall the following useful conditions that hold in the pocrim $\mathcal{A}$. For all $a, b, c \in A$, we have:

P1 $a \odot b \leq a, \quad a \odot b \leq b$;
P2 $a \odot (a \to b) \leq a \leq b \to (a \odot b)$ and $a \odot (a \to b) \leq b \leq a \to (a \odot b)$;
P3 If $a \leq b$, then $a \odot c \leq b \odot c$, $c \to a \leq c \to b$, and $b \to c \leq a \to c$;
P4 $a \to (b \to c) = b \to (a \to c)$;
P5 $(a \to b) \odot (b \to c) \leq a \to c$;
P6 $a \to b \leq (a \odot c) \to (b \odot c)$;
P7 $a \to b \leq (c \to a) \to (c \to b)$ and $a \to b \leq (b \to c) \to (a \to c)$.

Notation 1.2. *Given $(M, \leq)$ a poset and $a,\ b \in M$. $a \sqcup b$ denotes the set of multi-supremum of $\{a, b\}$ (a multi-supremum of $\{a, b\}$ is a minimal element of the set of upper bounds of $\{a, b\}$). The set of multi-infimum will be denoted by $a \sqcap b$ (a multi-infimum of $\{a, b\}$ is a maximal element of the set of lower bounds of $\{a, b\}$). Therefore, $\sqcup$ and $\sqcap$ are hyperoperations from $A \times A$ to $\mathcal{P}^*(A)$ (the power set of A minus the empty set).*

Definition 1.3 ([6])**.** A poset $(M, \leq)$, is a join-multisemilattice if, for all $a, b, c \in M$, $a \leq c$ and $b \leq c$ implies that there exists $x \in a \sqcup b$ such that $x \leq c$.

Dual property defines the concept of meet-multisemilattice.

A **multilattice** is a poset $(M, \leq)$ which is a meet and join-multisemilattice.

A multilattice is said to be full, if $a \sqcup b \neq \varnothing$ and $a \sqcap b \neq \varnothing$ for all $a, b \in M$.

Several authors have given definition of multilattice distributivity (m-distributivity), here we consider the definition in [5].

Definition 1.4 ([5])**.** A multilattice $(M, \sqcap, \sqcup)$ is said to be m-distributive if the following conditions hold, for all $a,\ b \in M$ with $a \leq b$ and $x \in M$, $a \sqcup x \subseteq (a \sqcup x) \sqcap (b \sqcup x)$ and $b \sqcap x \subseteq (a \sqcap x) \sqcup (b \sqcap x)$.

Definition 1.5 ([2])**.** A residuated multilattice is a pocrim whose underlying poset is a multilattice. If in addition, there exists a bottom element, the residuated multilattice is said to be bounded.

Lemma 1.6 ([2])**.** *Every residuated multilattice is full.*

Notation 1.7. *Given a residuated multilattice $(M, \leq, \odot, \to, \sqcap, \sqcup, T)$, $\odot$, $\to$, $\sqcap$ and $\sqcup$ can be extended to $\mathcal{P}^*(M)$ as follow: for all $A,\ B \in \mathcal{P}^*(M)$, $A \odot B = \{a \odot b; a \in A\, and\, b \in B\}$, $A \to B = \{a \to b; a \in A\, and\, b \in B\}$, $A \sqcup B = \bigcup\limits_{a \in A, b \in B} (a \sqcup b)$, $A \sqcap B = \bigcap\limits_{a \in A, b \in B} (a \sqcap b)$. Particularly, for all $z \in M$, $A \to z$, $A \odot z$, $A \sqcup z$ and $A \sqcap z$ will stand respectively for $A \to \{z\}$, $A \odot \{z\}$, $A \sqcup \{z\}$ and $A \sqcap \{z\}$.*

For all $a,\ b,\ c \in M$, $a \sqcup b = c$ and $a \sqcap b = c$ stand for $a \sqcup b = \{c\}$ and $a \sqcap b = \{c\}$.

Proposition 1.8 ([6])**.** *Let $(M, \leq, \sqcap, \sqcup)$ be a multilattice.*

$\sqcap$ and $\sqcup$ are idempotent, i.e., for all $a \in M$, $a \sqcap a = a$ and $a \sqcup a = a$.

$\sqcap$ and $\sqcup$ are m-associative, i.e, for all $a, b, c \in M$,

$a * b = b \Rightarrow (a * b) * c \subseteq a * (b * c)$, where $* \in \{\sqcap, \sqcup\}$.

In the following, we recall the Comparability properties which have an important role in multilattice theory.

Proposition 1.9 ([6]). *In any multilattice $(M, \leq, \sqcap, \sqcup)$, the following properties (called comparability laws) are satisfied*

- (C_1) $c \in a \sqcup b$ *implies that* $a \leq c$ *et* $b \leq c$.
- (C_2) $c \in a \sqcap b$ *implies that* $a \geq c$ *et* $b \geq c$.
- (C_3) $c, d \in a * b$ *and* $c \leq d$ *implies that* $c = d$, *where* $* \in \{\sqcap, \sqcup\}$.

Proposition 1.10 ([2]). *In a residuated multilattice M, the following inclusions hold for all $a, b, c \in M$:*

(i) $(a \to c) \sqcap (b \to c) \subseteq (a \sqcup b) \to c$.
(ii) $(c \to a) \sqcap (c \to b) \subseteq c \to (a \sqcap b)$.

Proposition 1.11 ([2]). *Let $(M, \leq, \odot, \to, \sqcap, \sqcup, T)$ be a residuated multilattice. For all $a, b, c \in M$, such that $a \leq b$, if $z \in b \sqcup c$ (resp. $w \in a \sqcup c$), then there exists $w \in a \sqcup c$ (resp. $z \in b \sqcup c$) such that $w \leq z$.*

Proposition 1.12 ([2]). *Let $(M, \leq, \odot, \to, \sqcap, \sqcup, T)$ be an m-distributive residuated multilattice. For all, $a, b, c \in M$, if $a \leq b$ and $w \in a \sqcup c$ (resp. $z \in b \sqcap c$), then there exists $z \in b \sqcup c$ (resp. $w \in a \sqcap c$) such that $w \leq z$.*

Due to the combination of two structures (pocrim and multilattice) on any residuated multilattice, we have the notion of filter of pocrim (p-filter) and the notion of filter of multilattice (m-filter).

Definition 1.13 ([2]). Given $\mathcal{A} = (A, \leq, \odot, \to, T)$ a pocrim, a non empty subset $F \subseteq A$ is said to be a p-filter if the following conditions hold:

(i) if $a, b \in F$, then $a \odot b \in F$;
(ii) if $a \leq b$ and $a \in F$, then $b \in F$.

F is a deductive system if it satisfies:

(1) $T \in F$;
(2) $a \to b \in F$ and $a \in F$, imply $b \in F$, for all $a, b \in A$.

Remark 1.14. Filters and deductive systems of a pocrim are equivalent.

Definition 1.15 ([2]). Let $(M, \sqcup, \sqcap)$ be a multilattice. A non-empty set $F \subseteq M$ is said to be a m-filter if the following conditions hold: for all $a, b \in M$,

(i) $a, b \in F$ implies $\varnothing \neq a \sqcap b \subseteq F$;
(ii) $a \in F$ implies $a \sqcup x \subseteq F$, for all $x \in M$;
(iii) If $(a \sqcup b) \cap F \neq \varnothing$, then $a \sqcup b \subseteq F$.

Let us now recall the definition of filter in a residuated multilattice.

Definition 1.16 ([2]). Let M be a residuated multilattice. A non-empty set $F \subseteq M$ is said to be a filter if it is a deductive system and the following condition holds: $a \to b \in F$ implies $(a \sqcup b) \to b \subseteq F$ and $a \to (a \sqcap b) \subseteq F$.

The following result is the link between, the notion of filter and m-filter.

Theorem 1.17 ([2]). *Let M be a residuated multilattice and F a deductive system of M, then F is a filter if and only if:*

(i) *F is a m-filter;*
and for all a, $b \in M$,

(ii) *For all $x, y \in a \sqcup b$, if $x \to y \in F$, then $y \to x \in F$;*
(iii) *For all $x, y \in a \sqcap b$, if $x \to y \in F$, then $y \to x \in F$.*

Definition 1.18. Let $(M, \leq, \odot, \to, T)$ be a residuated multilattice. A deductive system F is said to be consistent if for all $a, b, c \in M$ the following conditions hold:

(i) If $a \to c$, $b \to c \in F$, then $(a \sqcup b) \to c \subseteq F$;
(ii) If $c \to a$, $c \to b \in F$, then $c \to (a \sqcap b) \subseteq F$.

The notion of congruences and that of homomorphism between hyperstructures are studied in literature and applies in various field such as: logic-based approaches to uncertainty, computer science, fuzzy reasoning. We now recall the definition introduced by I.P. Cabrera et Al in the frame work of multilattices.

Let $\sim$ be a binary relation on M, then $\sim$ induces a binary relation $\widehat{\approx}$ on $\mathcal{P}^*(M)$ as follow: for all $A, B \in \mathcal{P}^*(M)$, $A \widehat{\approx} B$ if and only if, for all $a \in A$, there exists $b \in B$ such that $a \sim b$ and for all $b \in B$, there exists $a \in A$ such that $a \sim b$.

Definition 1.19 ([4])**.** Let $(M, \sqcup, \sqcap)$ be a multilattice. A congruence on M is any equivalence relation $\sim$ on M such that, if $a \sim b$, then $(a \sqcap c) \widehat{\approx} (b \sqcap c)$ and $(a \sqcup c) \widehat{\approx} (b \sqcup c)$, for all a, b, $c \in M$.

Theorem 1.20 ([4])**.** *Let $(M, \sqcup, \sqcap)$ be a multilattice and $\sim$ be a binary relation on M. Then $\sim$ is a congruence relation if and only if the following hold:*

(i) *$\sim$ is reflexive.*
(ii) *$a \sim b$ if and only if there exist $x \in a \sqcap b$ and $y \in a \sqcup b$ with $x \sim y$.*
(iii) *If $a \leq b \leq c$ with $a \sim b$ and $b \sim c$, then $a \sim c$.*
(iv) *If $a \leq b$ with $a \sim b$, then $(a \sqcap x) \widehat{\approx} (b \sqcap x)$ and $(a \sqcup x) \widehat{\approx} (b \sqcup x)$, for all $x \in M$.*

Definition 1.21 ([2])**.** Let M be a residuated multilattice. An equivalence relation $\sim$ on M is said to be a congruence on M, if for all $a, b, c \in M$, $a \sim b$ implies $(a \sqcup c) \widehat{\approx} (b \sqcup c)$, $(a \sqcap c) \widehat{\approx} (b \sqcap c)$, $(a \odot c) \sim (b \odot c)$, $(a \to c) \sim (b \to c)$ and $(c \to a) \sim (c \to b)$.

Definition 1.22 ([2])**.** Let $h : (M, \leq, \odot, \to, \sqcap, \sqcup, T) \to (M', \leq, \odot, \to, \sqcap, \sqcup, T')$ be a map between residuated multilattices, h is said to be a residuated multilattice homomorphism if h is a multilattice homomorphism i.e.,

$h(a \sqcup b) = (h(a) \sqcup h(b)) \cap h(M)$, $h(a \sqcap b) = (h(a) \sqcap h(b)) \cap h(M)$, $h(a \to b) = h(a) \to h(b)$ and $h(a \odot b) = h(a) \odot h(b)$, for all $a, b \in M$. One can observe that $h(T) = T'$.

Definition 1.23 ([7])**.** Let $h : M \to M'$ be a residuated multilattice homomorphism. Then h induced a congruence relation, namely Kernel relation $\sim_h$, defined as $a \sim_h b$ if and only if $h(a) = h(b)$.

$Ker(h) = \{x \in M, h(x) = T'\}$ is a filter called the kernel filter of M, where T' is the top element of M'.

2. Fuzzy filters of residuated multilattice

From the definition of deductive system, p-filter, m-filter, filter, given in [2] and recalled in the previous section, we introduce the notion of fuzzy p-filter, fuzzy m-filter, fuzzy filter as follow.

From now on, $[0, 1]$ would stand for the unit interval of reals.

Definition 2.1 ([10]). A **fuzzy subset** of a non-empty set M is a function $\mu : M \to [0,1]$.

Let μ be a fuzzy subset of M. For $\alpha \in [0,1]$, the set $\mu_\alpha = \{x \in M \mid \mu(x) \geq \alpha\}$ is called a $\alpha - levelsubset$ or $\alpha - cutset$ of μ.

Definition 2.2. Let $(M, \leq, \odot, \to, T)$ be a pocrim. A fuzzy subset μ of M is called a fuzzy p-filter of M if it satisfies,

(i) $\forall a, b \in M, \quad \mu(a \odot b) \geq \min\{\mu(a), \mu(b)\}$;
(ii) $\forall a, b \in M, \quad a \leq b \Rightarrow \mu(a) \leq \mu(b)$ (i.e., μ is order preserving).

A fuzzy subset μ of M is said to be a fuzzy deductive system if the following hold:

(1) $\forall a \in M, \quad \mu(T) \geq \mu(a)$;
(2) $\forall a, b \in M, \quad \mu(b) \geq \min\{\mu(a), \mu(a \to b)\}$.

Proposition 2.3. *Let $(M, \leq, \odot, \to, T)$ be a pocrim. A fuzzy subset μ of M is a fuzzy p-filter of M if and only if it is a fuzzy deductive system.*

Proof. Assume that μ satisfies conditions (1) and (2). Let $a, b \in M$.

If $a \leq b$, then $a \to b = T$, and so $\mu(b) \geq \min\{\mu(a), \mu(T)\} = \mu(a)$.

Since $b \leq a \to (a \odot b)$ (from P2), we have $\mu(b) \leq \mu(a \to (a \odot b))$ and then $\mu(a \odot b) \geq \min\{\mu(a), \mu(b)\}$.

Conversely suppose that μ is a fuzzy p-filter. Let $a, b \in M$.

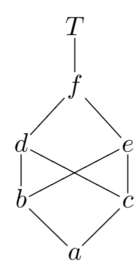
We have $a \leq T$, then $\mu(a) \leq \mu(T)$.

Since $a \odot (a \to b) \leq b$ (from P2), we have $\mu(b) \geq \mu(a \odot (a \to b))$. Thus $\mu(b) \geq \min\{\mu(a), \quad \mu(a \to b)\}$. This complete the proof. $\square$

Definition 2.4. Let $(M, \sqcup, \sqcap)$ be a multilattice. A fuzzy subset μ of M is said to be a fuzzy m-filter if the following conditions hold: For all a, $b \in M$,

(i) $\forall\ x \in a \sqcap b$, $\mu(x) \geq \min\{\mu(a), \mu(b)\}$;
(ii) $\forall\ x \in a \sqcup b$, $\mu(x) \geq \max\{\mu(a); \mu(b)\}$;
(iii) $\forall\ x, y \in a \sqcup b$, $\mu(x) = \mu(y)$.

Example 2.5. Let $(M, \sqcup, \sqcap)$ be a multilattice described in the following figure:



Let η be the a fuzzy subset of M given by: $\eta(x) = \begin{cases} 1, & if\ f \leq x \\ \frac{2}{5}, & otherwise \end{cases}$

η is a fuzzy m-filter.

Remark 2.6. If μ is a fuzzy m-filter of M, then μ is order preserving (i.e., $\forall\ x, y \in M$, $x \leq y \Rightarrow \mu(x) \leq \mu(y)$) and therefore, for all a, $b \in M$, $\mu(x) = \min\{\mu(a), \mu(b)\}$, $\forall\ x \in a \sqcap b$.

The following proposition is the characterization of fuzzy m-filter of M in term of α-cut set.

Proposition 2.7. *Let $(M, \sqcup, \sqcap)$ be a full multilattice. A fuzzy subset μ of M is a fuzzy m-filter of M iff, for all $\alpha \in [0,1]$ the level set μ_α is either empty or a m-filter.*

Proof. ($\Rightarrow$) Let μ be a fuzzy m-filter of M and $\alpha \in [0,1]$. If $\mu_\alpha = \varnothing$ there is nothing to prove.

Let us assume that $\mu_\alpha \neq \varnothing$. We will prove that:

(i) $a, b \in \mu_\alpha$ implies $a \sqcap b \subseteq \mu_\alpha$.
(ii) $a \in \mu_\alpha$ implies $a \sqcup b \subseteq \mu_\alpha$, for all $b \in M$.
(iii) For all $a, b \in M$, if $(a \sqcup b) \cap \mu_\alpha \neq \varnothing$, then $a \sqcup b \subseteq \mu_\alpha$.

For (i), let $a, b \in \mu_\alpha$. By the property (i) of the fuzzy m-filter, for all $x \in a \sqcap b$, $\mu(x) \geq \min\{\mu(a), \mu(b)\} \geq \alpha$, i.e., $x \in \mu_\alpha$. Hence $a \sqcap b \subseteq \mu_\alpha$.

For (ii), let $a \in \mu_\alpha$. By the properties of the fuzzy m-filter, for all $b \in M$, $\forall\ x \in a \sqcup b$, $\mu(x) \geq \max\{\mu(a); \mu(b)\} \geq \mu(a) \geq \alpha$. Hence $a \sqcup b \subseteq \mu_\alpha$, for all $b \in M$.

For (iii), let $a, b \in M$. Assume that $(a \sqcup b) \cap \mu_\alpha \neq \varnothing$. Then there exists $x_0 \in M$, such that $x_0 \in a \sqcup b$ and $\mu(x_0) \geq \alpha$. For all $x \in (a \sqcup b)$, $\mu(x) = \mu(x_0) \geq \alpha$, because μ is a fuzzy m-filter. Therefore, $a \sqcup b \subseteq \mu_\alpha$. Thus, μ_α is a m-filter of M.

($\Leftarrow$) Conversely, suppose that for any $\alpha \in [0,1]$, such that $\mu_\alpha \neq \varnothing$, μ_α is a m-filter.

Let $a, b \in M$.

For $\alpha = \min\{\mu(a), \mu(b)\}$, we have $a, b \in \mu_\alpha$. Because μ_α is a m-filter, $a \sqcap b \subseteq \mu_\alpha$. Therefore, for all $x \in a \sqcap b$, $\mu(x) \geq \alpha = \min\{\mu(a), \mu(b)\}$.

For $\alpha = \max\{\mu(a), \mu(b)\}$, we have $a \in \mu_\alpha$ or $b \in \mu_\alpha$. Then, $a \sqcup b \subseteq \mu_\alpha$, because μ_α is a m-filter of M. Hence, for all $x \in a \sqcup b$, $\mu(x) \geq \alpha = \max\{\mu(a), \mu(b)\}$.

Let $x, y \in a \sqcup b$. By hypothesis $\mu_{\mu(x)}$ and $\mu_{\mu(y)}$ are m-filters of M. Since $x \in (a \sqcup b) \cap \mu_{\mu(x)}$ and $y \in (a \sqcup b) \cap \mu_{\mu(y)}$, we have $y \in a \sqcup b \subseteq \mu_{\mu(x)}$ and $x \in a \sqcup b \subseteq \mu_{\mu(y)}$. Then, $\mu(y) \geq \mu(x)$ and $\mu(x) \geq \mu(y)$, i.e., $\mu(x) = \mu(y)$.

Thus, μ is a m-fuzzy filter of M. □

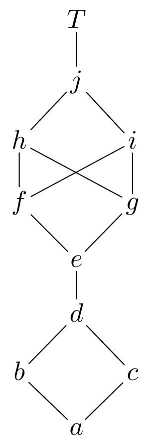
Let us now introduce the notion that we are interested in throughout this work. From now on, M will be a residuated multilattice.

Definition 2.8. A fuzzy subset μ of M is said to be a fuzzy filter if it is a fuzzy deductive system and for all $a, b \in M$,

$$\min\left\{\inf_{t \in (a \sqcup b) \to b} \mu(t), \inf_{t \in a \to (a \sqcap b)} \mu(t)\right\} = \mu(a \to b)$$

.

Example 2.9. Let $(M, \leq, \odot, \to, \sqcup, \sqcap, T)$ be the residuated multilattice described in the following figure and the operation $\odot$ and $\to$ defined as follows:



$\odot$	a	b	c	d	e	f	g	h	i	j	T
a	a	a	a	a	a	a	a	a	a	a	a
b	a	a	a	a	a	a	a	a	a	a	b
c	a	a	a	a	a	a	a	a	a	a	c
d	a	a	a	a	a	a	a	a	a	a	d
e	a	a	a	a	e	e	e	e	e	e	e
f	a	a	a	a	e	e	e	e	e	e	f
g	a	a	a	a	e	e	g	g	g	g	g
h	a	a	a	a	e	e	g	g	g	g	h
i	a	a	a	a	e	e	g	g	g	g	i
j	a	a	a	a	e	e	g	g	g	g	j
T	a	b	c	d	e	f	g	h	i	j	T

$\rightarrow$	a	b	c	d	e	f	g	h	i	j	T
a	T	T	T	T	T	T	T	T	T	T	T
b	j	T	j	T	T	T	T	T	T	T	T
c	j	j	T	j	T	T	T	T	T	T	T
d	j	j	j	T	T	T	T	T	T	T	T
e	d	d	d	d	T	T	T	T	T	T	T
f	d	d	d	d	j	T	j	T	T	T	T
g	d	d	d	d	f	f	T	T	T	T	T
h	d	d	d	d	f	f	j	T	j	T	T
i	d	d	d	d	f	f	j	j	T	T	T
j	d	d	d	d	f	f	j	j	j	T	T
T	a	b	c	d	e	f	g	h	i	j	T

let μ and ν be the two fuzzy subsets of M given by: $\mu(x) = \begin{cases} 1, & if\ e \leq x \\ \frac{1}{3}, & otherwise \end{cases}$ and

$\nu(x) = \begin{cases} \frac{1}{2}, & if\ g \leq x \\ \frac{1}{3}, & otherwise \end{cases}$.

μ is a fuzzy filter, a fuzzy m-filter and a fuzzy p-filter.

ν is fuzzy p-filter but it is not a fuzzy filter because $\nu(h \to i) = \nu(j) = \frac{1}{2}$ and $\inf\limits_{t \in h \to (h \sqcap i)} \nu(t) = \inf\{\nu(h \to f), \nu(h \to g)\} = \inf\{\nu(f), \nu(j)\} = \frac{1}{3}$ and then, $\inf\limits_{t \in h \to (h \sqcap i)} \nu(t) \neq \nu(h \to i)$.

The following result is the link between the notion of fuzzy filter and fuzzy m-filter.

Theorem 2.10. *Let μ be a fuzzy deductive system of M, then μ is a fuzzy filter of M if and only if,*

(i) μ *is a fuzzy m-filter;*
and for all $a,\, b \in M$,
(ii) *for all* $x, y \in a \sqcup b$, $\mu(y \to x) = \mu(x \to y)$;
(iii) *for all* $x, y \in a \sqcap b$, $\mu(y \to x) = \mu(x \to y)$.

Proof. ($\Rightarrow$) Assume that μ is a fuzzy filter of M. Firstly let us prove that μ is a fuzzy m-filter.

Let $a, b \in M$.

• For all $x \in a \sqcap b$, since μ is a fuzzy deductive system and because $\inf\limits_{t\in a\to(a\sqcap b)} \mu(t) = \mu(a \to b)$ and $a \to b \geq b$, we have
$\mu(x) \geq \min\{\mu(a), \mu(a \to x)\} \geq \min\{\mu(a), \mu(a \to b)\} \geq \min\{\mu(a), \mu(b)\}$.

• For all $x \in a \sqcup b$, we have $x \geq a$ and $x \geq b$. Therefore, $\mu(x) \geq \mu(a)$ and $\mu(x) \geq \mu(b)$.

Thus, $\mu(x) \geq \max\{\mu(a); \mu(b)\}$, $\forall x \in a \sqcup b$.

• We want to prove that, $\forall\, x,\, y \in a \sqcup b$, $\mu(x) = \mu(y)$.

If $a \sqcup b$ is a singleton, there is nothing to prove.

Otherwise let x and $y \in a \sqcup b$ such that $x \neq y$. As $a, b \leq x, y$ there exist two different elements $a', b' \in x \sqcap y$ such that $a \leq a'$ and $b \leq b'$. We also have $x, y \in a' \sqcup b'$. By P3 we have, $x \leq y \to x$ then, $\mu(x) \leq \mu(y \to x)$. Since μ is a fuzzy filter of M, we obtain $\inf\limits_{t\in y\to x\sqcap y} \mu(t) = \mu(y \to x) \geq \mu(x)$. Therefore, $\mu(y \to b') \geq \mu(x)$, because $b' \in x \sqcap y$. Using the fact that, $x \in a' \sqcup b'$ and $y \geq a'$, we have $\mu(x \to b') \geq \inf\limits_{t\in(a'\sqcup b')\to b'} \mu(t) = \mu(a' \to b') \geq \mu(y \to b') \geq \mu(x)$. From $b' \leq y$ we obtain $\mu(x \to b') \leq \mu(x \to y)$. Thus, $\mu(y) \geq \min\{\mu(x), \mu(x \to y)\} \geq \min\{\mu(x), \mu(x \to b')\} \geq \mu(x)$.

Similarly, we prove that, $\mu(x) \geq \mu(y)$.

Thus, $\mu(x) = \mu(y)$, for all $x,\, y \in a \sqcup b$.

We will now prove item (ii) in the statement. If $a \sqcup b$ is a singleton, there is nothing to prove.

let $x, y \in a \sqcup b$, such that $x \neq y$. As $a, b \leq x, y$ there exist two different elements $a', b' \in x \sqcap y$ such that $a \leq a'$ and $b \leq b'$ and also $x, y \in a' \sqcup b'$. Because μ is a fuzzy filter $\inf\limits_{t\in x\to x\sqcap y} \mu(t) = \mu(x \to y)$. $a' \in x \sqcap y$ then $x \to a' \in x \to x \sqcap y$. Thus, we have
$\mu(x \to a') \geq \inf\limits_{t\in x\to x\sqcap y} \mu(t) = \mu(x \to y) \qquad (*)$.

From $b' \leq x$ we obtain by P3 $x \to a' \leq b' \to a'$ by the properties of fuzzy filter and $(*)$, we obtain $\mu(b' \to a') \geq \mu(x \to y)$ and $\inf\limits_{t\in b'\sqcup a'\to a'} \mu(t) = \mu(b' \to a')$. $y \in a' \sqcup b'$ implies $\mu(y \to a') \geq \mu(b' \to a') \geq \mu(x \to y)$.

Since $a' \leq x$, we have $y \to a' \leq y \to x$. Once again by the properties of fuzzy filter, we have $\mu(y \to x) \geq \mu(y \to a') \geq \mu(x \to y)$. Finally $\mu(y \to x) \geq \mu(x \to y)$.

Following the same pattern we have $\mu(y \to x) \leq \mu(x \to y)$.

The proof for item (iii) is similar.

($\Leftarrow$) Conversely suppose now that μ is a fuzzy m-filter satisfying condition (ii) and (iii). As we are assuming that μ is a fuzzy deductive system, we have just to prove that $\forall\, a,\, b \in M$, $\min\left\{\inf\limits_{t\in(a\sqcup b)\to b} \mu(t), \quad \inf\limits_{t\in a\to(a\sqcap b)} \mu(t)\right\} = \mu(a \to b)$.

Let $a, b \in M$.

By proposition 1.10 we have,

$[(a \to b) \sqcap (b \to b)] = (a \to b) \sqcap T = \{a \to b\} \subseteq (a \sqcup b) \to b$.

Thus there exists $x_1 \in a \sqcup b$ such that $a \to b = x_1 \to b$. If $a \sqcup b$ is a singleton there is nothing to prove.

Otherwise, for all $t \in (a \sqcup b) \to b$, such that $t \neq x_1 \to b$, there exists $x_2 \in a \sqcup b$, such that $x_1 \neq x_2$ and $t = x_2 \to b$. By the hypothesis

$\mu(x_2 \to x_1) = \mu(x_1 \to x_2)$ $(**)$ and since μ is a fuzzy deductive system, we obtain $\mu(x_2 \to b) \geq \min\{\mu(x_1 \to b), \mu((x_1 \to b) \to (x_2 \to b))\}$.

By (P7), $(x_1 \to b) \to (x_2 \to b) \geq x_2 \to x_1$, then $\mu((x_1 \to b) \to (x_2 \to b)) \geq \mu(x_2 \to x_1)$ and $\mu(t) = \mu(x_2 \to b) \geq \min\{\mu(x_1 \to b), \mu(x_2 \to x_1)\} = \min\{\mu(x_1 \to b), \mu(x_1 \to x_2)\}$ by using $(**)$.

From $x_2 \geq b$, we have $x_1 \to x_2 \geq x_1 \to b$, then $\mu(t) = \mu(x_2 \to b) \geq \mu(x_1 \to b) = \mu(a \to b)$. Therefore, by (P7) $\mu(a \to b) \geq \inf\limits_{t \in (a \sqcup b) \to b} \mu(t) \geq \mu(a \to b)$ and $\inf\limits_{t \in (a \sqcup b) \to b} \mu(t) = \mu(a \to b)$.

Similarly, we prove the other condition, $\inf\limits_{t \in a \to (a \sqcap b)} \mu(t) = \mu(a \to b)$, finally

$$\forall a, b \in M, \min\left\{\inf_{t \in (a \sqcup b) \to b} \mu(t), \inf_{t \in a \to (a \sqcap b)} \mu(t)\right\} = \mu(a \to b). \qquad \square$$

Using the transfer principe for fuzzy set [8] we can obtain that, a fuzzy subset μ of M is a fuzzy filter of M if and only if, for all $\alpha \in [0, 1]$ the level set μ_α is either empty or a filter.

In the following, we are interested in the class of fuzzy filters and fuzzy deductive systems in residuated multilattice in which the implication behaves consistently with multiinfima and multisuprema.

Definition 2.11. A fuzzy deductive system μ is said to be consistent if for all $a, b, c \in M$, the following conditions hold:

(i) $\inf\limits_{t \in (a \sqcup b) \to c} \mu(t) \geq \min\{\mu(a \to c), \mu(b \to c)\}$.

(ii) $\inf\limits_{t \in c \to (a \sqcap b)} \mu(t) \geq \min\{\mu(c \to a), \mu(c \to b)\}$.

Proposition 2.12. *Every consistent fuzzy deductive system is a fuzzy filter.*

Proof. We have just to prove the specific condition in Definition 2.8. We have by (i), that $\inf\limits_{t \in (a \sqcup b) \to c} \mu(t) \geq \min\{\mu(a \to c), \mu(b \to c)\}$. Replacing c by b in (i) and by using $b \to b = T$ we obtain, $\inf\limits_{t \in (a \sqcup b) \to b} \mu(t) = \mu(a \to b)$. Similarly we prove that $\inf\limits_{t \in a \to (a \sqcap b)} \mu(t) = \mu(a \to b)$. $\square$

The converse of the previous proposition is not true, as the following example shows.

Example 2.13. The fuzzy filter μ in the Example 2.9, is consistent. Let η, be the fuzzy subset given by $\eta(x) = \begin{cases} 1 & if\ x = T \\ \frac{1}{5} & otherwise \end{cases}$, η is a fuzzy filter, but not fuzzy consistent. Because $\inf\limits_{t \in f \to (h \sqcap i)} \eta(t) = \inf\{\eta(j), \eta(T)\} = \eta(j) = \frac{1}{5}$,

$\min\{\eta(f \to h), \eta(f \to i)\} = \min\{\eta(T), \eta(T)\} = \eta(T) = 1$, and
$\inf\limits_{t \in f \to (h \sqcap i)} \eta(t) < \min\{\eta(f \to h), \eta(f \to i)\}$.

Several papers have been published on the relation between filters and congruences on different algebraic structures ([4], [6], [2]). In this paper, we now study the notion of fuzzy congruence on residuated multilattices setting.

3. Fuzzy congruence relation of residuated multilattice

In [2] I. P. Cabera et al. have studied some properties of congruence relation on multilattices and in [3] the authors have studied fuzzy congruence relation on non-deterministic groupoids. As we have seen above (notation 1.7), given a multilattice $(M; \leq, \odot, \to, \sqcap, \sqcup)$, the operations $\odot$, $\to$ and the hyperoperations $\sqcap$, $\sqcup$ can be extended to the power set 2^M. On the other hand, given A an arbitrary set and $\leq$ a preorder (reflexive and transitive relation) defined over A, it is possible to extend the preorder structure to the powerset 2^A by the so-called Hoare, Smyth, and Egli-Milner powerset preorders defined for all X, $Y \subseteq A$ by

$X \mathrel{\widehat{\leq}_H} Y \Leftrightarrow \forall\, x \in X, \exists\, y \in Y, x \leq y.$

$X \mathrel{\widehat{\leq}_S} Y \Leftrightarrow \forall\, y \in Y, \exists\, x \in X, x \leq y.$

$X \mathrel{\widehat{\leq}_{EM}} Y \Leftrightarrow \forall\, x \in X, \exists\, y \in Y, x \leq y$ and $\forall\, y \in Y, \exists\, x \in X, x \leq y.$

It is obvious that $X \mathrel{\widehat{\leq}_{EM}} Y \Rightarrow X \mathrel{\widehat{\leq}_S} Y$ and $X \mathrel{\widehat{\leq}_{EM}} Y \Rightarrow X \mathrel{\widehat{\leq}_H} Y$. One can see that none of the above preorders is antisymmetric like the following example proves.

Example 3.1. Let $(M, \sqcup, \sqcap)$ be the multilattice described in example 2.5. Consider the subsets $X = \{a; b; d; f\}$ and $Y = \{a; c; e; f\}$ of M. We have $X \mathrel{\widehat{\leq}_{EM}} Y$ and $Y \mathrel{\widehat{\leq}_{EM}} X$ but $X \neq Y$.

As we will see in this section, any fuzzy binary relation ρ on M can be extended to a fuzzy binary relation $\widehat{\rho}$ on the power set 2^M. We will now focus our interest on the compatibility of $\widehat{\rho}$ with $\odot$, $\to$, $\sqcap$ and $\sqcup$ as the extended operations on 2^M.

Let us First introduce the notations which will be useful hereafter.

Definition 3.2 ([9]).
i) Let X and Y be non-empty sets. A function $\rho : X \times Y \to [0, 1]$ (i.e., a fuzzy subset of $X \times Y$) is called a fuzzy relation between the set X and the set Y.
ii) Let M be a non-empty set. For every fuzzy relation ρ on M. The power set extension of ρ is defined as follows:

$$\widehat{\rho} : \mathcal{P}(M) \times \mathcal{P}(M) \longrightarrow [0,1] \text{ with } \widehat{\rho}(X, Y) = \min\left\{\left(\bigwedge_{x \in X} \bigvee_{y \in Y} \rho(x, y)\right); \left(\bigwedge_{y \in Y} \bigvee_{x \in X} \rho(x, y)\right)\right\},$$

for all non-empty subsets X and Y of M.
iii) Let ρ be a fuzzy relation on a non-empty set X. ρ is said to be:
- reflexive if, $\rho(x, x) = 1$, for all $x \in X$;
- symmetric if, $\rho(x, y) = \rho(y, x)$, for all $x, y \in X$;
- Transitive if, $\min\{\rho(x, y), \rho(y, z)\} \leq \rho(x, z)$ for all $x, y, z \in X$.

iv) A reflexive, symmetric and transitive fuzzy relation on X is called a fuzzy equivalence.

v) A fuzzy equivalence relation ρ on X is called a fuzzy equality if for any $x, y \in X$, $\rho(x, y) = 1$ implies $x = y$.

Definition 3.3. A fuzzy equivalence relation ρ on M is said to be a fuzzy congruence relation if and only if, for all $a, b, c, d \in M$ we have the following

- $\rho(a \odot b, c \odot d) \geq \min\{\rho(a, c); \rho(b, d)\}$;
- $\rho(a \to b, c \to d) \geq \min\{\rho(a, c); \rho(b, d)\}$;
- $\widehat{\rho}(a \sqcap b, c \sqcap d) \geq \min\{\rho(a, c); \rho(b, d)\}$;
- $\widehat{\rho}(a \sqcup b, c \sqcup d) \geq \min\{\rho(a, c); \rho(b, d)\}$.

Proposition 3.4. *Let ρ be a fuzzy equivalence relation on M. Then ρ is a fuzzy congruence relation on M, if the following condition are satisfied. $\forall a, b, c \in M$*

- $\rho(a \odot b, c \odot b) \geq \rho(a, c)$*;*
- $\rho(a \to b, c \to b) \geq \rho(a, c)$*;*
- $\widehat{\rho}(a \sqcap b, c \sqcap b) \geq \rho(a, c)$*;*
- $\widehat{\rho}(a \sqcup b, c \sqcup b) \geq \rho(a, c)$*.*

Proof. If ρ is a fuzzy congruence relation on M, then by replacing in definition 3.3 d by b, we have the result. The converse is proved by using the transitivity of ρ. □

In [3] the authors, proved that, if ρ is a fuzzy relation in a non empty set M and $\widehat{\rho}$ its power set extension, ρ is a fuzzy equivalence relation if and only if $\widehat{\rho}$ is also a fuzzy equivalence relation.

Theorem 3.5. *Let ρ be a fuzzy relation on a residuated multilattice $(M, \leq, \odot, \to, \sqcap, \sqcup, T)$. Then, ρ is a fuzzy congruence relation if and only if $\widehat{\rho}$ is a fuzzy equivalence relation on the power set 2^M and for all X, Y, $Z \in 2^M$, $\widehat{\rho}(X * Z, Y * Z) \geq \widehat{\rho}(X, Y)$.*

Proof. Since ρ is a fuzzy equivalence relation on M if and only if $\widehat{\rho}$ is a fuzzy equivalence relation over 2^M, we need only to prove that for all X, Y, $Z \in 2^M$, $\widehat{\rho}(X * Z, Y * Z) \geq \widehat{\rho}(X, Y) \Leftrightarrow \forall x, y, z \in M$, $\widehat{\rho}(x * z, y * z) \geq \rho(x, y)$.

Suppose that for all $X, Y, Z \in 2^M$, $\widehat{\rho}(X * Z, Y * Z) \geq \widehat{\rho}(X, Y)$. Then, particularly for $X = \{x\}$, $Y = \{y\}$, $Z = \{z\}$ we have, $\widehat{\rho}(x * z, y * z) = \widehat{\rho}(\{x\} * \{z\}, \{y\} * \{z\}) \geq \widehat{\rho}(\{x\}, \{y\}) = \rho(x, y)$.

Conversely, assume that $\forall x, y, z \in M$, $\widehat{\rho}(x * z, y * z) \geq \rho(x, y)$ with $* \in \{\odot, \to, \sqcap, \sqcup\}$. Then,

For $* \in \{\sqcap, \sqcup\}$ we have

$$\begin{aligned}
\widehat{\rho}(X * Z, Y * Z) &= \min\left\{ \bigwedge_{x*z \subseteq X*Z} \bigvee_{y*z' \subseteq Y*Z} \widehat{\rho}(x * z, y * z'); \bigwedge_{y*z' \subseteq Y*Z} \bigvee_{x*z \subseteq X*Z} \widehat{\rho}(x * z, y * z') \right\} \\
&\geq \min\left\{ \bigwedge_{x*z \subseteq X*Z} \bigvee_{y \in Y} \widehat{\rho}(x * z, y * z); \bigwedge_{y*z \subseteq Y*Z} \bigvee_{x \in X} \widehat{\rho}(x * z, y * z) \right\} \\
&\geq \min\left\{ \bigwedge_{x \in X} \bigvee_{y \in Y} \rho(x, y); \bigwedge_{y \in Y} \bigvee_{x \in X} \rho(x, y) \right\} = \widehat{\rho}(X, Y).
\end{aligned}$$

For $* \in \{\odot, \rightarrow\}$, $x * z$ is assume to be $\{x * z\}$. Following the same pattern as above we have $\widehat{\rho}(X * Z, Y * Z) \geq \widehat{\rho}(X, Y)$. □

Regarding the extension of the definition of fuzzy congruence to the non-deterministic case, the following definition of compatibility, in the case of an underlying hyperstructure, was introduced by Bakhshi and Borzooei [1].

Definition 3.6. Let $(M, \leq, \odot, \rightarrow, \sqcap, \sqcup, T)$ be a residuated multilattice. Then a fuzzy relation ρ on M is said to be compatible if for all $x \in a * c$ there exists $y \in b * c$ and for all $y \in b * c$ there exists $x \in a * c$ such that $\rho(x, y) \geq \rho(a, b)$, for all $a, b, c \in M$ and $* \in \{\odot, \rightarrow, \sqcap, \sqcup\}$.

In [1] Bakhshi and Borzooei have proved that a fuzzy relation that is compatible (in the sense of Definition 3.6) with a non-deterministic operation $*$ satisfies Conditions of proposition 3.4, but the converse is not in general true. To have the equivalence, we need the sup property given in the following definition.

Definition 3.7 ([3])**.** Let M be a non-empty set and ρ a fuzzy relation on M. We say that ρ satisfies the right (resp. left) sup property if for all $a \in M$ and for all non-empty $X \subseteq M$, there exists $y_0 \in Y$ (resp $x_0 \in X$) such that $\sup\limits_{y \in Y} \rho(a, y) = \rho(a, y_0)$ (resp. $\sup\limits_{x \in X} \rho(x, a) = \rho(x_0, a)$).

Lemma 3.8. *Let ρ be a fuzzy equivalence relation on a residuated multilattice $(M, \leq, \odot, \rightarrow, \sqcap, \sqcup, T)$ which satisfies the right and left sup property. Then ρ is a fuzzy congruence relation if and only if ρ is compatible with $\odot, \rightarrow, \sqcap, \sqcup$ (in the sense of Definition 3.6).*

Proof. Let us suppose that ρ is compatible. Let $x \in a \sqcup c$, then there exists $y \in b \sqcup c$ such that $\rho(x, y) \geq \rho(a, b)$. So $\bigvee\limits_{y \in b \sqcup c} \rho(x, y) \geq \rho(a, b)$ and $\bigwedge\limits_{x \in a \sqcup c} \bigvee\limits_{y \in b \sqcup c} \rho(x, y) \geq \rho(a, b)$. Analogously $\bigwedge\limits_{y \in b \sqcup c} \bigvee\limits_{x \in a \sqcup c} \rho(x, y) \geq \rho(a, b)$, therefore, $\widehat{\rho}(a \sqcup c, b \sqcup c) \geq \rho(a, b)$. Similarly, $\widehat{\rho}(a \sqcap c, b \sqcap c) \geq \rho(a, b)$.

Let $x = a \rightarrow c$, then there exists $y = b \rightarrow c$ such that $\rho(x, y) \geq \rho(a, b)$, that is, $\rho(a \rightarrow c, b \rightarrow c) \geq \rho(a, b)$. Similarly, $\rho(a \odot c, b \odot c) \geq \rho(a, b)$.

Conversely, let us suppose ρ is a fuzzy congruence relation. Then, $\widehat{\rho}(a \sqcup c, b \sqcup c) \geq \rho(a, b)$. In particular $\bigwedge\limits_{x \in a \sqcup c} \bigvee\limits_{y \in b \sqcup c} \rho(x, y) \geq \rho(a, b)$. By the right sup property, for all $x \in a \sqcup c$ there exists $y_0 \in b \sqcup c$ such that $\bigvee\limits_{y \in b \sqcup c} \rho(x, y) = \rho(x, y_0)$. Since $\bigwedge\limits_{x \in a \sqcup c} \rho(x, y_0) \leq \rho(x, y_0)$, we obtain $\rho(a, b) \leq \rho(x, y_0)$. Similarly, by the left sup property for all $y \in b \sqcup c$ there exists $x_0 \in a \sqcup c$ such that $\rho(a, b) \leq \rho(x_0, y)$. We only check one hyperoperation because the other ones follows the same scheme.

The prove is straightforward for $\odot$ and $\rightarrow$. □

Lemma 3.9. *Let ρ be a fuzzy congruence relation on residuated multilattice M, then for all a, $b \in M$, $\rho(a, b) = \min\{\rho(a, c); \rho(c, b)\}$ for all $c \in a * b$ with $* \in \{\sqcup, \sqcap\}$. Moreover, if the product is idempotent then, $\rho(a, b) = \min\{\rho(a, a \odot b); \rho(a \odot b, b)\}$.*

Proof. Let $a,\ b \in M$. Because M is full, $a \sqcup b \neq \varnothing$. Let $c \in a \sqcup b$, then $\rho(a,b) \geq \min\{\rho(a,c); \rho(c,b)\}$ by transitivity.

Since $a \sqcup a = \{a\}$ and $b \sqcup b = \{b\}$, $\rho(a,b) \leq \widehat{\rho}(a, a \sqcup b)$ and $\rho(a,b) \leq \widehat{\rho}(a \sqcup b, b)$, because, ρ is a fuzzy congruence relation on M. Therefore, $\rho(a,b) \leq \rho(a,c)$ and $\rho(a,b) \leq \rho(c,b)$. Thus $\rho(a,b) \leq \min\{\rho(a,c); \rho(c,b)\}$. Finally, $\rho(a,b) = \min\{\rho(a,c); \rho(c,b)\}$.

Similarly, we have $\rho(a,b) = \min\{\rho(a,c); \rho(c,b)\}$ for all $c \in a \sqcap b$.

Since ρ is a fuzzy congruence relation on M and $\odot$ is idempotent, $\rho(a,b) \leq \rho(a \odot a, a \odot b) = \rho(a, a \odot b)$ and similarly, $\rho(a,b) \leq \rho(a \odot b, b)$, which implies that $\rho(a,b) \leq \min\{\rho(a, a \odot b); \rho(a \odot b, b)\}$. By transitivity and symmetry, $\rho(a,b) \geq \min\{\rho(a, a \odot b); \rho(a \odot b, b)\}$. Thus, $\rho(a,b) = \min\{\rho(a, a \odot b); \rho(a \odot b, b)\}$. □

Theorem 3.10. *Let $(M, \leq, \odot, \rightarrow, \sqcap, \sqcup, T)$ be a residuated multilattice with idempotent product $\odot$, and let ρ be a fuzzy equivalence relation. Then ρ is a fuzzy congruence relation on M, if and only if for all $a, b, c \in M$ with $a \leq b$, the following condition holds:*

$$\widehat{\rho}(a * c; b * c) \geq \rho(a,b), \ \text{with} * \in \{\odot, \rightarrow, \sqcup, \sqcap\}. \tag{3.1}$$

Proof. Proposition 3.4 prove the necessity, thus we will just prove the sufficiency. Let $a,\ b \in M$.

For all $z \in a \sqcup b$, we have $a \leq z$ and $b \leq z$ and by Lemma 3.9, we have $\rho(a,b) = \min\{\rho(a,z); \rho(z,b)\}$. Then, by condition (3.1) $\widehat{\rho}(a \sqcup c; z \sqcup c) \geq \rho(a,z)$ and $\widehat{\rho}(z \sqcup c; b \sqcup c) \geq \rho(z,b)$. Therefore, by transitivity of $\widehat{\rho}$, we have

$\widehat{\rho}(a \sqcup c; b \sqcup c) \geq \min\{\widehat{\rho}(a \sqcup c; z \sqcup c); \widehat{\rho}(z \sqcup c; b \sqcup c)\} \geq \min\{\rho(a,z); \rho(z,b)\} = \rho(a,b)$.

For $\sqcap$, the prove is similar to the previous.

By P1 we have, $a \odot b \leq a$ and $a \odot b \leq b$. Then, by condition (3.1) $\rho((a \odot b) \odot c; a \odot c) \geq \rho(a \odot b; a)$ and $\rho((a \odot b) \odot c; b \odot c) \geq \rho(a \odot b; b)$. Therefore, by transitivity and reflexivity of ρ we have,

$$\begin{aligned} \rho(a \odot c; b \odot c) &\geq \min\{\rho(a \odot c, (a \odot b) \odot c),\ \rho((a \odot b) \odot c; b \odot c)\} \\ &\geq \min\{\rho(a, a \odot b), \rho(a \odot b, b)\} = \rho(a,b). \end{aligned}$$

Once again by P1 $a \odot b \leq a$ and $a \odot b \leq b$. By the hypothesis, we have

$\rho(a \odot b \rightarrow c; a \rightarrow c) \geq \rho(a \odot b, a)$ and $\rho(a \odot b \rightarrow c; b \rightarrow c) \geq \rho(a \odot b, b)$. Therefore, by transitivity of ρ, we have $\rho(a \rightarrow c; b \rightarrow c) \geq \min\{\rho(a; a \odot b); \rho(a \odot b; b)\} = \rho(a,b)$. □

Proposition 3.11. *Let $(M, \leq, \odot, \rightarrow, \sqcap, \sqcup, T)$ be a residuated multilattice and ρ be a fuzzy congruence relation of M.*

For all $a, b, c \in M$ such that $a \leq b$, if there exist $w \in a * c$ and $z \in b * c$, such that $w \leq z$ then $\rho(w,z) \geq \rho(a,b)$, where $* \in \{\sqcup, \sqcap, \odot, \rightarrow\}$.

Proof. Let $a, b, c \in M$ such that $a \leq b$. Suppose there exist $w \in a * c$ and $z \in b * c$, such that $w \leq z$.

Let $*$ be the hyperoperation $\sqcup$. Then, $w \sqcup z = z$ and $a \leq w$ (i.e., $w = a \sqcup w$) and since ρ is a fuzzy congruence relation,

$$\rho(a,b) \leq \widehat{\rho}(a \sqcup w; b \sqcup w) \leq \bigvee_{x \in a \sqcup w} \bigwedge_{y \in b \sqcup w} \rho(x,y) = \bigwedge_{y \in b \sqcup w} \rho(w,y). \tag{3.2}$$

Therefore, it is sufficient to prove that $z \in b \sqcup w$.

Because, $z \in b \sqcup c$ implies $b \leq z$ and $b \sqcup z = z$, we have $z = b \sqcup z \in b \sqcup (w \sqcup z) \subseteq (b \sqcup w) \sqcup z$, thus there exists $z' \in b \sqcup w$ such that $z \in z' \sqcup z$, then $z' \leq z$.

Similarly, using the inequalities $b \leq z'$ and $c \leq w \leq z'$, there exists $z'' \in b \sqcup c$ satisfying $z'' \leq z'$ and therefore $z'' \leq z$ and because $z \in b \sqcup c$, Property (C3) of Proposition 1.9 leads to $z'' = z$. Finally, we have $z' \leq z$ and $z \leq z'$, hence $z = z' \in b \sqcup w$. Thus applying (3.2), $\rho(w, z) \geq \rho(a, b)$.

The prove for the hyperoperation $\sqcap$ follows the same pattern as above.

Let $*$ be the operation $\to$. Let suppose there exist $w = a \to c$ and $z = b \to c$, with $w \leq z$ that is, $a \to c \leq b \to c$, since $a \leq b$ implies $a \to c \geq b \to c$ then $w = z$. Thus, $\rho(w, z) = \rho(w, w) = 1 \geq \rho(a, b)$.

Finally, let $*$ be the operation $\odot$. Then $w = a \odot c$ and $z = b \odot c$ with $w \leq z$. Therefore, $\rho(a \odot c, b \odot c) \geq \rho(a, b)$, that is $\rho(w, z) \geq \rho(a, b)$. ☐

Theorem 3.12. *The set* FCon(M) *of the fuzzy congruence relations on a m-distributive residuated multilattice M, is a sublattice of the set* FEq(M) *of the fuzzy equivalence relations on M, moreover is a complete lattice wrt the fuzzy inclusion ordering.*

Proof. Let $\{\rho_i\}_{i \in \Lambda}$ be a set of fuzzy congruence relations on M, consider $\rho_\cap$ to be their intersection. Since $\rho_\cap$ is a fuzzy congruence relation on M, by Theorem 3.10 we have just to check that, for every $a, b, c \in M$ with $a \leq b$, $\widehat{\rho}_\cap(a \sqcup c, b \sqcup c) \geq \rho_\cap(a, b)$.

From Proposition 1.11, if $z \in b \sqcup c$ then there exists $w \in a \sqcup c$ such that $w \leq z$ and, then, Theorem 3.11 implies $\rho_i(w, z) \geq \rho_i(a, b)$ for all $i \in \Lambda$. So,

$$\bigvee_{x \in a \sqcup c} \rho_\cap(x, z) \geq \bigvee_{w \leq z, w \in a \sqcup c} \rho_\cap(w, z) = \bigvee_{w \leq z, w \in a \sqcup c} \bigwedge_{i \in \Lambda} \rho_i(w, z) \geq \bigvee_{w \leq z, w \in a \sqcup c} \bigwedge_{i \in \Lambda} \rho_i(a, b) = \rho_\cap(a, b).$$

Analogously, from Propositions 1.11 and 3.11, if $w \in a \sqcup c$ then there exists $z \in b \sqcup c$ such that $w \leq z$

$$\bigvee_{y \in b \sqcup c} \rho_\cap(w, y) \geq \bigvee_{z \geq w, z \in b \sqcup c} \rho_\cap(w, z) = \bigvee_{z \geq w, z \in b \sqcup c} \bigwedge_{i \in \Lambda} \rho_i(w, z) \geq \bigvee_{z \geq w, z \in b \sqcup c} \bigwedge_{i \in \Lambda} \rho_i(a, b) = \rho_\cap(a, b).$$

Therefore, $\widehat{\rho}_\cap(a \sqcup c, b \sqcup c) \geq \rho_\cap(a, b)$.

The proof for transitive closure of union follows by a routine calculation. ☐

4. Cosets of fuzzy filter

In this section, M will stand for the residuated multilattice $(M, \leq, \odot, \to, \sqcap, \sqcup, T)$.

Let μ be a fuzzy filter of M. For all $a \in M$, μ^a is the fuzzy subset of M, called a coset of the fuzzy filter μ and defined by $\mu^a(x) = \min\{\mu(a \to x), \mu(x \to a)\}$.

Lemma 4.1. *Let μ be a fuzzy filter of M, then $\mu^a = \mu^b$, if and only if $\mu(a \to b) = \mu(b \to a) = \mu(T)$.*

Proof. If $\mu^a = \mu^b$, then $\mu^a(a) = \mu^b(a)$. That is $\mu(a \to a) = \mu(T) = \min\{\mu(a \to b), \mu(b \to a)\}$. Which implies that $\mu(T) \leq \mu(a \to b)$ and $\mu(T) \leq \mu(b \to a)$. So $\mu(a \to b) = \mu(b \to a) = \mu(T)$.

Conversely let $\mu(a \to b) = \mu(b \to a) = \mu(T)$. Let $z \in M$, since μ is a fuzzy filter of M, we have $\mu(z \to a) \geq \min\{\mu(z \to b), \mu((z \to b) \to (z \to a))\}$. Therefore, $\mu(z \to a) \geq \min\{\mu(z \to b), \mu(b \to a)\}$, because $(z \to b) \to (z \to a) \geq b \to a$. Thus, $\mu(z \to a) \geq \min\{\mu(z \to b), \mu(T)\} \geq \mu(z \to b)$.

Similarly we have $\mu(z \to b) \geq \mu(z \to a)$. Therefore, $\mu(z \to b) = \mu(z \to a)$. We also have $\mu(b \to z) = \mu(a \to z)$. Thus, $\mu^a(z) = \mu^b(z)$, for all $z \in M$. i.e., $\mu^a = \mu^b$. □

Let μ be a fuzzy filter of M and $\alpha \in [0,1]$. Consider on M the relation $\sim_\alpha$ defined by, $a \sim_\alpha b$ if and only if $\mu^a(b) \geq \alpha$, for all a, $b \in M$. Therefore, from Lemma 4.1, $\mu^a = \mu^b$, if and only if $a \sim_{\mu(T)} b$.

Proposition 4.2. *Let μ be a fuzzy filter of M. Then, for all $\alpha \in [0, \mu(T)]$, $\sim_\alpha$ is an equivalence relation on M.*

Proof. Let $\alpha \in [0, \mu(T)]$. Let x, y, $z \in M$.
- $x \sim_\alpha x$, because $\mu(x \to x) = \mu(T) \geq \alpha$.
- Obviously, we have $x \sim_\alpha y$ if and only if $y \sim_\alpha x$.
- Suppose that $x \sim_\alpha y$ and $y \sim_\alpha z$.

$x \sim_\alpha y$ implies $\min\{\mu(x \to y), \mu(y \to x)\} \geq \alpha$ and , $y \sim_\alpha z$ implies $\min\{\mu(y \to z), \mu(z \to y)\} \geq \alpha$. Since μ is a deductive system,

$$\mu(x \to z) \geq \min\{\mu(y \to z), \mu((y \to z) \to (x \to z))\} \geq \min\{\mu(y \to z), \mu(x \to y)\} \geq \alpha.$$

Similarly, we have $\mu(z \to x) \geq \alpha$. Then $x \sim_\alpha z$.

Therefore, $\sim_\alpha$ is an equivalence relation on M. □

Proposition 4.3. *Let μ be a fuzzy filter of M and $\alpha \in [0, \mu(T)]$.*
Then
(i) $(\forall x, y, z \in M)$, $(x \sim_\alpha y \Rightarrow ((x \to z) \sim_\alpha (y \to z)$ *and* $(z \to x) \sim_\alpha (z \to y)))$;
(ii) $(\forall x, y, a, b \in M)$, $((x \sim_\alpha y)$ *and* $(a \sim_\alpha b) \Rightarrow (x \to a) \sim_\alpha (y \to b))$.

Proof. Straightforward. □

Proposition 4.4. *Let μ be a fuzzy filter of M and $\alpha \in [0, \mu(T)]$, then $\sim_\alpha$ is congruence relation of pocrim.*

Proof. By Proposition 4.2 and 4.3 we already know that $\sim_\alpha$ is an equivalence relation compatible with the operation $\to$. We only have to prove the compatibility with the product.

Let $x \sim_\alpha y$. By P6, we have $(x \odot z) \to (y \odot z) \geq x \to y$ and $(y \odot z) \to (x \odot z) \geq y \to x$. Since μ is a fuzzy p-filter $\mu((x \odot z) \to (y \odot z)) \geq \mu(x \to y)$ and $\mu((y \odot z) \to (x \odot z)) \geq \mu(y \to x)$. Thus, $(x \odot z) \sim_\alpha (y \odot z)$. □

Theorem 4.5. *Let μ be a fuzzy filter of M then $\sim_{\mu(T)}$ is a congruence relation on M.*

Proof. By Proposition 4.4 $\sim_{\mu(T)}$ is a congruence of pocrims. We have to prove that $\sim_{\mu(T)}$ is a congruence of multilattice by using the Theorem 1.20. Since $\sim_{\mu(T)}$ is an equivalence relation, the items (i) and (iii) of Theorem 1.20 are satisfy. We will only prove items (ii) and (iv) of that Theorem.

For item (ii), we should prove that for all $a, b \in M$, $a \sim_{\mu(T)} b$ if and only if there exist $z \in a \sqcap b$ and $w \in a \sqcup b$ such that $z \sim_{\mu(T)} w$.

Let $a, b \in M$, then for all $z \in a \sqcap b$ and $w \in a \sqcup b$, we have $z \leq w$, so $z \to w = T$ and $\mu(z \to w) = \mu(T)$ $(\star)$.

Suppose that $a \sim_{\mu(T)} b$, then we have $\mu(b \to a) = \mu(T)$. Since μ is a fuzzy filter, $\inf\limits_{t \in (b \sqcup a) \to a} \mu(t) \geq \mu(b \to a)$ and particularly, for $w \in a \sqcup b$, $\mu(w \to a) = \mu(T)$.

Analogously, $z \in a \sqcap b$ implies $\mu(a \to z) \geq \mu(a \to b) = \mu(T)$. Since μ is a fuzzy filter of M, $\mu(w \to z) \geq \min\{\mu(w \to a), \mu((w \to a) \to (w \to z))\} \geq \min\{\mu(w \to a), \mu(a \to z)\} \geq \mu(T)$. Which implies $\mu(w \to z) = \mu(T)$ $(\star\star)$.

$(\star)$ and $(\star\star)$ ensure that $w \sim_{\mu(T)} z$.

Conversely, assume that there exist $z \in a \sqcap b$ and $w \in a \sqcup b$ such that $z \sim_{\mu(T)} w$. Then $\mu(w \to z) = \mu(T)$. From $z \leq b$ and P3 we obtain $w \to z \leq w \to b$, then $\mu(w \to b) = \mu(T)$. Likewise, from $w \geq a$ we have $w \to b \leq a \to b$. Then, $\mu(T) = \mu(w \to b) \leq \mu(a \to b)$. Analogously we obtain $\mu(b \to a) = \mu(T)$. Thus, $a \sim_{\mu(T)} b$.

Now, for item (iv), let $a, b \in M$ such that $a \leq b$ and $a \sim_{\mu(T)} b$. We have to prove that, for all $c \in M$, $(a \sqcap c) \widehat{\sim}_{\mu(T)} (b \sqcap c)$ and $(a \sqcup c) \widehat{\sim}_{\mu(T)} (b \sqcup c)$.

For $x \in a \sqcap c$, since $x \leq a \leq b$ and $x \leq c$, $\exists y \in b \sqcap c$ such that $x \leq y$.

On one hand $\mu(x \to y) = \mu(T)$ (1).

On the other hand, since $b \geq y$, we have $b \to a \leq y \to a$, which implies $\mu(T) = \mu(b \to a) \leq \mu(y \to a)$. Because μ is a fuzzy filter $\inf\limits_{t \in y \to (a \sqcap y)} \mu(t) \geq \mu(y \to a) = \mu(T)$. $x \in a \sqcap y$ and, hence, $\mu(y \to x) = \mu(T)$ (2).

(1) and (2) implies $x \sim_{\mu(T)} y$. □

Lemma 4.6. *Let μ be a fuzzy filter of M and $x, y, a, b \in M$. If $\mu^x = \mu^a$ and $\mu^y = \mu^b$, then $\mu^{x \to y} = \mu^{a \to b}$, $\mu^{x \odot y} = \mu^{a \odot b}$, $\mu^{x \sqcup y} = \mu^{a \sqcup b}$ and $\mu^{x \sqcap y} = \mu^{a \sqcap b}$. Where $\mu^{a \sqcup b} = \{\mu^c, c \in a \sqcup b\}$ and $\mu^{a \sqcap b} = \{\mu^c, c \in a \sqcap b\}$.*

Proof. Suppose $\mu^x = \mu^a$ and $\mu^y = \mu^b$, then $x \sim_{\mu(T)} a$, $y \sim_{\mu(T)} b$. Because $\sim_{\mu(T)}$ is a congruence relation on M, we have $(x \to y) \sim_{\mu(T)} (a \to b)$, $(x \odot y) \sim_{\mu(T)} (a \odot b)$, $(x \sqcup y) \widehat{\sim}_{\mu(T)} (a \sqcup b)$ and $(x \sqcap y) \widehat{\sim}_{\mu(T)} (a \sqcap b)$. Therefore, by Lemma 4.1, $\mu^{x \to y} = \mu^{a \to b}$ and $\mu^{x \odot y} = \mu^{a \odot b}$ and by the definition of $\widehat{\sim}_{\mu(T)}$ and Lemma 4.1, we have $\mu^{x \sqcup y} = \mu^{a \sqcup b}$ and $\mu^{x \sqcap y} = \mu^{a \sqcap b}$. □

Let μ be a fuzzy filter of M. Let $M_{/_\mu}$ denote the set of all cosets of μ, i.e., $M_{/_\mu} = \{\mu^a, a \in M\}$. For any $\mu^a, \mu^b \in M_{/_\mu}$, we defined $\mu^a \underline{\sqcup} \mu^b = \mu^{a \sqcup b}$, $\mu^a \overline{\sqcap} \mu^b = \mu^{a \sqcap b}$, $\mu^a \to \mu^b = \mu^{a \to b}$, $\mu^a \odot \mu^b = \mu^{a \odot b}$. Therefore, Lemma 4.6 proves that $\odot$, $\to$, $\overline{\sqcap}$ and $\underline{\sqcup}$ are well defined on $M_{/_\mu}$.

Lemma 4.7. *Let μ be a fuzzy filter of M and $\preceq$ be the relation on $M_{/_\mu}$ defined by $\mu^a \preceq \mu^b$ if and only if $a \to b = T$.*

Then, $\preceq$ is a partial order on M.

Lemma 4.8. *Let μ be a fuzzy filter of M. Then $(M_{/_\mu}, \preceq, \odot, \to, \overline{\sqcap}, \underline{\sqcup}, \mu^T)$ is a residuated multilattice.*

Proof. Since $(M_{/_\mu}, \preceq, \odot, \to, \mu^T)$ satisfies items (1) and (2) of Definition 1.1 and $(M_{/_\mu}, \overline{\sqcap}, \underline{\sqcup}, \mu^T)$ is a multilattice, we should prove only the adjointness condition.

Let μ^a, μ^b, $\mu^c \in M_{/_\mu}$. Then $\mu^a \odot \mu^b \preceq \mu^c \Leftrightarrow (a \odot b) \to c = T \Leftrightarrow a \odot b \leq c \Leftrightarrow a \leq b \to c \Leftrightarrow a \to (b \to c) = T \Leftrightarrow \mu^a \preceq \mu^b \to \mu^c$. □

Theorem 4.9. *Let μ be a fuzzy filter of M and define the mapping $\widetilde{\mu} : M \to M/_\mu$, by $\widetilde{\mu}(a) = \mu^a$, for all $a \in M$. Then,*

(i) $\widetilde{\mu}$ is a surjective homomorphism of residuated multilattice;

(*ii*) $\ker(\widetilde{\mu}) = \mu_{\mu(T)}$;
(*iii*) $M/_{\mu} \cong M/_{\mu_{\mu(T)}}$.

Proof. For (*i*), let a, $b \in M$. We have $\widetilde{\mu}(a \odot b) = \mu^{a\odot b} = \mu^a \odot \mu^b = \widetilde{\mu}(a) \odot \widetilde{\mu}(b)$, $\widetilde{\mu}(a \to b) = \mu^{a\to b} = \mu^a \to \mu^b = \widetilde{\mu}(a) \to \widetilde{\mu}(b)$, $\widetilde{\mu}(a \sqcup b) = \mu^{a\sqcup b} = \mu^a \underline{\sqcup} \mu^b = \widetilde{\mu}(a) \underline{\sqcup} \widetilde{\mu}(b)$, $\widetilde{\mu}(a \sqcap b) = \mu^{a\sqcap b} = \mu^a \overline{\sqcap} \mu^b = \widetilde{\mu}(a) \overline{\sqcap} \widetilde{\mu}(b)$, and $\widetilde{\mu}(T) = \mu^T$. This show that $\widetilde{\mu}$ is a surjective homomorphism.

(*ii*) $x \in \ker(\widetilde{\mu})$ if and only if $\widetilde{\mu}(x) = \mu^T$ if and only if $\mu^x = \mu^T$ if and only if $x \sim_{\mu(T)} T$ if and only if $x \in \mu_{\mu(T)}$. Hence, $\ker(\widetilde{\mu}) = \mu_{\mu(T)}$.

(*iii*) By previous items we have $M/_{\mu} \cong M/_{\mu_{\mu(T)}}$. □

5. Fuzzy homomorphism

Definition 5.1 ([7])**.** Let ρ and σ be a fuzzy equalities defined on the sets A and B, respectively. A partial fuzzy function φ from A to B is a mapping $\varphi : A\times B \longrightarrow [0, 1]$ satisfying the following conditions for all $a, a' \in A$ and $b, b' \in B$:

- $\min\{\varphi(a, b), \rho(a, a')\} \leq \varphi(a', b)$;
- $\min\{\varphi(a, b), \sigma(b, b')\} \leq \varphi(a, b')$;
- $\min\{\varphi(a, b), \varphi(a, b')\} \leq \sigma(b, b')$;

If in addition, the following condition holds: for all $a \in A$ there is $b \in B$ such that $\varphi(a, b) = 1$, we say that φ is a perfect fuzzy function.

Definition 5.2. Let $(M, \leq, \odot, \to, \sqcup, \sqcap, T)$ and $(M', \leq, \odot, \to, \sqcup, \sqcap, T')$ be residuated multilattice endowed with fuzzy equalities ρ and σ, respectively. A perfect fuzzy function $\varphi \in [0, 1]^{M\times M'}$ is said to be fuzzy homomorphism if for all $a_1, a_2 \in M$ and $b_1, b_2 \in M'$ the following condition hold:

- $\varphi(a_1 \odot a_2, b_1 \odot b_2) \geq \min\{\varphi(a_1, b_1); \varphi(a_2, b_2)\}$;
- $\varphi(a_1 \to a_2, b_1 \to b_2) \geq \min\{\varphi(a_1, b_1); \varphi(a_2, b_2)\}$;
- $\widehat{\varphi}(a_1 \sqcup a_2, b_1 \sqcup b_2) \geq \min\{\varphi(a_1, b_1); \varphi(a_2, b_2)\}$;
- $\widehat{\varphi}(a_1 \sqcap a_2, b_1 \sqcap b_2) \geq \min\{\varphi(a_1, b_1); \varphi(a_2, b_2)\}$;
- $\varphi(T, T') = 1$.

Moreover, φ is said to be complete if the following conditions hold:

(1) if $\bigvee_{y\in Y} \varphi(a, y) = 1$, then there exists $y \in Y$ such that $\varphi(a, y) = 1$
(2) if $\bigvee_{x\in X} \varphi(x, b) = 1$, then there exists $x \in X$ such that $\varphi(x, b) = 1$.

Theorem 5.3. *Let $h : M \to M'$ be a homomorphism between residuated multilattices, and μ a fuzzy filter of M', then the inverse image of μ denoted by $h^{-1}(\mu)$ is a fuzzy filter of M, where $\forall\ x \in M$, $h^{-1}(\mu)(x) = \mu(h(x))$.*

Proof. • For all $a \in M$, we have $h(a) \leq T' = h(T)$. Because μ is order preserving, we have $\mu(h(a)) \leq \mu(T') = \mu(h(T))$. i.e., $h^{-1}(\mu)(a) \leq h^{-1}(\mu)(T)$.

• Let $a, b \in M$. $\min\{h^{-1}(\mu)(a), h^{-1}(\mu)(a \to b)\} = \min\{\mu(h(a)), \mu(h(a \to b))\} = \min\{\mu(h(a)), \mu(h(a) \to h(b))\} \leq \mu(h(b)) = h^{-1}(\mu)(b)$.

• Let $a, b \in M$. We will prove that $\min\left\{\inf_{t\in(a\sqcup b)\to b} h^{-1}(\mu)(t), \inf_{t\in a\to(a\sqcap b)} h^{-1}(\mu)(t))\right\} = h^{-1}(\mu)(a \to b)$.

Because μ is a fuzzy filter of M', we have $h^{-1}(\mu)(a \to b) = \mu(h(a) \to h(b)) = \min\left\{\inf\limits_{s\in(h(a)\sqcup h(b))\to h(b)} \mu(s), \inf\limits_{s\in h(a)\to(h(a)\sqcap h(b))} \mu(s)\right\}$. Since, $(h(a) \sqcup h(b)) \to h(b) \supseteq h((a \sqcup b) \to b)$ and $h(a) \to (h(a) \sqcap h(b)) \supseteq h(a \to (a \sqcap b))$, we have $\inf\limits_{s\in(h(a)\sqcup h(b))\to h(b)} \mu(s) \le \inf\limits_{s\in h((a\sqcup b)\to b)} \mu(s) = \inf\limits_{t\in(a\sqcup b)\to b} \mu(h(t))$ and $\inf\limits_{s\in h(a)\to(h(a)\sqcap h(b))} \mu(s) \le \inf\limits_{s\in h(a\to(a\sqcap b))} \mu(s) = \inf\limits_{t\in a\to(a\sqcap b)} \mu(h(t))$. Then, $h^{-1}(\mu)(a \to b) \le \min\left\{\inf\limits_{t\in(a\sqcup b)\to b} h^{-1}(\mu)(t), \inf\limits_{t\in a\to(a\sqcap b)} h^{-1}(\mu)(t))\right\}$. By the Proposition 1.10, $a \to b \in (a \sqcup b) \to b$ and $a \to b \in a \to (a \sqcup b)$. Then, $\inf\limits_{t\in(a\sqcup b)\to b} h^{-1}(\mu)(t) \le h^{-1}(\mu)(a \to b)$ and $\inf\limits_{t\in a\to(a\sqcup b)} h^{-1}(\mu)(t) \le h^{-1}(\mu)(a \to b)$. Therefore, $\min\left\{\inf\limits_{t\in(a\sqcup b)\to b} h^{-1}(\mu)(t), \inf\limits_{t\in a\to(a\sqcap b)} h^{-1}(\mu)(t))\right\} = h^{-1}(\mu)(a \to b)$. □

Let us concentrate now on the relationship between fuzzy homomorphism and congruences.

Definition 5.4 ([7]). Let φ be a fuzzy homomorphism and h a homomorphism from M to M'. A fuzzy kernel relation induced by φ on M, denoted by $\rho_\varphi \in [0,1]^{M\times M}$, is defined as $\rho_\varphi(a, a') = \varphi(a, h(a'))$, where h is the crisp description of φ.

We adopt here the term kernel as an extension of the crisp case because of the inequality $\rho_\varphi(a, a') \ge \min\{\varphi(a, b), \varphi(a', b)\}$.

Proposition 5.5 ([7]). *Let φ a perfect function from A to B. For all a, $a' \in A$, $\rho_\varphi(a, a') = \bigvee\limits_{b\in B} \min\{\varphi(a, b),\ \varphi(a', b)\}$.*

The following Lemma shows that the inequality $\rho_\varphi(a, a') \ge \min\{\varphi(a, b), \varphi(a', b)\}$ is still valid for the power set extension of ρ_φ.

Lemma 5.6. *For all $A, A' \in 2^M$ and $B \in 2^{M'}$, $\widehat{\rho}_\varphi(A, A') \ge \min\{\widehat{\varphi}(A, B), \widehat{\varphi}(A', B)\}$.*

Proof. Let $A, A' \in 2^M$ and $B \in 2^{M'}$. We have

$$\begin{aligned}\min\{\widehat{\varphi}(A, B), \widehat{\varphi}(A', B)\} &= \min\left\{\bigwedge_{a\in A}\bigvee_{b\in B} \varphi(a, b), \bigwedge_{b\in B}\bigvee_{a\in A} \varphi(a, b), \bigwedge_{a'\in A'}\bigvee_{b\in B} \varphi(a', b), \bigwedge_{b\in B}\bigvee_{a'\in A'} \varphi(a', b)\right\}\\ &= \min\left\{\bigwedge_{a\in A}\bigvee_{b\in B} \varphi(a, b), \bigwedge_{b\in B}\bigvee_{a'\in A'} \varphi(a', b)\right\} \wedge \min\left\{\bigwedge_{b\in B}\bigvee_{a\in A} \varphi(a, b), \bigwedge_{a'\in A'}\bigvee_{b\in B} \varphi(a', b)\right\}.\end{aligned}$$

Now, by idempotency and distributivity, we have that $\min\{\widehat{\varphi}(A, B), \widehat{\varphi}(A', B)\}$ equals

$$\bigwedge_{a\in A}\bigvee_{b\in B}\left(\min\{\varphi(a, b), \bigwedge_{b'\in B}\bigvee_{a'\in A'} \varphi(a', b')\}\right) \wedge \bigvee_{b\in B}\bigwedge_{a'\in A'}\left(\min\{\varphi(a', b), \bigwedge_{b'\in B}\bigvee_{a\in A} \varphi(a, b')\}\right).$$

As $\bigwedge\limits_{b'\in B}\bigvee\limits_{a'\in A'} \varphi(a', b') \le \bigvee\limits_{a'\in A'} \varphi(a', b)$ and $\bigwedge\limits_{b'\in B}\bigvee\limits_{a\in A} \varphi(a, b') \le \bigvee\limits_{a\in A} \varphi(a, b)$, for all $b \in B$, we have that

$$\begin{aligned}\min\{\widehat{\varphi}(A, B), \widehat{\varphi}(A', B)\} &\le \bigwedge_{a\in A}\bigvee_{b\in B}\left(\min\{\varphi(a, b), \bigvee_{a'\in A'} \varphi(a', b)\}\right) \wedge \bigvee_{b\in B}\bigwedge_{a'\in A'}\left(\min\{\varphi(a', b), \bigvee_{a\in A} \varphi(a, b)\}\right)\\ &= \bigwedge_{a\in A}\bigvee_{b\in B}\bigvee_{a'\in A'} (\min\{\varphi(a, b), \varphi(a', b)\}) \wedge \bigvee_{b\in B}\bigwedge_{a'\in A'}\bigvee_{a\in A} \min\{\varphi(a', b), \varphi(a, b)\}\end{aligned}$$

$$= \bigwedge_{a\in A}\bigvee_{a'\in A'}\left(\bigvee_{b\in B}\min\{\varphi(a,b),\varphi(a',b)\}\right)\wedge\bigwedge_{a'\in A'}\bigvee_{a\in A}\left(\bigvee_{b\in B}\min\{\varphi(a',b),\varphi(a,b)\}\right).$$

Since $\rho_\varphi(a,a') = \bigvee_{b\in B}\min\{\varphi(a,b),\ \varphi(a',b)\}$ we obtain

$\min\{\widehat{\varphi}(A,B),\widehat{\varphi}(A',B)\} \leq \rho_\varphi(A,A')$. □

Theorem 5.7. *Let φ be a fuzzy homomorphism from M and M'. The fuzzy kernel relation ρ_φ is a fuzzy congruence relation.*

Proof. Let us see the compatibility with the hyperoperation $\sqcup$.

Firstly, we will prove that $\rho_\varphi(a,a) = 1$. From Proposition 5.5 $\rho_\varphi(a,a) = \bigvee_{b\in B}\varphi(a,b)$ because φ is a perfect fuzzy function, there exists $b_0 \in B$ such that $\varphi(a,b_0) = 1$. Then, $\rho_\varphi(a,a) = 1$.

$$\begin{aligned}
\widehat{\rho}_\varphi(a_1\sqcup a_3,\ a_2\sqcup a_4) &\geq \min\{\widehat{\varphi}(a_1\sqcup a_3,\ h(a_2)\sqcup h(a_4)),\widehat{\varphi}(a_2\sqcup a_4,\ h(a_2)\sqcup h(a_4))\} \text{ by Lemma 5.6}\\
&\geq \min\{\varphi(a_1,h(a_2)),\ \varphi(a_2,h(a_2)),\ \varphi(a_3,h(a_4)),\ \varphi(a_4,h(a_4))\}\\
&\geq \min\{\rho_\varphi(a_1,a_2),\ \rho_\varphi(a_2,a_2),\ \rho_\varphi(a_3,a_4),\ \rho_\varphi(a_4,a_4)\}\ \textit{as}\ \rho_\varphi(a,a') = \varphi(a,h(a'))\\
&\geq \min\{\rho_\varphi(a_1,a_2),\rho_\varphi(a_3,a_4)\}.
\end{aligned}$$

The compatibility with $\sqcup$ is similar. Let us see the compatibility with $\odot$.

$$\begin{aligned}
\rho_\varphi(a_1\odot a_3,a_2\odot a_4) &= \varphi(a_1\odot a_3,h(a_2\odot a_4)),\\
&= \varphi(a_1\odot a_3,h(a_2)\odot h(a_4)),\\
&\geq \min\{\varphi(a_1,h(a_2)),\ \varphi(a_3,h(a_4))\}\\
&\geq \min\{\rho_\varphi(a_1,a_2),\rho_\varphi(a_3,a_4)\}.
\end{aligned}$$

The compatibility with $\to$ is similar.

Thus, ρ_φ is a fuzzy congruence relation on M. □

Conclusion

In this paper, we have initiated the study of fuzzy filters in residuated multilattices and established many important properties.

As future work, given a residuated multilattices $(M,\leq,\odot,\to,\sqcup,\sqcap,T)$, one can study the nature and properties of the induced power set 2^M, endowed with the extended operation $\odot$, $\to$, $\sqcup$ and $\sqcap$.

Lattices are the most general algebraic structure of truth-values considered in the theory of fuzzy concept analysis to evaluate the attributes and objets. It will also be interesting to use the residuated multilattice as underlying set of truth-values for these attributes and objets.

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