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On fuzzy Baire spaces and fuzzy semi-closed sets

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Abstract. In this paper we discuss several characterizations of fuzzy Baire spaces in terms of fuzzy semi-closed and fuzzy semi-open sets. The conditions for fuzzy first category sets to be fuzzy semi-closed sets in a fuzzy topological space, are also established.

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1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by L.A.Zadeh [16] in his classical paper in the year 1965. Thereafter the paper of C.L.Chang [6] in 1968 paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed.

In 1899, Rene Louis Baire [3] introduced the concepts of first and second category sets in his doctoral thesis. To define first category sets, Baire relied on Cantor's definition of dense sets and Paul du Bois-Reymond's definition of nowhere dense sets. In 1913, A.Denjoy introduced residual sets which are the complements of first category sets. In classical topology, Baire space named in honor of Rene Louis Baire, was first introduced in Bourbaki's [5] Topologie Generale Chapter IX. The concepts of Baire spaces have been studied extensively in classical topology in [2, 7, 8, 9, 10, 11]. The concept of Baire spaces in fuzzy setting was introduced and studied by G.Thangaraj and S.Anjalmose in [12]. In 1981, K. K. Azad [1] introduced and studied the concepts of fuzzy semi-open sets and fuzzy semi-closed sets in fuzzy topological spaces. The purpose of this paper is to study several characterizations of fuzzy Baire spaces in terms of fuzzy semi-closed and fuzzy semi-open sets. The conditions for fuzzy first category sets to be fuzzy semi-closed sets in a fuzzy topological space, are also established in this paper.

2. Preliminaries

By a fuzzy topological space we shall mean a non-empty set X together with a fuzzy topology T (in the sense of Chang) and denote it by (X, T).

Definition 2.1 ([6]). Let λ and μ be any two fuzzy sets in a fuzzy topological space (X, T). Then we define

(i) $\lambda \vee \mu : X \to [0, 1]$ as follows : $(\lambda \vee \mu)(x) = Max\{\lambda(x), \mu(x)\}$ where $x \in X$,
(ii) $\lambda \wedge \mu : X \to [0, 1]$ as follows : $(\lambda \wedge \mu)(x) = Min\{\lambda(x), \mu(x)\}$ where $x \in X$,
(iii) $\mu = \lambda^c \Leftrightarrow \mu(x) = 1 - \lambda(x)$ where $x \in X$.

More generally, for a family $\{\lambda_i / i \in I\}$ of fuzzy sets in (X, T), the union $\psi = \vee_i \lambda_i$ and intersection $\delta = \wedge_i \lambda_i$ are defined respectively as $\psi(x) = sup_i\{\lambda_i(x), x \in X\}$ and $\delta(x) = inf_i\{\lambda_i(x), x \in X\}$.

Definition 2.2 ([1]). Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). We define the interior $int(\lambda)$ and the closure $cl(\lambda)$ of λ as follows:

(i) $int(\lambda) = \vee\{\mu / \mu \leq \lambda, \mu \in T\}$
(ii) $cl(\lambda) = \wedge\{\mu / \lambda \leq \mu, 1 - \mu \in T\}$.

Lemma 2.3 ([1]). *Let λ be any fuzzy set in a fuzzy topological space (X, T). Then $1 - cl(\lambda) = int(1 - \lambda)$ and $1 - int(\lambda) = cl(1 - \lambda)$.*

Definition 2.4 ([14]). A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy dense if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$. That is., $cl(\lambda) = 1$.

Definition 2.5 ([14]). A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is., $intcl(\lambda) = 0$.

Definition 2.6 ([4]). Let (X, T) be a fuzzy topological space and λ be a fuzzy set in X. Then λ is called a fuzzy G_δ-set if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, for each $\lambda_i \in T$.

Definition 2.7 ([4]). Let (X, T) be a fuzzy topological space and λ be a fuzzy set in X. Then λ is called a fuzzy F_σ-set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, for each $1 - \lambda_i \in T$.

Definition 2.8 ([14]). A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy first category set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy second category.

Definition 2.9 ([12]). Let λ be a fuzzy first category set in a fuzzy topological space (X, T). Then $1 - \lambda$ is called a fuzzy residual set in (X, T).

Definition 2.10 ([14]). A fuzzy topological space (X, T) is called a fuzzy first category space if $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Any fuzzy topological space which is not of fuzzy first category, is said to be of fuzzy second category.

Lemma 2.11 ([1]). *For a family $\mathscr{A} = \{\lambda_\alpha\}$ of fuzzy sets of a fuzzy space X, $\vee\big(cl(\lambda_\alpha)\big) \leq cl\big(\vee(\lambda_\alpha)\big)$. In case $\mathscr{A}$ is a finite set, $\vee\big(cl(\lambda_\alpha)\big) = cl\big(\vee(\lambda_\alpha)\big)$. Also $\vee\big(int(\lambda_\alpha)\big) \leq int\big(\vee(\lambda_\alpha)\big)$.*

Definition 2.12 ([1]). A fuzzy set λ in a fuzzy topological space is called

(1) fuzzy semi-closed if and only if $int[cl(\lambda)] \leq \lambda$;
(2) fuzzy semi-open if and only if $\lambda \leq cl[int(\lambda)]$.

Definition 2.13 ([4]). A fuzzy topological space (X,T) is called a fuzzy submaximal space if $cl(\lambda) = 1$, for any non-zero fuzzy set λ in (X,T), then $\lambda \in T$.

Definition 2.14 ([15]). A fuzzy topological space (X,T) is said to be a fuzzy strongly irresolvable space if $cl[int(\lambda)] = 1$, for each fuzzy dense set λ in (X,T).

3. Fuzzy baire spaces and fuzzy semi-closed sets

Definition 3.1 ([12]). Let (X,T) be a fuzzy topological space. Then (X,T) is called a fuzzy Baire space if $int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T).

Theorem 3.2 ([12]). *If a non-zero fuzzy set λ in a fuzzy topological space (X,T), is a fuzzy nowhere dense set, then λ is a fuzzy semi-closed set in (X,T).*

Theorem 3.2 enables us to characterize a fuzzy first category set in terms of fuzzy semi-closed sets in a fuzzy topological space.

Proposition 3.3. *If λ is a fuzzy first category set in a fuzzy topological space (X,T), then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where (λ_i)'s are fuzzy semi-closed sets in (X,T).*

Proof. Let λ be a fuzzy first category set in (X,T). Then, $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). By Theorem 3.2, the fuzzy nowhere dense sets (λ_i)'s are fuzzy semi-closed sets in (X,T) and hence we have $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy semi-closed sets in (X,T). □

Theorem 3.2 enables us to characterize a fuzzy residual set in terms of fuzzy semi-open sets in a fuzzy topological space.

Proposition 3.4. *If λ is a fuzzy residual set in a fuzzy topological space (X,T), then $\lambda = \wedge_{i=1}^{\infty}(\mu_i)$, where (μ_i)'s are fuzzy semi-open sets in (X,T).*

Proof. Let λ be a fuzzy residual set in (X,T).Then, $1-\lambda$ is a fuzzy first category set in (X,T) and hence $1-\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). Then, $\lambda = 1 - \vee_{i=1}^{\infty}(\lambda_i) = \wedge_{i=1}^{\infty}(1-\lambda_i)$. By Theorem 3.2, the fuzzy nowhere dense sets (λ_i)'s are fuzzy semi-closed sets in (X,T) and hence $(1-\lambda_i)$'s are fuzzy semi-open sets in (X,T). Let $\mu_i = 1-\lambda_i$. Then we have $\lambda = \wedge_{i=1}^{\infty}(\mu_i)$, where (μ_i)'s are fuzzy semi-open sets in (X,T). □

Proposition 3.5. *Let λ be a fuzzy semi-closed set in a fuzzy topological space (X,T). If λ is not a fuzzy nowhere dense set in (X,T), then $int(\lambda) \neq 0$.*

Proof. Since λ is a fuzzy semi-closed set in (X,T), we have int$[cl(\lambda)] \leq \lambda$. If λ is not a fuzzy nowhere dense set in (X,T), then $int[cl(\lambda)] \neq 0$. Let $int[cl(\lambda)] = \mu$ and hence μ is a fuzzy open set in (X,T). Now $\mu \leq \lambda$, implies that $int(\lambda) \neq 0$. □

Proposition 3.6. *Let (X, T) be a fuzzy topological space. Then a fuzzy set λ is a fuzzy nowhere dense set in (X, T) if and only if $1 - cl(\lambda)$ is a fuzzy open and fuzzy dense set in (X, T).*

Proof. Let λ be a fuzzy nowhere dense set in (X, T). Then we have $int[cl(\lambda)] = 0$. Now $cl(\lambda)$ is a fuzzy closed set in (X, T), implies that $1 - cl(\lambda)$ is a fuzzy open set in (X, T). Also $cl[1 - cl(\lambda)] = 1 - int[cl(\lambda)] = 1 - 0 = 1$, implies that $1 - cl(\lambda)$ is a fuzzy dense set in (X, T). Thus, $1 - cl(\lambda)$ is a fuzzy open and fuzzy dense set in (X, T).

Conversely, let $1 - cl(\lambda)$ be a fuzzy open and fuzzy dense set in (X, T). Then, $cl[1 - cl(\lambda)] = 1$, implies that $1 - int[cl(\lambda)] = 1$ and hence we have $int[cl(\lambda)] = 0$. Therefore λ is a fuzzy nowhere dense set in (X, T). □

Proposition 3.7. *Let (X, T) be a fuzzy topological space. If $[1 - cl(\lambda)]$ is a fuzzy open and fuzzy dense set, for a fuzzy set λ in (X, T), then λ is a fuzzy semi-closed set in (X, T).*

Proof. Let $1 - cl(\lambda)$ be a fuzzy open and fuzzy dense set, for a fuzzy set λ in (X, T). Then, by Proposition 3.6, λ is a fuzzy nowhere dense set in (X, T) and hence by Theorem 3.2, λ is a fuzzy semi-closed set in (X, T). □

Theorem 3.8 ([12])**.** *Let (X, T) be a fuzzy topological space. Then the following are equivalent:*
(1) (X, T) is a fuzzy Baire space.
(2) $int(\lambda) = 0$ for every fuzzy first category set λ in (X, T).
(3) $cl(\mu) = 1$ for every fuzzy residual set μ in (X, T).

The following propositions give the conditions for fuzzy first category sets to be fuzzy semi-closed sets in a fuzzy topological space.

Proposition 3.9. *If a fuzzy topological space (X, T) is a fuzzy Baire and fuzzy submaximal space and if λ is a fuzzy first category set in (X, T), then λ is a fuzzy semi-closed set in (X, T).*

Proof. Let λ be a fuzzy first category set in (X, T). Then, $1-\lambda$ is a fuzzy residual set in (X, T). Since (X, T) is a fuzzy Baire space, by Theorem 3.8, we have $cl(1-\lambda) = 1$ in (X, T). Again since (X, T) is a fuzzy submaximal space, the fuzzy dense set $1 - \lambda$ is a fuzzy open set in (X, T). Then λ is a fuzzy closed set in (X, T) and hence $cl(\lambda) = \lambda$ in (X, T). Since (X, T) is a fuzzy Baire space, by Theorem 3.8, we have $int(\lambda) = 0$. Now $intcl(\lambda) = int(\lambda) = 0$ and hence λ is a fuzzy nowhere dense set in (X, T). Then by Theorem 3.2, the fuzzy nowhere dense set λ is a fuzzy semi-closed set in (X, T). □

Proposition 3.10. *If a fuzzy topological space (X, T) is a fuzzy Baire and fuzzy strongly irresolvable space and if λ is a fuzzy first category set in (X, T), then λ is a fuzzy semi-closed set in (X, T).*

Proof. Let λ be a fuzzy first category set in (X, T). Then, $1-\lambda$ is a fuzzy residual set in (X, T). Since (X, T) is a fuzzy Baire space, by Theorem 3.8, we have $cl(1-\lambda) = 1$ in (X, T). Again since (X, T) is a fuzzy strongly irresolvable space, for the fuzzy dense set $1-\lambda$, we have $cl(int(1-\lambda)) = 1$ in (X, T). Then $1-int[cl(\lambda)] = 1$, implies that $int(cl(\lambda)) = 0$ and hence λ is a fuzzy nowhere dense set in (X, T). Then, by Theorem 3.2, the fuzzy nowhere dense set λ is a fuzzy semi-closed set in (X, T). □

Proposition 3.11. *If a fuzzy topological space (X, T) is a fuzzy Baire space, then $int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy semi-closed sets in (X, T).*

Proof. Let (X, T) be a fuzzy Baire space. Then, by Theorem 3.8, we have $int(\lambda) = 0$ for a fuzzy first category set λ in (X, T) and by Proposition 3.3, we have $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$ where (λ_i)'s are fuzzy semi-closed sets in (X, T). Therefore $int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy semi-closed sets in (X, T). □

Proposition 3.12. *If a fuzzy topological space (X, T) is a fuzzy Baire space, then $cl(\wedge_{i=1}^{\infty}(\mu_i)) = 1$, where (μ_i)'s are fuzzy semi-open sets in (X, T).*

Proof. Let (X, T) be a fuzzy Baire space. Then, by Theorem 3.8, we hace $cl(\lambda) = 1$ for a fuzzy residual set λ in (X, T) and by Proposition 3.4, we have $\lambda = \wedge_{i=1}^{\infty}(\mu_i)$,where (μ_i)'s are fuzzy semi-open sets in (X, T). Therefore $cl(\wedge_{i=1}^{\infty}(\mu_i)) = 1$ where (μ_i)'s are fuzzy semi-open sets in (X, T). □

Proposition 3.13. *If a fuzzy topological space (X, T) is a fuzzy Baire space, then $cl(\wedge_{i=1}^{\infty}(\mu_i)) = 1$, where (μ_i)'s are fuzzy dense sets in (X, T).*

Proof. Let (X, T) be a fuzzy Baire space. Then by Proposition 3.12, $cl(\wedge_{i=1}^{\infty}(\mu_i)) = 1$, where (μ_i)'s are fuzzy semi-open sets in (X, T). Now $cl(\wedge_{i=1}^{\infty}(\mu_i)) \leq \wedge_{i=1}^{\infty}cl(\mu_i)$, implies that $1 \leq \wedge_{i=1}^{\infty}cl(\mu_i)$. That is, $\wedge_{i=1}^{\infty}cl(\mu_i) = 1$ and hence we have $cl(\mu_i) = 1$. Therefore $cl(\wedge_{i=1}^{\infty}(\mu_i)) = 1$, where (μ_i)'s are fuzzy dense sets in (X, T). □

Proposition 3.14. *If a fuzzy topological space (X, T) is a fuzzy Baire space, then $cl(\wedge_{i=1}^{\infty}(\mu_i)) = 1$, where the fuzzy sets (μ_i)'s are such that $cl[int(\mu_i)] = 1$ in (X, T).*

Proof. Let (X, T) be a fuzzy Baire space. Then, by Proposition 3.12, $cl(\wedge_{i=1}^{\infty}(\mu_i)) = 1$, where (μ_i)'s are fuzzy semi-open sets in (X, T). Now $cl(\wedge_{i=1}^{\infty}(\mu_i)) \leq \wedge_{i=1}^{\infty}cl(\mu_i)$, implies that $1 \leq \wedge_{i=1}^{\infty}cl(\mu_i)$. That is, $\wedge_{i=1}^{\infty}cl(\mu_i) = 1$ and hence we have $cl(\mu_i) = 1$. Since (μ_i)'s are fuzzy semi-open sets in (X, T), $\mu_i \leq cl[int(\mu_i)]$. Then $cl(\mu_i) \leq cl[cl[int(\mu_i)]]$ and hence we have $1 \leq cl[int(\mu_i)]$. That is, $cl[int(\mu_i)] = 1$. Therefore, we have $cl(\wedge_{i=1}^{\infty}(\mu_i)) = 1$, where the fuzzy sets (μ_i)'s are such that $cl[int(\mu_i)] = 1$ in (X, T). □

Proposition 3.15. *If a fuzzy topological space (X, T) is a fuzzy first category space, then $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where (λ_i)'s are fuzzy semi-closed sets in (X, T).*

Proof. Let (X, T) be a fuzzy first category space. Then, $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). By Theorem 3.2, the fuzzy nowhere dense sets (λ_i)'s are fuzzy semi-closed sets in (X, T) and hence we have $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where (λ_i)'s are fuzzy semi-closed sets in (X, T). □

Theorem 3.16 ([13])**.** *If $\lambda \leq \mu$ and μ is a fuzzy nowhere dense set in a fuzzy topological space (X, T), then λ is also a fuzzy nowhere dense set in (X, T).*

Proposition 3.17. *If $\lambda \leq \mu$ and μ is a fuzzy first category set in a fuzzy topological space (X, T), then λ is a fuzzy first category set in (X, T).*

Proof. Let μ be a fuzzy first category set in (X, T). Then, $\mu = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Now $\lambda \leq \mu$, implies that $\lambda = \lambda \wedge \mu = \lambda \wedge [\vee_{i=1}^{\infty}(\lambda_i)] = \vee_{i=1}^{\infty}[\lambda \wedge (\lambda_i)]$. Since $\lambda \wedge \lambda_i \leq \lambda_i$ and (λ_i)'s are fuzzy nowhere dense sets in (X, T), by Theorem 3.16, we have $[\lambda \wedge (\lambda_i)]$'s are fuzzy nowhere dense sets in (X, T). Hence $\lambda = \vee_{i=1}^{\infty}[\lambda \wedge (\lambda_i)]$ where $[\lambda \wedge (\lambda_i)]$'s are fuzzy nowhere dense sets in (X, T), implies that λ is a fuzzy first category set in (X, T). □

Proposition 3.18. *If $\lambda \leq \mu$ and λ is a fuzzy residual set in a fuzzy topological space (X, T), then μ is also a fuzzy residual set in (X, T).*

Proof. Let λ be a fuzzy residual set in (X, T). Then, $1 - \lambda$ is a fuzzy first category set in (X, T). Now $\lambda \leq \mu$, implies that $1 - \lambda \geq 1 - \mu$. By Proposition 3.17, $1 - \mu$ is a fuzzy first category set in (X, T) and hence μ is a fuzzy residual set in (X, T). □

Proposition 3.19. *Let (X, T) be a fuzzy topological space. Then the following are equivalent:*

(i) *(X, T) is a fuzzy Baire space.*
(ii) *Each non-zero fuzzy open set is a fuzzy second category set in (X, T).*

Proof. (1) $\Longrightarrow$ (2). Let (X, T) be a fuzzy Baire space. Suppose that λ is a non-zero fuzzy open set in (X, T) such that $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Then, $int(\lambda) = int[\vee_{i=1}^{\infty}(\lambda_i)]$. Since λ is a non-zero fuzzy open set in (X, T), we have $int(\lambda) = \lambda \neq 0$ and hence $int[\vee_{i=1}^{\infty}(\lambda_i)] \neq 0$, a contradiction to (X, T) being a fuzzy Baire space. Hence, for the fuzzy open set λ in (X, T), we have $\lambda \neq \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). That is, no non-zero fuzzy open set is a fuzzy first category set in (X, T) and hence each non-zero fuzzy open set is a fuzzy second category set in (X, T).

(2) $\Longrightarrow$ (1). Let (X, T) be a fuzzy topological space in which each non-zero fuzzy open set is a fuzzy second category set in (X, T). We claim that (X, T) is a fuzzy Baire space. Suppose not. Then, by Theorem 3.8, $int(\lambda) \neq 0$, for a fuzzy first category set λ in (X, T) and hence there exists a non-zero fuzzy open set μ in (X, T) such that $\mu \leq \lambda$. Since λ is a fuzzy first category set in (X, T) and $\mu \leq \lambda$, by Proposition 3.17, μ is also a fuzzy first category set in (X, T), a contradiction to the hypothesis. Hence we must have $int(\lambda) = 0$, for each fuzzy first category set λ in (X, T) and therefore by Theorem 3.8, (X, T) is a fuzzy Baire space. □

Theorem 3.20 ([13]). *If λ is a fuzzy dense and fuzzy G_δ-set in a fuzzy topological space (X, T), then λ is a fuzzy residual set in (X, T).*

Proposition 3.21. *If a fuzzy topological space (X, T) is a fuzzy Baire space, then a fuzzy set λ in (X, T) is a fuzzy residual set if and only if there exists a fuzzy dense and fuzzy G_δ-set μ in (X, T) such that $\mu \leq \lambda$.*

Proof. Let (X, T) be a fuzzy Baire space and λ be a fuzzy residual set in (X, T). Then $1-\lambda$ is a fuzzy first category set in (X, T) and hence $1-\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Let $\mu = \wedge_{i=1}^{\infty}[1 - cl(\lambda_i)]$. Then μ is a fuzzy G_δ-set in (X, T). Now $1 - cl(\lambda_i) \leq 1 - (\lambda_i)$, implies that $\wedge_{i=1}^{\infty}[1 - cl(\lambda_i)] \leq \wedge_{i=1}^{\infty}(1 - \lambda_i)$. Hence, we have $\wedge_{i=1}^{\infty}[1 - cl(\lambda_i)] \leq 1 - \vee_{i=1}^{\infty}(\lambda_i)$. Thus $\mu \leq 1 - (1 - \lambda)$. That is, $\mu \leq \lambda$. Since (λ_i)'s are fuzzy nowhere dense sets in (X, T), $[cl(\lambda_i)]$'s are fuzzy nowhere dense sets in (X,T). [Since $int[cl[cl(\lambda)]] = int[cl(\lambda)] = 0$ and $cl[cl(\lambda)] = cl(\lambda)$]. Now $\mu = \wedge_{i=1}^{\infty}[1 - cl(\lambda_i)]$, implies that $1 - \mu = \vee_{i=1}^{\infty} cl(\lambda_i)$ and hence $1 - \mu$ is a fuzzy first category set in (X,T). Since (X,T) is a fuzzy Baire space, by theorem 3.8, $int(1 - \mu) = 0$ and $1 - cl(\mu) = int(1 - \mu)$, implies that $1 - cl(\mu) = 0$. Thus, μ is a fuzzy dense and fuzzy G_δ-set in (X,T) such that $\mu \leq \lambda$.

Conversely, let μ be a fuzzy dense and fuzzy G_δ-set in (X,T) such that $\mu \leq \lambda$. By Theorem 3.20, μ is a fuzzy residual set in (X,T). Since $\mu \leq \lambda$ and μ is a fuzzy residual set in (X,T), by proposition 3.18, λ is a fuzzy residual set in (X,T). □

Proposition 3.22. *If a fuzzy topological space (X,T) is a fuzzy Baire space, then a fuzzy set δ in (X,T) is a fuzzy first category set if and only if there exists a fuzzy F_σ-set γ with $int(\gamma) = 0$, in (X,T) such that $\delta \leq \gamma$.*

Proof. Let (X,T) be a fuzzy Baire space and δ be a fuzzy first category set in (X,T). Then, $1 - \delta$ is a fuzzy residual set in (X,T). By Proposition 3.21, there exists a dense and fuzzy G_δ-set μ in (X,T) such that $\mu \leq 1 - \delta$. Then we have $\delta \leq 1 - \mu$. Since μ is a fuzzy G_δ-set in (X,T), $1 - \mu$ is a fuzzy F_σ-set in (X,T). Also $cl(\mu) = 1$, implies that $int(1 - \mu) = 1 - cl(\mu) = 1 - 1 = 0$. Let $\gamma = 1 - \mu$, then γ is a F_σ-set with $int(\gamma) = 0$, in (X,T) such that $\delta \leq \gamma$.

Conversely, let γ be a fuzzy F_σ-set with $int(\gamma) = 0$, in (X,T) such that $\delta \leq \gamma$. Then $1 - \gamma$ is a fuzzy dense (since $cl(1 - \gamma) = 1 - int(\gamma) = 1 - 0 = 1$) and fuzzy G_δ-set in (X,T) such that $1 - \gamma \leq 1 - \delta$. Then, by Proposition 3.21, $1 - \delta$ is a fuzzy residual set in (X,T) and hence δ is a fuzzy first category set in (X,T). □

Remark 3.23. If λ is a fuzzy semi-open set in a fuzzy topological space (X,T), then $cl(\lambda)$ is a fuzzy regular closed set in (X,T).

For, λ is a fuzzy semi-open set in (X,T), implies that $\lambda \leq cl[int(\lambda)]$. Then, we have $cl(\lambda) \leq cl(cl[int(\lambda)]) = cl[int(\lambda)] \leq cl[intcl(\lambda)]$ and $cl[intcl(\lambda)] \leq cl[cl(\lambda)] = cl(\lambda)$. This implies that $cl[int(cl(\lambda))] = cl(\lambda)$ and hence $cl(\lambda)$ is a fuzzy regular closed set in (X,T).

Proposition 3.24. *If a fuzzy topological space (X,T) is a fuzzy Baire space, then $\wedge_{i=1}^{\infty}(cl(\mu_i)) = 1$, where $(cl(\mu_i))$'s are fuzzy regular closed sets in (X,T).*

Proof. Let (X,T) be a fuzzy Baire space. Then, $int[\vee_{i=1}^{\infty}(\lambda_i)] = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). By Theorem 3.2, the fuzzy nowhere dense sets (λ_i)'s are fuzzy semi-closed sets in (X,T) and hence $(1 - \lambda_i)$'s are fuzzy semi-open sets in (X,T) and by remark 3.23, $cl(1 - \lambda_i)$'s are fuzzy regular closed sets in (X,T). Now $int[\vee_{i=1}^{\infty}(\lambda_i)] = 0$, implies that $1 - int[\vee_{i=1}^{\infty}(\lambda_i)] = 1$. Then, $cl[\wedge_{i=1}^{\infty}(1 - \lambda_i)] = 1$. But $cl[\wedge_{i=1}^{\infty}(1 - \lambda_i)] \leq \wedge_{i=1}^{\infty}[cl(1 - \lambda_i)]$, implies that $\wedge_{i=1}^{\infty}[cl(1 - \lambda_i)] = 1$ and hence we have $cl(1 - \lambda_i) = 1$. Let $\mu_i = 1 - \lambda_i$. Then, $\wedge_{i=1}^{\infty}(cl(\mu_i)) = 1$, where $(cl(\mu_i))$'s are fuzzy regular closed sets in (X,T). □

4. Conclusions

In this paper, characterizations of fuzzy Baire spaces in terms of fuzzy semi-closed and fuzzy semi-open sets are studied. Also the characterization of fuzzy first category set in terms of fuzzy semi-closed sets and a characterization of fuzzy residual set in terms of fuzzy semi-open sets are established. The conditions for a fuzzy first category set to a fuzzy semi-closed set in a topological space are obtained. Characterization of a fuzzy residual set in a fuzzy Baire space is also established.

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