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Generalized smooth proximity spaces

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Abstract. In this paper, we define a smooth K-proximity space, a Leader and a Lodato smooth proximity spaces and study some of its properties. Furthermore, we construct Leader and Lodato smooth proximity structure using a given smooth K-proximity.

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1. Introduction

Šostak [21], introduced the fundamental concept of a 'fuzzy topological structure', as an extension of both crisp topology and Chang's fuzzy topology [3], indicating that not only the object were fuzzified, but also the axiomatics. Subsequently, Badard [2], introduced the concept of 'smooth topological space'. Chattopadhyay et al. [4] and Chattopadhyay and Samanta [5] re-introduced the same concept, calling it 'gradation of openess'. Ramadan [19] and his colleagues introduced a similar definition, namely, smooth topological space for lattice $L = [0, 1]$. Following Ramadan, several authors have re-introduced and further studied smooth topological space (cf. [4, 5, 6, 22]). Thus, the terms 'fuzzy topology', in Šostak's sense, 'gradation of openness' and 'smooth topology' are essentially referring to the same concept. In our paper, we adopt the term smooth topology.

The concept of proximity space was first described by Frigyes Riesz (1909) but ignored at the time. It was rediscovered and axiomatized by Efremovič under the name of infinitesimal space [7]. In addition to, Leader [14, 15] and Lodato [16, 17] have introduced a weaker axioms than those of Efremovič space. Kim et al. [12] introduced the concept of K-proximity as a generalization of the concept of proximity. Katsaras [10, 11] introduced and studied fuzzy proximity spaces. Park [18] introduced the concept of fuzzy K-proximity. Samanta [20] introduced the concept of gradations of fuzzy proximity. It was shown that this fuzzy proximity is more general than that of Artico and Moresco [1]. On the other hand, Ghanim et al. [9] introduced fuzzy proximity spaces with somewhat different definition of Samanta [20]. Fuzzy proximity theory was studied in many directions (cf. [8, 13, 23, 24, 25]). In this paper we introduce some generalization of the concept of the smooth proximity, precisely a 'smooth K-proximity', a 'Leader smooth proximity' and a 'Lodeto smooth proximity'. We also try to study some of its properties. In addition we introduced the fuzzy K-proximally continuous based on the smooth K-proximity. Furthermore, we construct Leader and Lodato smooth proximity structure using a given smooth K-proximity.

2. Preliminaries

Throughout this paper, let X be a non-empty set, $I = [0, 1]$, $I_0 = (0, 1]$, $I_1 = [0, 1)$ and I^X be the family of all fuzzy sets on X. For any $\mu_1, \mu_2 \in I^X$, $(\mu_1 \wedge \mu_2)(x) = \min\{\mu_1(x), \mu_2(x) : x \in X\}$ and $(\mu_1 \vee \mu_2)(x) = \max\{\mu_1(x), \mu_2(x) : x \in X\}$. For $\lambda \in I^X$, $\bar{1} - \lambda$ denotes the complement of λ. For $\alpha \in I$, $\bar{\alpha}(x) = \alpha \ \forall x \in X$. By $\bar{0}$ and $\bar{1}$, we denote constant maps on X with value 0 and 1, respectively. For $x \in X$ and $t \in I_0$, a fuzzy point x_t which takes t if $x = y$ and 0 otherwise, for all $y \in X$. Let $Pt(X)$ be a family of all fuzzy points in X. The fuzzy point x_t is said to be contained in a fuzzy set λ iff $\lambda(x) \geq t$. A fuzzy point x_t is said to be quasi-coincident with a fuzzy set λ, denoted by $x_t \ q \ \lambda$ if and only if $\lambda(x) + t > 1$. For $\mu, \lambda \in I^X$, μ is called quasi-coincident with λ, denoted by $\mu \ q \ \lambda$, if $\mu(x) + \lambda(x) > 1$ for some $x \in X$, otherwise we write $\mu \ \bar{q} \ \lambda$. For $\lambda_1, \lambda_2 \in I^X$, $\lambda_1 \leq \lambda_2$ if and only if $\lambda_1 \ \bar{q} \ \bar{1} - \lambda_2$. Also, $\lambda_1 \leq \lambda_2$ if and only if $(\forall x_t \in Pt(X))$ $(x_t \ q \ \lambda_1 \Longrightarrow x_t \ q \ \lambda_2)$.

Definition 2.1 ([2, 4, 19, 21]). A smooth topology on X is a mapping $\tau : I^X \to I$ which satisfies the following properties:

(1) $\tau(\bar{0}) = \tau(\bar{1}) = 1$,
(2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, $\forall \ \mu_1, \mu_2 \in I^X$,
(3) $\tau(\bigvee_{i \in J} \mu_i) \geq \bigwedge_{i \in J} \tau(\mu_i)$, for any $\{\mu_i : i \in J\} \subseteq I^X$.

The pair (X, τ) is called a smooth topological space. For $r \in I_0$, μ is an r-open fuzzy set of X if $\tau(\mu) \geq r$, and μ is an r-closed fuzzy set of X if $\tau(\bar{1} - \mu) \geq r$. Note, Šostak [21] used the term 'fuzzy topology' and Chattopadhyay [4], the term 'gradation of openness' for a smooth topology τ.

Subsequently, the fuzzy closure (resp. interior) for any fuzzy set in smooth topological space is given as follows:

Definition 2.2 ([5]). Let (X, τ) be a smooth topological space. For $\lambda \in I^X$ and $r \in I_0$, a fuzzy closure of λ is a mapping $C_\tau : I^X \times I_0 \to I^X$ defined as

$$C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X | \ \mu \geq \lambda, \ \tau(\bar{1} - \mu) \geq r\}. \tag{2.1}$$

And, a fuzzy interior of λ is a mapping $I_\tau : I^X \times I_0 \to I^X$ define as

$$I_\tau(\lambda, r) = \bigvee\{\mu \in I^X | \ \mu \leq \lambda, \ \tau(\mu) \geq r\}. \tag{2.2}$$

Definition 2.3 ([5]). A mapping $C : I^X \times I_0 \to I^X$ is called a fuzzy closure operator if, for $\lambda, \mu \in I^X$ and $r, s \in I_0$, the mapping C satisfies the following conditions:

(C1) $C(\bar{0}, r) = \bar{0}$,
(C2) $\lambda \leq C(\lambda, r)$,
(C3) $C(\lambda, r) \vee C(\mu, r) = C(\lambda \vee \mu, r)$,
(C4) $C(\lambda, r) \leq C(\lambda, s)$ if $r \leq s$,
(C5) $C(C(\lambda, r), r) = C(\lambda, r)$.

The fuzzy closure operator C generates a smooth topology $\tau_C : I^X \longrightarrow I$ given by

$$\tau_C(\lambda) = \bigvee \{r \in I |\ C(\bar{1} - \lambda, r) = \bar{1} - \lambda\}. \tag{2.3}$$

Definition 2.4 ([9])**.** A mapping $\delta : I^X \times I^X \longrightarrow I$ is said to be a gradation of proximity on X if it satisfies the following axioms:
(FP1) $\delta(\mu, \rho) = \delta(\rho, \mu)$.
(FP2) $\delta(\mu \vee \rho, \lambda) = \delta(\mu, \lambda) \vee \delta(\rho, \lambda)$.
(FP3) $\delta(\bar{1}, \bar{0}) = 0$.
(FP4) $\delta(\mu, \rho) < r \Longrightarrow \exists \lambda \in I^X$ such that $\delta(\mu, \lambda) < r, \delta(\bar{1} - \lambda, \rho) < r$.
(FP5) $\delta(\mu, \rho) \neq 1 \Longrightarrow \mu\ \bar{q}\ \rho$.
The pair (X, δ) is called a fuzzy proximity space.

Remark 2.5. In this paper the gradation of proximity δ on X referee to smooth proximity on X and the fuzzy proximity space (X, δ) referee to smooth proximity space.

Lemma 2.6 ([9])**.** *Let (X, δ) be a smooth proximity space. If $\delta(\mu, \lambda) \geq r$, $\mu \leq \mu_1$ and $\lambda \leq \lambda_1$, then $\delta(\mu_1, \lambda_1) \geq r$.*

Definition 2.7 ([19])**.** Let (X, τ) and (Y, τ^*) be smooth topological spaces. A mapping $f : (X, \tau) \longrightarrow (Y, \tau^*)$ is called fuzzy continuous if $\tau(f^{-1}(\mu)) \geq \tau^*(\mu)$ for all $\mu \in I^Y$.

Theorem 2.8 ([4])**.** *Let (X, τ) and (Y, τ^*) be smooth topological spaces. Then, a mapping $f : (X, \tau) \longrightarrow (Y, \tau^*)$ is fuzzy continuous map iff $f(C_\tau(\mu, r)) \leq C_{\tau^*}(f(\mu), r)$, for all $\mu \in I^X$, for all $r \in I_0$.*

3. Smooth K-proximity

Definition 3.1. A mapping $\delta : Pt(X) \times I^X \longrightarrow I$ is said to be a smooth K-proximity on X if it satisfies the following axioms:
(FK1) $\delta(x_t, \mu \vee \rho) = \delta(x_t, \mu) \vee \delta(x_t \rho)$.
(FK2) $\delta(x_t, \bar{0}) = 0 \ \ \forall\, x_t \in Pt(X)$.
(FK3) $x_t\ q\ \mu$ implies $\delta(x_t, \mu) = 1$.
(FK4) $\delta(x_t, \rho) < r$ implies there exists $\eta \in I^X$ such that $\delta(x_t, \eta) < r$ and $\delta(y_s, \rho) < r$ for all $y_s \in \bar{1} - \eta$.

The pair (X, δ) is called smooth K-proximity space.

One can easily show that the smooth proximity on X implies the smooth K-proximity on X.

Proposition 3.2. *Every smooth proximity on X is a smooth K-proximity on X.*

Proof. (FP1) and (FP2) implies (FK1), (FP3) implies (FK2), and (FP5) implies (FK3). If $\mu = \{x_t\}$ and $\delta(\mu, \rho) < r$, then from (FP4) there exists a $\eta \in I^X$ with $\delta(x_t, \eta) < r$ and $\delta(\bar{1} - \eta, \rho) < r$. By Lemma 2.6, we get for each $y_s \in \bar{1} - \eta$, we have $\delta(y_s, \rho) < r$. This means (FK4) holds, and thus δ a smooth K-proximity on X. □

Lemma 3.3. *Let (X, δ) be a smooth K-proximity, if $\delta(x_t, \mu) \geq r$ and $\mu \leq \rho$, then $\delta(x_t, \rho) \geq r$.*

Proof. The result follows immediately from (FK1) axiom. □

Theorem 3.4. *Let (X, δ) be a smooth K-proximity. Then, a mapping $C_\delta : I^X \times I_1 \longrightarrow I^X$ given by*

$$x_t \; q \; C_\delta(\lambda, r) \text{ if and only if } \delta(x_t, \lambda) \geq 1 - r \tag{3.1}$$

is a fuzzy closure operator and induced a smooth topology on X called $\tau_\delta : I^X \longrightarrow I$ defined as

$$\tau_\delta(\lambda) = \bigvee \{r \in I | \; C_\delta(\bar{1} - \lambda, r) = \bar{1} - \lambda\}. \tag{3.2}$$

Proof. (C1) Since $\delta(x_t, \bar{0}) = 0$ for all $x_t \in Pt(X)$, it follows that $x_t \; \bar{q} \; C_\delta(\bar{0}, r)$ for all $x_t \in Pt(X)$ and for all $r \in I_1$, thus $C_\delta(\bar{0}, r) = \bar{0}$.

(C2) Let $x_t \; q \; \lambda$, then by (FK3) axiom we get $\delta(x_t, \lambda) = 1 \geq 1 - r$ for all $r \in I_1$ this implies $x_t \; q \; C_\delta(\lambda, r)$. Hence $\lambda \leq C_\delta(\lambda, r)$.

(C3)

$$\begin{aligned} x_t \; q \; C_\delta(\lambda \vee \mu, r) &\iff \delta(x_t, \lambda \vee \mu) \geq 1 - r \\ &\iff \delta(x_t, \lambda) \geq 1 - r \;\; \vee \;\; \delta(x_t, \mu) \geq 1 - r \\ &\iff x_t \; q \; C_\delta(\lambda, r) \;\; \vee \;\; x_t \; q \; C_\delta(\mu, r) \\ &\iff x_t \; q \; [C_\delta(\lambda, r) \vee C_\delta(\mu, r)]. \end{aligned}$$

Hence, $C_\delta(\lambda \vee \mu, r) = C_\delta(\lambda, r) \vee C_\delta(\mu, r)$.

(C4) Let $r, s \in I_1$ such that $r \leq s$ and $x_t \; q \; C_\delta(\lambda, r)$. Then, $\delta(x_t, \lambda) \geq 1 - r \geq 1 - s$ and this means $x_t \; q \; C_\delta(\lambda, s)$. Hence, if $r \leq s$, we have $C_\delta(\lambda, r) \leq C_\delta(\lambda, s)$.

(C5) Let $x_t \; q \; C_\delta(C_\delta(\lambda, r), r)$ and $x_t \; \bar{q} \; C_\delta(\lambda, r)$. Then, $\delta(x_t, \lambda) < 1 - r$. By (FK4) axiom there exists $\eta \in I^X$ such that $\delta(x_t, \eta) < 1 - r$ and $\delta(y_s, \lambda) < 1 - r$ for all $y_s \in \bar{1} - \eta$. Therefore $y_s \; \bar{q} \; C_\delta(\lambda, r)$ for all $y_s \in \bar{1} - \eta$, and gives $\bar{1} - \eta \; \bar{q} \; C_\delta(\lambda, r)$. So $C_\delta(\lambda, r) \leq \eta$. Again, by Lemma 3.3 we have $\delta(x_t, C_\delta(\lambda, r)) < 1 - r$ which means that $x_t \; \bar{q} \; C_\delta(C_\delta(\lambda, r), r)$. So, we have a contradiction. The other inclusion follows from (C2) it follows $C_\delta(\lambda, r) \leq C_\delta(C_\delta(\lambda, r), r)$. Hence $C_\delta(C_\delta(\lambda, r), r) = C_\delta(\lambda, r)$. Therefore, C_δ is a fuzzy closure operator. From Definition 2.3, we get τ_δ is a smooth topology on X.

□

Definition 3.5. If on a set X there is a smooth topology τ and a smooth K-proximity δ such that $\tau = \tau_\delta$, then τ and δ are said to compatible, denoted $\tau \sim \delta$, or we say that the smooth topological space (X, τ) is a smooth K-proximal space.

In the following theorem we show that every smooth topological space induced a compatible smooth K-proximity space.

Theorem 3.6. *Let (X,τ) be a smooth topological space. Let $\delta : Pt(X)\times I^X \longrightarrow I$ defined by*

$$\delta(x_t,\mu)=\bigvee\{1-r|\ x_t\ q\ C_\tau(\mu,r), r\in I_0\}. \tag{3.3}$$

Then δ is a compatible smooth K-proximity on X.

Proof. First, we prove that δ satisfies $(FK1)-(FK4)$.

(FK1)

$$\begin{aligned}\delta(x_t,\mu\vee\rho)=1-r &\iff x_t\ q\ C_\tau(\mu\vee\rho,r)\\ &\iff x_t\ q\ C_\tau(\mu,r)\vee x_t\ q\ C_\tau(\rho,r)\\ &\iff \delta(x_t,\mu)=1-r\ \vee\ \delta(x_t,\rho)=1-r\\ &\iff [\delta(x_t,\mu)\vee\delta(x_t,\rho)]=1-r.\end{aligned}$$

Hence, $\delta(x_t,\mu\vee\rho)=\delta(x_t,\mu)\vee\delta(x_t,\rho)$.

(FK2) Since $x_t\ \bar{q}\ \bar{0}=C_\tau(\bar{0},r)$ for all $r\in I_0$. Then, $\delta(x_t,\bar{0})=0$ for all $x_t\in Pt(X)$.

(FK3) Let $x_t\ q\ \mu$, then $x_t\ q\ C_\tau(\mu,r)$ for all $r\in I_0$. This implies $\delta(x_t,\mu)=1$.

(FK4)

$$\begin{aligned}\delta(x_t,\mu)<1-r &\iff x_t\ \bar{q}\ C_\tau(\mu,r)\\ &\iff x_t\ \bar{q}\ C_\tau(C_\tau(\mu,r),r)\\ &\iff \delta(x_t,C_\tau(\mu,r))<1-r\\ &\iff \text{if } \eta=C_\tau(\mu,r),\ \text{then } \delta(x_t,\eta)<1-r\\ &\qquad \text{and } \delta(y_s,\mu)<1-r \text{ for all } y_s\in\bar{1}-C_\tau(\mu,r).\end{aligned}$$

Hence, (X,δ) is a smooth K-proximity space.

To show $\tau_\delta=\tau$, $x_t\ q\ C_\delta(\mu,r)\iff\delta(x_t,\mu)\geq 1-r\iff x_t\ q\ C_\tau(\mu,r)$. Thus, $\tau_\delta=\tau$.

□

Definition 3.7. Let (X,δ) be a smooth K-proximity space. For $\mu\in I^X$, we say that μ is a δ-neighborhood of a fuzzy point $x_t\in Pt(X)$, denote $x_t\ll\mu$ if $\delta(x_t,\bar{1}-\mu)=0$.

Some basic properties of δ-neighborhood.

Proposition 3.8. *Let (X,δ) be a smooth K-proximity space. Then:*

(1) $x_t\ll\bar{1}$, *for all* $x_t\in Pt(X)$.

(2) $x_t\ll\mu\Longrightarrow x_t\in\mu$.

(3) $x_t\ll\mu$ *and* $\mu\leq\rho\Longrightarrow x_t\ll\rho$.

(4) $x_t\ll\mu_i$, *for* $i=1,...,n\Longrightarrow x_t\ll\bigwedge_{i=1,\dots,n}\mu_i$, $x_t\ll\bigvee_{i=1,\dots,n}\mu_i$.

(5) $x_t\ll\mu\Longleftrightarrow x_t\ll I_\delta(\mu,r)\Longrightarrow x_t\in I_\delta(\mu,r)$, *for all* $r\in I_0$.

Proof. The results of parts $(1)-(4)$ follows immediately from Definition 3.7, (FK2), (FK3), Lemma 3.3 and (FK1).

(5)

$$\begin{aligned} x_t \ll \mu &\iff \delta(x_t, \bar{1} - \mu) = 0 \\ &\iff x_t \; \bar{q} \; C_\delta(\bar{1} - \mu, r), \; \forall \, r \in I_1 \\ &\iff x_t \; \bar{q} \; C_\delta(C_\delta(\bar{1} - \mu, r), r) \\ &\iff \delta(x_t, C_\delta(\bar{1} - \mu, r)) = 0 \\ &\iff \delta(x_t, \bar{1} - I_\delta(\mu, r)) = 0 \\ &\iff x_t \ll I_\delta(\mu, r). \end{aligned}$$

□

Now we introduce the concept of fuzzy K-proximity (or fuzzy K-proximally continuous) mapping.

Definition 3.9. Let (X, δ_1) and (Y, δ_2) be two smooth K-proximity spaces. A mapping $f : X \longrightarrow Y$ is said to be a fuzzy K-proximity mapping if $\delta_1(x_t, \mu) \leq \delta_2(f(x_t), f(\mu))$ for each $x_t \in Pt(X)$ and $\mu \in I^X$. Equivalently, f is a fuzzy K-proximity mapping if $\delta_2(x_t, \mu) < r$ implies $\delta(f^{-1}(x_t), f^{-1}(\mu)) < r$ or $x_t \ll_2 \mu$ implies $f^{-1}(x_t) \ll_1 f^{-1}(\mu)$ for each $x_t \in Pt(Y)$ and $\mu \in I^Y$.

Theorem 3.10. *If $f : X \longrightarrow Y$ is an injective mapping. Then, a fuzzy K-proximity mapping $f : (X, \delta_1) \longrightarrow (Y, \delta_2)$ is fuzzy continuous with respect to τ_{δ_1} and τ_{δ_2}.*

Proof. Let $x_t \; q \; f(C_{\delta_1}(\mu, r))$. Then, $f^{-1}(x_t) \; q \; f^{-1}(f(C_{\delta_1}(\mu, r)))$ implies $f^{-1}(x_t) \; q \; C_{\delta_1}(\mu, r)$. It follows $\delta_1(f^{-1}(x_t), \mu) \geq 1 - r$. Since f is a fuzzy K-proximity, then $\delta_2(f(f^{-1}(x_t)), f(\mu)) \geq 1 - r$ implies $\delta_2(x_t, f(\mu)) \geq 1 - r$. Thus $x_t \; q \; C_{\delta_2}(f(\mu), r)$. Hence, f is fuzzy continuous. □

Theorem 3.11. *Let $f : (X, \delta_1) \longrightarrow (Y, \delta_2)$ and $g : (Y, \delta_2) \longrightarrow (Z, \delta_3)$ be two fuzzy K-proximity mapping. Then $g \circ f$ is fuzzy K-proximity mapping.*

Proof. The result follows directly from Definition 3.9. □

4. Some generalized smooth proximities induced by smooth K-proximity

In this section we define a Leader and a Lodato smooth proximity and construct Leader and Lodato smooth proximity structure using a given smooth K-proximity.

Definition 4.1. Let $\delta : I^X \times I^X \longrightarrow I$ be a mapping. For any μ, ρ and $\lambda \in I^X$, consider the following axioms:

(FL1) $\delta(\mu, \rho) = \delta(\rho, \mu)$.
(FL2) $\delta(\mu, \rho \vee \lambda) = \delta(\mu, \rho) \vee \delta(\mu, \lambda)$ and
$\delta(\mu \vee \rho, \lambda) = \delta(\mu, \lambda) \vee \delta(\rho, \lambda)$.
(FL3) $\delta(\bar{1}, \bar{0}) = 0$.
(FL4) $\delta(\mu, \rho) \geq r$, $\delta(x_t, \lambda) \geq r$ for every $x_t \; q \; \rho$ implies $\delta(\mu, \lambda) \geq r$.
(FL5) $\mu \; q \; \rho$ implies $\delta(\mu, \rho) = 1$.

Then, δ is said to be:

(1) A Leader smooth proximity on X, if its satisfies (FL2), (FL3), (FL4) and (FL5) axioms.
(2) A Lodato smooth proximity on X, if its satisfies $(FL1)-(FL5)$ axioms.

If δ is a Leader (resp. Lodato) smooth proximity on X, then (X,δ) is called a Leader (resp. Lodato) smooth proximity space.

Lemma 4.2. *Let (X,δ) be a Leader (resp. Lodato) smooth proximity space, $\delta(\mu,\rho)\geq r$ and $\rho\leq\lambda$, then $\delta(\mu,\lambda)\geq r$.*

Proof. The result follows immediately from (FL2) axiom. □

Theorem 4.3. *Let (X,δ) be a Leader (resp. Lodato) smooth proximity space. Then, the mapping $C_\delta: I^X\times I_1\longrightarrow I^X$ defined by*

$$x_t\ q\ C_\delta(\lambda,r) \ \text{ if and only if } \ \delta(x_t,\lambda)\geq 1-r \tag{4.1}$$

is a fuzzy closure operator and induced a smooth topology on X called $\tau_\delta: I^X\longrightarrow I$ defined as

$$\tau_\delta(\lambda)=\bigvee\{r\in I|\ C_\delta(\bar{1}-\lambda,r)=\bar{1}-\lambda\}. \tag{4.2}$$

Proof. Similar to the proof of Theorem 3.4. □

Theorem 4.4. *Let (X,δ) be a Leader (resp. Lodato) smooth proximity space. Then,*

$$\delta(\mu,\rho)=\delta(\mu,C_\delta(\rho,r)),\ \ \forall\, r\in I_1. \tag{4.3}$$

Proof. Let $\delta(\mu,C_\delta(\rho,r))\geq 1-r$ and for all $x_t\ q\ C_\delta(\rho,r)$, we have $\delta(x_t,\rho)\geq 1-r$. Thus from (FL4) implies $\delta(\mu,\rho)\geq 1-r$. The other inclusion follows from Theorem 4.3 and Lemma 4.2. □

Theorem 4.5. *Let (X,δ) be a smooth K-proximity space. Then, $\pi: I^X\times I^X\longrightarrow I$ given by*

$$\pi(\mu,\rho)=r \iff \exists x_t\in\mu \ \text{ such that } \ \delta(x_t,\rho)=r \tag{4.4}$$

is a Leader smooth proximity.

Proof. (FL2)

$$\begin{aligned}\pi(\mu,\rho\vee\lambda)=r &\iff \exists x_t\in\mu;\ \delta(x_t,\rho\vee\lambda)=r\\ &\iff \exists x_t\in\mu;\ \delta(x_t,\rho)=r\vee\exists x_t\in\mu;\ \delta(x_t,\lambda)=r\\ &\iff \pi(\mu,\rho)=r\vee\pi(\mu,\lambda)=r\\ &\iff [\pi(\mu,\rho)\vee\pi(\mu,\lambda)]=r\end{aligned}$$

Hence, $\pi(\mu,\rho\vee\lambda)=\pi(\mu,\rho)\vee\pi(\mu,\lambda)$.

Similarly one can prove that $\pi(\mu\vee\rho,\lambda)=\pi(\mu,\lambda)\vee\pi(\rho,\lambda)$.

(FL3) Since $\delta(x_t,\bar{0})=0$, for all $x_t\in Pt(X)$, then $\pi(\bar{1},\bar{0})=0$.

(FL4) Let $\pi(\mu,\rho)\geq 1-r$ and $\pi(y_s,\lambda)\geq 1-r$ for all $y_s\ q\ \rho$. So there exists $x_t\in\mu$ such that $\delta(x_t,\rho)\geq 1-r$ and $\delta(y_s,\lambda)\geq 1-r$ for all $y_s\ q\ \rho$ which implies that there exists $x_t\in\mu$ such that $x_t\ q\ C_\delta(\rho,r)$ and $y_s\ q\ C_\delta(\lambda,r)$ for all $y_s\ q\ \rho$. So, $x_t\in\mu$ such that $x_t\ q\ C_\delta(\rho,r)$ and $\rho\ q\ C_\delta(\lambda,r)$, then $x_t\in\mu$ such that $x_t\ q\ C_\delta(\rho,r)$ and $C_\delta(\rho,r)\ q\ C_\delta(C_\delta(\lambda,r),r)=C_\delta(\lambda,r)$. Therefore $x_t\in\mu$ such that $x_t\ q\ C_\delta(\lambda,r)$, then $x_t\in\mu$ such that $\delta(x_t,\lambda)\geq 1-r$. Therefore, $\pi(\mu,\lambda)\geq 1-r$.

(FL5) Let $\mu \, q \, \rho$. Then, $\exists \, x_t \in \mu$, $x_t \, q \, \rho$. Then, from (FK3), $\delta(x_t, \rho) = 1$ implies $\pi(\mu, \rho) = 1$.

□

Theorem 4.6. *Let (X, δ) be a smooth K-proximity space and π be a Leader smooth proximity on X as defined in (4.4). Then $\beta : I^X \times I^X \longrightarrow I$ given by*

$$\beta(\mu, \rho) = r \iff \pi(\mu, \rho) \wedge \pi(\rho, \mu) = r \tag{4.5}$$

is a Lodato smooth proximity.

Proof. (FL1)

$$\begin{aligned} \beta(\mu, \rho) = r &\iff \pi(\mu, \rho) \wedge \pi(\rho, \mu) = r \\ &\iff \pi(\rho, \mu) \wedge \pi(\mu, \rho) = r \\ &\iff \beta(\rho, \mu) = r. \end{aligned}$$

(FL2)

$$\begin{aligned} \beta(\mu, \rho \vee \lambda) \geq r &\iff [\pi(\mu, \rho \vee \lambda) \ \ \wedge \ \ \pi(\rho \vee \lambda, \mu)] \geq r \\ &\iff \pi(\mu, \rho \vee \lambda) \geq r \ \ \wedge \ \ \pi(\rho \vee \lambda, \mu) \geq r \\ &\iff [\pi(\mu, \rho) \vee \pi(\mu, \lambda)] \geq r \ \ \wedge \\ &\qquad [\pi(\rho, \mu) \vee \pi(\lambda, \mu)] \geq r \\ &\iff [\pi(\mu, \rho) \geq r \ \wedge \ \pi(\rho, \mu) \geq r] \vee \\ &\qquad [\pi(\mu, \rho) \geq r \ \wedge \ \pi(\lambda, \mu) \geq r] \vee \\ &\qquad [\pi(\mu, \lambda) \geq r \ \wedge \ \pi(\rho, \mu) \geq r] \vee \\ &\qquad [\pi(\mu, \lambda) \geq r \ \wedge \ \pi(\lambda, \mu)] \geq r] \\ &\iff \beta(\mu, \rho) \geq r \ \vee \\ &\qquad [\pi(\mu, \rho) \geq r \ \wedge \ \pi(\lambda, \mu) \geq r] \vee \\ &\qquad [\pi(\mu, \lambda) \geq r \ \wedge \ \pi(\rho, \mu) \geq r] \vee \\ &\qquad \beta(\mu, \lambda) \geq r \\ &\iff [\beta(\mu, \rho) \ \ \vee \ \ \beta(\mu, \lambda)] \geq r. \end{aligned}$$

(FL3) Since $\pi(\bar{1}, \bar{0}) = 0$, then $\pi(\bar{1}, \bar{0}) \wedge \pi(\bar{0}, \bar{1}) = 0$. Therefore $\beta(\bar{1}, \bar{0}) = 0$.

(FL4) Let $\beta(\mu, \rho) \geq 1 - r$ and $\beta(y_s, \lambda) \geq 1 - r$ for all $y_s \, q \, \rho$, then from (4.5), $\pi(\mu, \rho) \geq 1 - r$, $\pi(\rho, \mu) \geq 1 - r$, $\pi(y_s, \lambda) \geq 1 - r$ and $\pi(\lambda, y_s) \geq 1 - r$ for each $y_s \, q \, \rho$.

Since $\pi(\mu, \rho) \geq 1 - r$ and $\pi(y_s, \lambda) \geq 1 - r$ for each $y_s \, q \, \rho$. Then $\pi(\mu, \lambda) \geq 1 - r$.

Since $\pi(\lambda, y_s) \geq 1 - r$ for each $y_s \, q \, \rho$ and $\pi(\rho, \mu) \geq 1 - r$. Then $\pi(\lambda, \mu) \geq 1 - r$. Hence $\beta(\mu, \lambda) \geq 1 - r$.

(FL5) Let $\beta(\mu, \rho) \neq 1$, then $\pi(\mu, \rho) \neq 1$ or $\pi(\rho, \mu) \neq 1$, from definition of π we have $\mu \, \bar{q} \, \rho$.

□

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References

[1] G. Artico and R. Moresco, Fuzzy proximities compatible with Lowen fuzzy uniformities, Fuzzy Sets and Systems 21 (1987) 85–98.

[2] R. Badard, Smooth axiomatics, First IFSA Congress Palma de Mallorca 1986.

[3] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.

[4] K. C. Chattopadhyay, R. N. Hazra and S. K. Samanta, Gradation of openness: Fuzzy topology, Fuzzy Sets and Systems 94 (1992) 237–242.

[5] K. C. Chattopadhyay and S. K. Samanta, Fuzzy topology: Fuzzy closure operator, fuzzy compactness and fuzzy connectedness, Fuzzy Sets and Systems 54 (1993) 207–212.

[6] M. K. El Gayyar, E. E. Kerre and A. A. Ramadan, Almost compactness and near compactness in smooth topological spaces, Fuzzy Sets and Systems 92 (1994) 193–202.

[7] V. A. Efremovic, Infinitesimal spaces, Doklady Akad. Nauk SSSR (N.S.) (in Russian) 76 (1951) 341–343.

[8] H. Hazra, P. Majumdar and S. K. Samanta, Soft proximity,Ann. Fuzzy Math. Inform. 7 (6) (2014) 867–877.

[9] M. H. Ghanim, O. A. Tantawy and F.M. Selim, Gradations of uniformity and gradation of proximity, Fuzzy Sets and Systems 79 (1996) 373–382.

[10] A. K. Katsaras, Fuzzy proximity spaces, J. Math. Anal. Appl. 68 (1) (1979) 100–110.

[11] A. K. Katsaras, On fuzzy proximity spaces, J. Math. Anal. Appl. 75 (2) (1980) 671–583.

[12] C. Y. Kim, K. N. Choi and Y. S. Shin, On the K-proximities, J. Kyungpook. Math. 13 (1) (1973) 69–80.

[13] E. S. Kim, S. H. Ahn and D. H. Park, The relationship between L-fuzzy proximities and L-fuzzy quasi-uniformities,Iran. J. Fuzzy Syst. 9 (6) (2012) 101–111.

[14] S. Leader, Local proximity spaces, Math. Annalen 169 (1957) 275–281.

[15] S. Leader, Extension based on proximity boundedness, Math. Z. 108 (1969) 137–144.

[16] M. W. Lodato, On topologically induced generalized proximity relations I, Proc. Amer. J. Math. soc. 15 (1964) 417–422.

[17] M. W. Lodato, On topologically induced generalized proximity relations II, Pacific J. Mat. 17 (1966) 131–135.

[18] Kuo-Duok Park, Note on the fuzzy proximity spaces, Kangweon-Kyungki Math. Jour. 10 (2) (2002) 131–140.

[19] A. A. Ramadan, Smooth topological spaces, Fuzzy Sets and Systems 48 (1992) 371–375.

[20] S. K. Samanta, Fuzzy proximities and fuzzy uniformities, Fuzzy sets and Systems 70 (1995) 97–105.

[21] A. P. Šostak, On a fuzzy topological structure, Suppl. Rend. Circ. Matem. Palermo, Ser.II 11 (1985) 89–103.

[22] A. P. Šostak, Basic structures of fuzzy topology, J. Math. Sci 78 (1996) 662-701.

[23] O. A. Tantawy, S. A. El-Sheikh and R. N. Majeed, Smooth biproximity spaces and P-smooth quasi-proximity spaces,(submitted).

[24] A. M. Zahran, M. A. Abd-Allah, K. El-Saady and A. Ghareeb, Double fuzzy preproximity spaces,Int. J. Fuzzy Logic Intell. Syst. 7 (4) (2007) 249–255.

[25] A. H. Zakari, A. Ghareeb and Y. C. Kim, Fuzzy $k_{\circ}$-preproximities related to fuzzy closure spaces,Arab. J. Math. 4 (2015) 141–151.

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