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A study of prime ideals of $\Gamma-$semiring generated by a translational invariant fuzzy subset and an element

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Abstract. We introduce the notion of units, associates, prime elements with respect to a fuzzy subset, an ideal of a $\Gamma-$semiring generated by translational invariant fuzzy subset and an element. We study the properties of image and pre-image of translational invariant fuzzy subset under the $\Gamma-$semiring homomorphism and we prove that every homomorphic image of an ideal of $\Gamma-$semiring generated by $\mu-$prime element and translational invariant, homomorphism - invariant fuzzy subset μ is a prime ideal of $\Gamma-$semiring.

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1. Introduction

An universal algebra $(S, +, \cdot)$ is called a semiring if and only if $(S, +), (S, \cdot)$ are semigroups which are connected by distributive laws, *i.e.*, $a(b + c) = ab + ac$, $(a + b)c = ac + bc$, for all $a, b, c \in S$. Semiring was first introduced by H. S. Vandiver [9] in 1934. Though semiring is a generalization of a ring, ideals of semiring do not coincide with ring ideals. For example an ideal of semiring need not be the kernel of some semiring homomorphism. To solve this problem Herniksen [1] defined $k-$ideals and Iizuka [2] defined $h-$ideals in semirings to obtain analogous of ring results for semrings. The theory of rings and the theory of semigroups have considerable impact on the development of the theory of semirings.

Semiring is very useful for solving problems in applied mathematics and information science because semiring provides an algebraic frame work for modeling. Semiring plays an important role in studying matrices and determinants. The theory of fuzzy sets is the most appropriate theory for dealing with uncertainty and was first introduced by L. A. Zadeh [10]. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to Logic, Set theory, Group theory, Ring theory, Real analysis, Topology, Measure theory etc.

Notion of $\Gamma-$semiring was introduced by M. Murali Krishna Rao [3] not only generalizes the notion of semiring and $\Gamma-$ring but also the notion of ternary semiring. The natural growth of $\Gamma-$semiring is influenced by two things. One is the generalization of results of $\Gamma-$rings and another is the generalization of results of semirings and ternary semirings. This notion provides an algebraic back ground to the non positive cones of the totally ordered rings.

A. K. Ray [7] introduced the notion of translational invariant fuzzy subset and A. K. Ray and Ali [8] generalized the results of ring theory by using the notion of translational invariant fuzzy subsets. In this paper, we introduce the notion of units, associates, prime elements with respect to a fuzzy subset, an ideal of a $\Gamma-$semiring generated by translational invariant fuzzy subset and an element. We study the properties of image and pre-image of translational invariant fuzzy subset under the $\Gamma-$semiring homomorphism. We prove that every homomorphic image of an ideal of $\Gamma-$semiring generated by $\mu-$prime element and translational invariant, homomorphism - invariant fuzzy subset μ is a prime ideal of $\Gamma-$semiring.

2. Preliminaries

In this section, we recall some of the fundamental concepts and definitions which are necessary for this paper.

Definition 2.1 ([3])**.** A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called a semiring provided

(i) addition is a commutative operation.
(ii) multiplication distributes over addition both from the left and from the right.
(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for each $x \in S$.

Definition 2.2 ([3])**.** Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. If there exists a mapping $M \times \Gamma \times M \to M$ (images to be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) satisfying the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$,
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$,

then M is called a $\Gamma-$semiring.

Definition 2.3 ([3])**.** A $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0,$ for all $x \in M$.

Example 2.1. Every semiring M is a $\Gamma-$semiring with $\Gamma = M$ and ternary operation is defined as the usual semiring multiplication

Example 2.2. Let M be the additive seimgroup of all $m \times n$ matrices over the set of non negative rational numbers and Γ be the additive semigroup of all $n \times m$ matrices over the set of non negative integers, then with respect to usual matrix multiplication M is a $\Gamma-$semiring.

Definition 2.4 ([3]). Let M be a $\Gamma-$semiring and A be a non-empty subset of M. A is called a $\Gamma-$subsemiring of M if A is a sub-semigroup of $(M, +)$ and $A\Gamma A \subseteq A$.

Definition 2.5 ([3]). Let M be a $\Gamma-$semiring. A subset A of M is called a left(right) ideals of M if A is closed under addition and $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$. A is called an ideal of M if it is both a left ideal and right ideal.

Definition 2.6 ([6]). Let M be a $\Gamma-$semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.7 ([6]). In a $\Gamma-$semiring M with unity 1, an element $a \in M$ is said to be left invertible (right invertible) if there exist $b \in M, \alpha \in \Gamma$ such that $b\alpha a = 1$ $(a\alpha b = 1)$.

Definition 2.8 ([6]). In a $\Gamma-$semiring M with unity 1, an element $a \in M$ is said to be invertible if there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.

Definition 2.9 ([4]). A function $f : R \to M$ where R and M are $\Gamma-$semirings is said to be $\Gamma-$semiring homomorphism if $f(a+b) = f(a)+f(b)$ and $f(a\alpha b) = f(a)\alpha f(b)$ for all $a, b \in R, \alpha \in \Gamma$.

Definition 2.10 ([10]). Let M be a nonempty set. A mapping $f : S \to [0, 1]$ is called a fuzzy subset of M.

Definition 2.11 ([4]). Let S and T be two sets and $\phi : S \to T$ be any function. A fuzzy subset f of S is called $\phi-$invariant if $\phi(x) = \phi(y) \Rightarrow f(x) = f(y)$.

Definition 2.12 ([8]). Let μ be a fuzzy subset of M. μ is said to be left translational invariant with respect to a binary operation $*$ on M if

$$\mu(x) = \mu(y) \Rightarrow \mu(a * x) = \mu(a * y), \quad x, y \in M, \quad \text{for all } a \in M.$$

Definition 2.13 ([8]). Let μ be a fuzzy subset of M. μ is said to be right translational invariant with respect to a binary operation $*$ on M if

$$\mu(x) = \mu(y) \Rightarrow \mu(x * a) = \mu(y * a), \ x, y \in M, \quad \text{for all } a \in M.$$

Definition 2.14 ([8]). A fuzzy subset μ of M is said to be translational invariant with respect to a binary operation $*$ on M if μ is both left and translational invariant with respect to μ.

Proposition 2.1 ([8]). *If a fuzzy subset μ is commutative with respect to a binary operation $*$ on M, i.e., $\mu(x*y) = \mu(y*x)$, for all $x, y \in M$ then μ is a translational invariant.*

Definition 2.15 ([5]). Let M be a $\Gamma-$semiring. An ideal P of M is called a prime ideal of Mif for any $a, b \in M$ and $\gamma \in \Gamma, a\gamma b \in P \Rightarrow a$ or $b \in P$.

3. Ideal of a $\Gamma-$semiring M generated by an element and translational invariant fuzzy subset of M

In this section, we introduce the notion of left and right translational invariant fuzzy subset of $\Gamma-$semiring M, the notion of unit with respect to fuzzy subset and study their properties. And also we prove that if μ is a translational invariant fuzzy subset of a commutative $\Gamma-$semiring with unity then principal ideal generated by an element and μ contains an unity element is not a proper ideal of $\Gamma-$semiring.

The set $\{r \in M \mid \mu(r) = \mu(x\alpha a), \quad a \in M, \text{ for some } x \in M, \quad \text{for all } \alpha \in \Gamma\}$ is denoted by $L(a, \mu)$ and the set $\{r \in M \mid \mu(r) = \mu(a\alpha x), \quad a \in M, \text{ for some } x \in M, \quad \text{for all } \alpha \in \Gamma\}$ is denoted by $R(a, \mu)$.

Definition 3.1. Let M be a $\Gamma-$semring and μ be a fuzzy subset of M. Then μ is said to be left translational invariant if

(i) $\mu(x) = \mu(y) \Rightarrow \mu(a + x) = \mu(a + y), x, y \in M, \text{ for all } a \in M$
(ii) $\mu(x) = \mu(y) \Rightarrow \mu(a\alpha x) = \mu(a\alpha y), \ x, y \in M, \text{ for all } a \in M, \ \alpha \in \Gamma.$

Definition 3.2. Let M be a $\Gamma-$semring and μ be a fuzzy subset of M. Then μ is said to be right translational invariant if

(i) $\mu(x) = \mu(y) \Rightarrow \mu(x + a) = \mu(y + a), x, y \in M, \text{ for all } a \in M$
(ii) $\mu(x) = \mu(y) \Rightarrow \mu(x\alpha a) = \mu(y\alpha a), \ x, y \in M, \text{ for all } a \in M, \ \alpha \in \Gamma.$

Example 3.1. Let M be the additive commutative semigroup of all non negative integers and Γ be the additive commutative semigroup of all commutative non negative even integers . Then M is a $\Gamma-$semiring if $a\gamma b$ is defined as usual multiplication of integers a, γ, b where $a, b \in M$ and $\gamma \in \Gamma$. Let μ be a fuzzy subset of M, defined by $\mu(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x \text{ is even} \\ 0.1, & \text{if } x \text{ is odd .} \end{cases}$

$$\begin{aligned} <2> &= \{0, 2, 4, \cdots\} \\ <6> &= \{0, 6, 12, \cdots\} \\ I(6, \mu) &= \{x \mid \mu(x) = \mu(y\alpha 6), \text{ for all } \alpha \in \Gamma, \text{ for some } y \in M\}. \\ &= \text{All even integers} \\ <6> &\subseteq I(6, \mu) \subseteq M. \end{aligned}$$

Theorem 3.1. *Let μ be a left translational invariant fuzzy subset of $\Gamma-$semiring M. Then for any $a \in M$, the set $L(a, \mu)$ is a left ideal of $\Gamma-$semiring M.*

Proof. Let μ be a left translational invariant fuzzy subset of $\Gamma-$semiring M, $s, r \in L(a, \mu)$. Then

$$\begin{aligned} &\mu(s) = \mu(y\alpha a), \text{and } \mu(r) = \mu(x\alpha a), \text{ for some } \ x, y \in M, \text{ for all } \ \alpha \in \Gamma \\ \Rightarrow &\mu(r + s) = \mu(x\alpha a + s) \text{ and } \mu(x\alpha a + s) = \mu(x\alpha a + y\alpha a) \\ \Rightarrow &\mu(r + s) = \mu\big((x + y)\alpha a\big). \end{aligned}$$

Therefore $r + s \in L(a, \mu)$.

Suppose $s \in L(a, \mu), \ r \in M, \ \beta \in \Gamma$.

Then $\mu(s) = \mu(y\alpha a)$, for some $y \in M$, for all $\alpha \in \Gamma$.

$$\begin{aligned} &\mu(r\beta s) = \mu\big(r\beta(y\alpha a)\big), \quad \text{for all } \alpha \in \Gamma \\ \Rightarrow &\mu(r\beta s) = \mu\big((r\beta y)\alpha a\big), \quad \text{for all } \alpha \in \Gamma. \end{aligned}$$

Therefore$r\beta s \in L(a,\mu)$. Hence $L(a,\mu)$ is a left ideal of $\Gamma-$semiring M. □

The proof of the following theorem is similar as that of Theorem 3.1

Theorem 3.2. *Let M be a $\Gamma-$semiring and μ be a right translational invariant fuzzy subset of M and $a \in M$. Then the set $R(a,\mu)$ is a right ideal of $\Gamma-$semiring M.*

Corollary 3.1. *Let M be a commutative $\Gamma-$semiring and μ be a translational invariant fuzzy subset of M. Then for any $a \in M$, $L(a,\mu)$ is an ideal of $\Gamma-$semiring M.*

If $L(a,\mu) = R(a,\mu)$ then the ideal $L(a,\mu)$ is denoted by $I(a,\mu)$ and $I(a,\mu)$ is called $\mu-$principal ideal of M generated by a and μ.

Theorem 3.3. *Let μ be a translational invariant fuzzy subset of $\Gamma-$semiring M.*

(i) *If $a \in L(b,\mu)$ then $L(a,\mu) \subseteq L(b,\mu)$.*
(ii) *If $a \in R(b,\mu)$ then $R(a,\mu) \subseteq R(b,\mu)$.*

Proof.

$$\begin{aligned}
\text{Suppose } a \in L(b,\mu) &\Rightarrow \mu(a) = \mu(x\alpha b) \text{ and for some } x \in M, \quad \text{for all } \alpha \in \Gamma.\\
\text{Let } t \in L(a,\mu) &\Rightarrow \mu(t) = \mu(y\alpha a) \text{ and for some } y \in M, \quad \text{for all } \alpha \in \Gamma.\\
\text{Now } \mu(a) = \mu(x\alpha b) &\Rightarrow \mu(y\beta a) = \mu(y\beta x\alpha b),\ \beta \in \Gamma,, \quad \text{for all } \alpha \in \Gamma.\\
&\Rightarrow \mu(t) = \mu(y\beta x\alpha b), \beta \in \Gamma,, \quad \text{for all } \alpha \in \Gamma.\\
&\Rightarrow t \in L(b,\mu).
\end{aligned}$$

Hence $L(a,\mu) \subseteq L(b,\mu)$. Similarly we can prove (ii). □

Theorem 3.4. *Let μ be a translational invariant fuzzy subset of a $\Gamma-$semiring M and $a, b \in M$. If $\mu(a) = \mu(b)$ then $L(a,\mu) = L(b,\mu)$ and $R(a,\mu) = R(b,\mu)$.*

Proof. Let μ be a left translational invariant fuzzy subset of a $\Gamma-$semiring M, $\mu(a) = \mu(b)$ and $x \in L(a,\mu)$. Then $\mu(x) = \mu(r\alpha a)$, for some $r \in M$, for all $\alpha \in \Gamma$.

$$\begin{aligned}
\text{Since } \mu(a) = \mu(b) &\Rightarrow \mu(r\alpha a) = \mu(r\alpha b)\\
&\Rightarrow \mu(x) = \mu(r\alpha b), \text{ for all } \alpha \in \Gamma\\
&\Rightarrow x \in L(b,\mu).
\end{aligned}$$

Hence $L(a,\mu) \subseteq L(b,\mu)$. Suppose $y \in L(b,\mu)$.

$$\begin{aligned}
\text{Then } \mu(y) = \mu(s\alpha b),& \quad \text{for some } \ s \in M, \text{ for all } \alpha \in \Gamma\\
&\Rightarrow \mu(y) = \mu(s\alpha b) = \mu(s\alpha a).\\
&\Rightarrow y \in L(a,\mu).\\
\text{Therefore } L(b,\mu) &\subseteq L(a,\mu).
\end{aligned}$$

Hence $L(a,\mu) = L(b,\mu)$. Similarly we can prove $R(a,\mu) = R(b,\mu)$. □

The proof of the following theorems are straightforward verification.

Theorem 3.5. *Let M be a $\Gamma-$semiring and μ be a translational invariant fuzzy subset of M. For any $a \in M$, the left ideal $M\Gamma a = \{r\alpha a \mid r \in M, \alpha \in \Gamma\}$ of M is contained in left ideal $L(a,\mu)$ and the right ideal $a\Gamma M = \{a\alpha r \mid r \in M, \alpha \in \Gamma\}$ is contained in right ideal $R(a,\mu)$.*

Theorem 3.6. *If M is a commutative $\Gamma-$semiring with unity, μ is a translational invariant fuzzy subset of M and $a \in M$ then the principal ideal $\langle a\rangle = I(a,\mu)$.*

Definition 3.3. Let M be a $\Gamma-$semiring with unity element e, μ be a translational invariant fuzzy subset of a $\Gamma-$semiring M and $\mu(0) \neq \mu(e)$. An element $a \in M$ with $\mu(a) \neq \mu(0)$ is called a $\mu-$unit of M if there exists an element $u \in M$ such that $\mu(u) \neq \mu(0)$ and $\mu(a\alpha u) = \mu(u\alpha a) = \mu(e)$, for all $\alpha \in \Gamma$.

Example 3.2. Let $M = [0,\ 1]$ and $\Gamma = N$ and let $+$ and the ternary operation be defined as $x + y = max\{x,y\},\ \alpha + \beta = max\{\alpha,\beta\},\ \ x\gamma y = min\{x,\gamma,y\}$ for all $x,y \in M, \alpha,\beta,\gamma \in \Gamma$ then M is a $\Gamma-$semiring with unity 1. Let μ be a fuzzy subset of M defined by

$$\mu(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x \neq 0. \end{cases}$$

Then all non zero elements of M are $\mu-$units.

Theorem 3.7. *Let M be a $\Gamma-$semiring with unity e and a be $\mu-$unit of M. Then $L(a,\mu) = R(a,\mu) = M$.*

Proof. Suppose a is a $\mu-$unit of $\Gamma-$semiring M. Then there exists $u \in M$ such that $\mu(u) \neq \mu(0),\ \mu(a\alpha u) = \mu(u\alpha a) = \mu(e)$, for all $\alpha \in \Gamma$. Let $x \in M$. Then there exists $\gamma \in \Gamma$ such that $x\gamma e = e\gamma x = x$. Now

$$\begin{aligned} &\mu(e) = \mu(a\alpha u), \text{ for all } \ \alpha \in \Gamma \\ \Rightarrow &\mu(e\beta x) = \mu(a\alpha u\beta x), \text{ for all } \ \alpha,\beta \in \Gamma \\ \Rightarrow &\mu(x) = \mu(a\alpha u\gamma x),\ \gamma \in \Gamma, \text{ for all } \ \alpha \in \Gamma \\ \Rightarrow &x \in R(a,\mu) \end{aligned}$$

Therefore $M \subseteq R(a,\mu)$.

Similarly we can prove $M \subseteq L(a,\mu)$. Hence $L(a,\mu) = R(a,\mu) = M$. □

Theorem 3.8. *Let M be a $\Gamma-$semiring with unity $e, a \in M$ and μ be a right translational invariant fuzzy subset of M. If $e \in R(a,\mu)$ then $R(a,\mu) = M$.*

Proof. Suppose e is the unity element of $\Gamma-$semiring $M, x \in M$ and $e \in R(a,\mu)$. Since $x \in M$, by definition of unity, there exists $\gamma \in \Gamma$ such that $e\gamma x = x$. Then $e \in R(a,\mu)$

$$\begin{aligned} \Rightarrow &\mu(e) = \mu(a\alpha y) \text{ for some } y \in M, \ \text{ for all } \alpha \in \Gamma \\ \Rightarrow &\mu(e\gamma x) = \mu(a\alpha y\gamma x), \text{ for all } \alpha,\gamma \in \Gamma \\ \Rightarrow &\mu(x) = \mu(a\alpha y\gamma x), \text{ for all } \alpha \in \Gamma. \end{aligned}$$

Therefore $x \in R(a,\mu)$. Hence $R(a,\mu) = M$. □

Corollary 3.2. *Let M be a commutative $\Gamma-$semiring with unity $e, a \in M$ and μ be a translational invariant fuzzy subset of M. If $e \in I(a,\mu)$ then $I(a,\mu) = M$.*

Theorem 3.9. *Let M be a $\Gamma-$semiring with unity $e, a \in M$ and μ be a right translational invariant fuzzy subset of M. If $x \in R(a,\mu)$ is an invertible then $R(a,\mu) = M$.*

Proof. Let M be a $\Gamma-$semiring with unity $e, a \in M$ and μ be a right translational invariant fuzzy subset of M. Suppose $x \in R(a,\mu)$ is invertible. Then there exist $y \in M, \alpha \in \Gamma$ such that $x\alpha y = e$. Since by Theorem 3.2, $R(a,\mu)$ is a right ideal of $\Gamma-$semiring M. Therefore $e = x\alpha y \in R(a,\mu)$. By Theorem 3.8, $R(a,\mu) = M$. Hence the Theorem . □

Corollary 3.3. *Let M be a commutative $\Gamma-$semiring with unity, $a \in M$ and μ be a translational invariant fuzzy subset of M. If $x \in I(a,\mu)$ is invertible then $I(a,\mu) = M$.*

4. Prime ideals

In this section, we introduce the notion of associates, prime elements with respect to a fuzzy subset, an ideal of a $\Gamma-$semiring generated by translational fuzzy subset and an element. We study the properties of image and pre-image of translational invariant fuzzy subset under the $\Gamma-$semiring homomorphism. We prove that every homomorphic image of an ideal of $\Gamma-$semiring generated by $\mu-$prime element and fuzzy translational invariant, homomorphism-invarinat fuzzy subset μ is a prime ideal of $\Gamma-$semiring. Throughout in this section M is a commutative $\Gamma-$semiring.

Definition 4.1. Let M be a $\Gamma-$semiring, $a, b \in M$ and $\mu(a) \neq \mu(0)$. a is said to be $\mu-$divisor of b if there exists $c \in M$ such that

$$\mu(b) = \mu(a\alpha c), \text{ for all } \alpha \in \Gamma.$$

It is denoted by $(a/_b)_\mu$.

Definition 4.2. Let M be a $\Gamma-$semiring, $a, b \in M$ and $\mu(a) \neq \mu(0)$. a and b are said to be $\mu-$associates if $(a/_b)_\mu$ and $(b/_a)_\mu$

Theorem 4.1. *Let M be a $\Gamma-$semiring, $a, b \in M$ and $\mu(a) \neq \mu(0)$. If $(a/_b)_\mu$ then $I(b,\mu) \subseteq I(a,\mu)$.*

Proof. Suppose $(a/_b)_\mu$.

$$\Rightarrow \mu(b) = \mu(a\alpha c), \quad \text{for some } c \in M, \quad \text{for all } \alpha \in \Gamma$$
$$\Rightarrow b \in I(a,\mu).$$

By Theorem 3.3, $I(b,\mu) \subseteq I(a,\mu)$.

Hence the Theorem. □

Definition 4.3. Let $f : M \to S$ be a homomorphism of $\Gamma-$semirings M, S and μ be a fuzzy subset of M. We define a fuzzy subset $f(\mu)$ of S by

$$f(\mu)(x) = \begin{cases} \sup\limits_{y \in f^{-1}(x)} y, & \text{if } f^{-1}(x) \neq \phi \\ 0\ , & \text{otherwise}\ . \end{cases}$$

Theorem 4.2. *Let f be a homomorphism from $\Gamma-$semiring M onto $\Gamma-$semiring S and μ be a $f-$invariant fuzzy subset of M. If $x = f(a)$ then $f(\mu)(x) = \mu(a),\ a \in M$.*

Proof. Let f be a homomorphism from $\Gamma-$semiring M onto $\Gamma-$semiring S and μ be a $f-$invariant fuzzy subset of M. Suppose $x = f(a)$. Then $f^{-1}(x) = a$. Let $t \in f^{-1}(x)$. Then $x = f(t) \Rightarrow f(a) = x = f(t)$. Since μ is a $f-$invariant fuzzy subset of $M \Rightarrow \mu(a) = \mu(t)$. Therefore $f(\mu)(x) = \sup\limits_{t\in f^{-1}(x)} \mu(t) = \mu(a)$. □

Theorem 4.3. *Let f be a homomorphism from $\Gamma-$semiring M onto $\Gamma-$semiring S. If μ is a translational invariant and $f-$invariant fuzzy subset of M then $f(\mu)$ is a translational invariant fuzzy subset of S.*

Proof. Let f be a homomorphism from $\Gamma-$semiring M onto $\Gamma-$semiring S and μ be a translational invariant and $f-$invariant fuzzy subset of M. Let $x, y \in S$ and $\alpha \in \Gamma$. $f(\mu)(x) = f(\mu)(y)$. Since f is onto, there exist $a, b \in M$ such that $f(a) = x, f(b) = y$. By Theorem 4.2, we have $f(\mu)(x) = \mu(a)$ and $f(\mu)(y) = \mu(b) \Rightarrow \mu(a) = \mu(b)$. Let $z \in S$. Then there exists $c \in M$ such that $f(c) = z$.

$$\begin{aligned} x + z =& f(a) + f(c) \\ =& f(a + c) \\ \text{and } y + z =& f(b) + f(c) \\ =& f(b + c). \end{aligned}$$

Hence $f(\mu)(x + z) = \mu(a + c)$ and $f(\mu)(y + z) = \mu(b + c)$.
Since μ is a translation invariant fuzzy subset of M.

$$\begin{aligned} \mu(a) =& \mu(b) \\ \Rightarrow \mu(a + c) =& \mu(b + c) \\ \Rightarrow f(\mu)(x + z) =& f(\mu)(y + z). \\ x\alpha z =& f(a)\alpha f(c) \\ =& f(a\alpha c) \\ y\alpha z =& f(b)\alpha f(c) \\ =& f(b\alpha c) \\ \Rightarrow f(\mu)(x\alpha z) =& \mu(a\alpha c) \\ \text{and } f(\mu)(y\alpha z) =& \mu(b\alpha c). \\ \text{Since } \ \mu \text{ is translational } \ \text{fuzzy invariant}, \ & \mu(a) = \mu(b) \\ \Rightarrow \mu(a\alpha c) =& \mu(b\alpha c), \text{ for all } \alpha \in \Gamma \\ \Rightarrow f(\mu)(x\alpha z) =& f(\mu)(y\alpha z), \text{ for all } \alpha \in \Gamma. \end{aligned}$$

Hence $f(\mu)$ is a translational invariant fuzzy subset of $\Gamma-$semiring S. □

Definition 4.4. Let $\phi : M \rightarrow M'$ be a homomorphism of $\Gamma-$semiring and μ be a fuzzy subset of M'. We define a fuzzy subset $\phi^{-1}(\mu)$ of M by $\phi^{-1}(\mu)(x) = \mu(\phi(x)),$ for all $x \in M$. We call $\phi^{-1}(\mu)$ is a pre image of μ.

Theorem 4.4. *Let f be a homomorphism from $\Gamma-$semiring M into $\Gamma-$semiring S and μ be a translational invariant fuzzy subset of S. Then $f^{-1}(\mu)$ is a translational invariant fuzzy subset of $\Gamma-$semiring M.*

Proof. Let f be a homomorphism from $\Gamma-$semiring M into $\Gamma-$semiring S and μ be a translational invariant fuzzy subset of S. Let $a, b \in M$ and $f^{-1}(\mu)(a) = f^{-1}(\mu)(b)$. Then $\mu(f(a)) = \mu(f(b))$. Suppose $x \in M$ then $f(x) = y \in S$, $\alpha \in \Gamma$. Since μ is translational invariant fuzzy subset of S, we have

$$\begin{aligned}
&\mu(f(a) + y) = \mu(f(b) + y)\\
\Rightarrow&\mu(f(a) + f(x)) = \mu(f(b) + f(x))\\
\Rightarrow&\mu(f(a + x)) = \mu(f(b + x))\\
\Rightarrow&f^{-1}(\mu)(a + x) = f^{-1}(\mu)(b + x).\\
&\mu(f(a)\alpha y) = \mu(f(b)\alpha y)\\
\Rightarrow&\mu(f(a)\alpha f(x)) = \mu(f(b)\alpha f(x))\\
\Rightarrow&\mu(f(a\alpha x)) = \mu(f(b\alpha x))\\
\Rightarrow&f^{-1}(\mu)(a\alpha x) = f^{-1}(\mu)(b\alpha x).
\end{aligned}$$

Hence $f^{-1}(\mu)$ is a translational invariant fuzzy subset of $\Gamma-$semiring M. □

Theorem 4.5. *Let f be a homomorphism from $\Gamma-$semiring M onto $\Gamma-$semiring S. If μ is a $f-$invariant, translational invariant fuzzy subset of M then*

$$f(I(a, \mu)) = I(f(a), f(\mu)), \textit{ for all } a \in M.$$

Proof. Let f be a homomorphism from $\Gamma-$semiring M onto $\Gamma-$semiring S and μ be a $f-$invariant, translational invariant fuzzy subset of M.

$$\begin{aligned}
&\text{Let } \ y \in I(f(a), f(\mu))\\
&\Leftrightarrow f(\mu)(y) = f(\mu)(s\alpha f(a)), \quad \text{for some } s \in S, \quad \text{for all } \alpha \in \Gamma.\\
&\Leftrightarrow y, s \in S, f \text{ is onto, there exist } x, r \in M \text{ such that } \ f(x) = y, f(r) = s.\\
&\Leftrightarrow f(\mu)f(x) = f(\mu)\big(f(r)\alpha f(a)\big) \ \text{ for all } \alpha \in \Gamma\\
&\Leftrightarrow f(\mu)f(x) = f(\mu)\big(f(r\alpha a)\big)\\
&\Leftrightarrow \mu(x) = \mu(r\alpha a) \ \text{ for all } \alpha \in \Gamma\\
&\Leftrightarrow x \in I(a, \mu)\\
&\Leftrightarrow y = f(x) \in f\big(I(a, \mu)\big).
\end{aligned}$$

Hence $I\big(f(a), f(\mu)\big) = f(I(a, \mu)\big)$. □

Theorem 4.6. *Let M be a $\Gamma-$semiring and $a, b \in M, \mu(a),\ \mu(b) \neq \mu(0)$. If a and b are $\mu-$assoicates then $I(a, \mu) = I(b, \mu)$.*

Proof. Let M be a $\Gamma-$semiring and $a, b \in M, \mu(a),\ \mu(b) \neq \mu(0)$. Suppose a and b are $\mu-$assoicates. Then by Definition 4.2, $(a/b)_\mu$ and $(b/a)_\mu$. By Theorem 4.1,

$$\begin{aligned}
I(b, \mu) &\subseteq I(a, \mu)\\
\text{and } I(a, \mu) &\subseteq I(b, \mu)\\
\text{Hence } \ I(b, \mu) &= I(a, \mu).
\end{aligned}$$

Hence the Theorem. □

Theorem 4.7. *Let a, b be in $\Gamma-$semiring M and $\mu(a), \mu(b) \neq \mu(0)$. If $\mu(a) = \mu(b\alpha u)$ for all $\alpha \in \Gamma$, for some $\mu-$unit $u \in M$ then a and b are $\mu-$associates.*

Proof. Let a, b be in $\Gamma-$semiring M and $\mu(a), \mu(b) \neq \mu(0)$. Suppose $\mu(a) = \mu(b\alpha u)$ for all $\alpha \in \Gamma$, for some $\mu-$unit $u \in M$. Then $(b/_a)_\mu$ Since u is a $\mu-$unit, there exists $v \in M$ and $\mu(v) \neq \mu(0)$ such that $\mu(u\beta v) = \mu(e)$, for all $\beta \in \Gamma$.

$$\begin{aligned} \mu(a) &= \mu(b\alpha u), \text{ for all } \alpha \in \Gamma. \\ \Rightarrow \mu(a\beta v) &= \mu(b\alpha u\beta v), \text{ for all } \alpha, \beta \in \Gamma \\ &= \mu(b\alpha e), \text{ for all } \alpha, \beta \in \Gamma \\ &= \mu(b), \text{ for all } \beta \in \Gamma \\ \Rightarrow & (a/_b)_\mu. \end{aligned}$$

Hence a and b are $\mu-$associates. □

Definition 4.5. Let M be a $\Gamma-$semiring and μ be a translational invariant fuzzy subset of M. Suppose element a is not a unit and $\mu(a) \neq \mu(0)$. Then a is said to be $\mu-$prime element if $\left(a/_{b\alpha c}\right)_\mu \Rightarrow \left(a/_b\right)_\mu$ or $\left(a/_c\right)_\mu$ for all $b, c \in M, \alpha \in \Gamma$.

Theorem 4.8. *Let M be a $\Gamma-$semiring, μ be a translational invariant fuzzy subset of M, $a \in M, \mu(a) \neq \mu(0)$ and $I(a, \mu) \neq M$. Then a is $\mu-$prime element if and only if the ideal $I(a, \mu)$ is a prime ideal of $\Gamma-$semiring M.*

Proof. Let M be a $\Gamma-$semiring, μ be a translational invariant fuzzy subset of M, $a \in M, \mu(a) \neq \mu(0)$ and $I(a, \mu) \neq M$. Suppose a is a $\mu-$prime element. By Corollary 3.1, $I(a, \mu)$ is an ideal of $\Gamma-$semiring M. Let $x, y \in M, \alpha \in \Gamma$ and $x\alpha y \in I(a, \mu)$.

$$\begin{aligned} &\Rightarrow \mu(x\alpha y) = \mu(a\beta r), \text{ for some } r \in M, \text{ for all } \beta \in \Gamma. \\ &\Rightarrow \left(a/_{x\alpha y}\right)_\mu \\ &\Rightarrow \left(a/_x\right)_\mu \text{ or } \left(a/_y\right)_\mu, \quad \text{since } a \text{ is a } \mu\text{−prime.} \\ \text{If } \left(a/_x\right)_\mu &\text{ then } \mu(x) = \mu(a\alpha c), \text{ for all } \alpha \in \Gamma, \text{ for some } c \in M \\ &\Rightarrow x \in I(a, \mu) \\ \text{If } \left(a/_y\right)_\mu &\text{ then } y \in I(a, \mu). \end{aligned}$$

Hence $I(a, \mu)$ is a prime ideal of $\Gamma-$semiring M.

Conversely suppose that $I(a, \mu)$ is a prime ideal of $\Gamma-$semiring M. Let $x, y \in M, \alpha \in \Gamma$ and $\left(a/_{(x\alpha y)}\right)_\mu$

$$\begin{aligned} &\Rightarrow \mu(x\alpha y) = \mu(a\beta c), \text{ for all } \beta \in \Gamma, \text{ for some } c \in M \\ &\Rightarrow x\alpha y \in I(a, \mu). \\ &\Rightarrow x \in I(a, \mu) \text{ or } y \in I(a, \mu), \quad \text{since } I(a, \mu) \text{ is a prime ideal.} \\ \text{If } x \in I(a, \mu) &\text{ then} \mu(x) = \mu(y\beta a), \text{ for all } \beta \in \Gamma, \text{ for some } y \in M \\ &\Rightarrow \left(a/_x\right)_\mu. \end{aligned}$$

Similarly we can show that if $y \in I(a,\mu)$ then $a/_{y})_{\mu}$. Hence a is a $\mu-$prime element of $\Gamma-$semiring M. □

Definition 4.6. A fuzzy subset μ is called homomorphism - invariant ($f-$invariant) if f is a $\Gamma-$semring homomorphism.

Theorem 4.9. *Let f be a homomorphism from $\Gamma-$semiring M onto $\Gamma-$semiring S and μ be a translational invariant and $f-$invariant fuzzy subset of M. If a is a $\mu-$prime element of $\Gamma-$semiring M then $f(a)$ is a $f(\mu)-$prime element of $\Gamma-$semiring S.*

Proof. Let μ be a translational invariant and $f-$invariant fuzzy subset of $\Gamma-$semiring M and f be a homomorphism from $\Gamma-$semiring M onto $\Gamma-$semiring S. By Theorem 4.3, $f(\mu)$ is a translational invariant fuzzy subset of S. Suppose a is a $\mu-$prime element of $\Gamma-$semiring M and $\left(f(a)/_{y\alpha z}\right)_{f(\mu)}, y,z \in S, \alpha \in \Gamma$. Since f is an onto homomorphism there exist $b,c \in M$ such that $f(b) = y, f(c) = z$. So $f(b\alpha c) = f(b)\alpha f(c) = y\alpha z$.

$$\begin{aligned}\left(f(a)/_{y\alpha z}\right)_{f(\mu)} &\Rightarrow \text{ there exists } d \in M \text{ such that } f(\mu)\big(f(a)\beta f(d)\big) = f(\mu)f(b\alpha c), \text{ for all } \beta \in \Gamma\\ &\Rightarrow \mu(a\beta d) = \mu(b\alpha c), \text{ for all } \beta \in \Gamma\\ &\Rightarrow \left(a/_{b\alpha c}\right)_{\mu}.\end{aligned}$$

Since a is $\mu-$prime element, we have $\left(a/_{b}\right)_{\mu}$ or $\left(a/_{c}\right)_{\mu}$.

$$\begin{aligned}&\Rightarrow \mu(a\alpha s) = \mu(b) \text{ or } \mu(a\alpha r) = \mu(c), \text{ for all } \alpha \in \Gamma, \text{ for some } s,r \in M\\ &\Rightarrow f(\mu)\Big(f(a\alpha s)\Big) = f(\mu)\Big(f(b)\Big) \text{ or } f(\mu)\Big(f(a\alpha r)\Big) = f(\mu)\Big(f(c)\Big), \text{ for all } \alpha \in \Gamma\\ &\Rightarrow f(\mu)\Big(f(a)\alpha f(s)\Big) = f(\mu)\Big(f(b)\Big) \text{ or } f(\mu)\Big(f(a)\alpha f(r)\Big) = f(\mu)\Big(f(c)\Big), \text{ for all } \alpha \in \Gamma.\\ &\Rightarrow \left(f(a)/_{f(b)}\right)_{f(\mu)} \text{ or } \left(f(a)/_{f(c)}\right)_{f(\mu)}\\ &\Rightarrow \left(f(a)/_{y}\right)_{f(\mu)} \text{ or } \left(f(a)/_{z}\right)_{f(\mu)}.\end{aligned}$$

Hence $f(a)$ is a $f(\mu)-$prime element of $\Gamma-$semiring S. □

Theorem 4.10. *Let f be a homomorphism from $\Gamma-$semiring M onto $\Gamma-$semiring S and μ be a translational invariant and $f-$invarianat fuzzy subset of $\Gamma-$semiring M. If a is a $\mu-$prime element of $\Gamma-$semiring M then the homomorphic image of $I(a,\mu)$ is a prime ideal of $\Gamma-$semiring S.*

Proof. Let f be a homomorphism from $\Gamma-$semiring M onto $\Gamma-$semiring S and μ be a translational invariant and $f-$invarianat fuzzy subset of $\Gamma-$semiring M. By Theorem 4.3, $f(\mu)$ is a fuzzy translational invariant fuzzy subset of $\Gamma-$semiring S. By Theorem 4.8, the $\mu-$principal ideal $I(a,\mu)$ is a prime ideal of $\Gamma-$semiring M where a is a $\mu-$prime element and μ is a $f-$invariant, translational invariant fuzzy subset of M. By Theorem 4.5, $f(I(a,\mu)) = I(f(a), f(\mu))$. By Theorem 4.9, $f(a)$ is $f(\mu)-$prime element $\Gamma-$semiring S. Therefore, by Theorem 4.8, $I(f(a), f(\mu))$ is a prime ideal of $\Gamma-$semiring S. Hence the Theorem. □

5. Conclusion

In this paper, we introduced the notion of left and right translational invariant fuzzy subset of $\Gamma-$semiring M, the notion of unit with respect to fuzzy subset and studied their properties. We proved that if μ is a translational invariant fuzzy subset of a commutative $\Gamma-$semiring with unity then principal ideal generated by an element and μ, contains an unity element is not a proper ideal of $\Gamma-$semiring.

we introduced the notion of associates, prime elements with respect to a fuzzy subset, an ideal of a $\Gamma-$semiring generated by translational fuzzy subset and an element. We studied the properties of image and pre-image of translational invariant fuzzy subset under the $\Gamma-$semiring homomorphism. We proved that every homomorphic image of an ideal of $\Gamma-$semiring generated by $\mu-$prime element and fuzzy translational invariant fuzzy subset μ is a prime ideal of $\Gamma-$semiring.

Our future work on this topic, we will extend these results to other algebraic structures and ordered $\Gamma-$semirings.

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