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Metric and s-metric induced by fuzzy metric and some of its topological properties

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Abstract. In this paper, we attempt to derive a metric (or s-metric) form a fuzzy metric in a sense of A. George and P. Veeramani [4], where the infimum of a D- induced interval is used to evaluate distance between two points. It is proved that every fuzzy-metric space $(X, D, *)$ with its continuous t-norm $*$ between $\tau_e(=$ Einstein product) and $\tau_m(=$ minimum) induces at least an s-metric space (X, d). The topological properties such as the compactness and metrizabilty (fuzzy metrizabilty) of a fuzzy metric induced topological space (X, τ_D) are investigated using its D-induced metric (or s-metric) induced topological space (X, τ_d). Similarly, we investigated the completeness and completion of a fuzzy metric space $(X, D, *)$ using the completion and completeness of its corresponding D-induced(or D-induced s-) metric space (X, d).

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1. Introduction

Since the theory of fuzzy sets, introduced by L.A. Zadeh [35] appeared in 1965 it has been used in a range of areas of mathematics and many authors developed the theory of fuzzy sets and their application in different ways(see, [1,18,21,27,29,30,36]). Fuzzy topology is one example of use of Zadeh's theory. Authors of this field have pursued the definition of a fuzzy metric space from different points of view (see for instance [2, 7, 10, 11, 20, 26]) so that the distance between different elements can be established according to degrees of closeness and remoteness. Authors continued to characterise the theory of fuzzy metric space and apply it in fixed point theory in fuzzy settings(see, [3, 5, 8–10, 12–17, 23, 24, 28, 31–34]).

Completeness and completion are very useful properties in the context of fuzzy and metric spaces. Sherwood proved in [25] in the framework of probabilistic metric spaces that every fuzzy metric space in a sense of Kramisol and Michelak [11] has a completion which is unique up to isometry, with the help of the completeness properties of Lęvy's metric. Unfortunately, there is a fuzzy metric space in a sense George and Veeramani (see [4]) with no completion (see [8]). We will take advantage of this desirable characteristic in order to investigate the completeness of a fuzzy metric space in a sense of George and Veeramani using the completeness of its corresponding induced metric or s -metric space.

The objectives of this paper are:

- to extract a metric or s-metric from a fuzzy metric introduced by George and Veeramani ([4]);
- to investigate the completeness and completion of a fuzzy metric spaces in relation to its induced metric or s- space and ;
- to study the (fuzzy) metrizabilty of a pre-determined topological space with respect to a fuzzy metric space, $(X, D, *)$ using its notion of induced metric space or s -metrics space.

2. Preliminaries

In this section, we briefly summarize important points in fuzzy metric spaces.

Notation. Throughout this paper the following notations are used.
τ_s = Topology derived by s -mapping; τ = topology;
$\mathbb{R}_+$ = the set of non negative real numbers; $a \wedge b = \min(a, b)$;
τ_D = Topology derived by fuzzy -metric, D; $a \vee b = \max(a, b)$;
τ_d = Topology derived by induced-metric (or induced s-metric), d;
T_m = minimum t-norm; T_e = Einstein product t-norm;
$\phi_\alpha = \phi(\_, \alpha)$ and $\phi^t = \phi(t, \_)$; where $\phi : [0, \infty) \times (0, 1) \longrightarrow (0, 1]$.

Definition 2.1 (See [19])**.** Let X be a non-empty set. A function $d : X^2 \to [0, \infty]$ is said to be metric on X if it satisfies:
(1) $d(x, y) = 0$ iff $x = y$
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$
(3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Definition 2.2 (B. Schweizer and A. Sklar [22])**.** A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if $*$ satisfies the following conditions:
(1) $*$ is associative;
(2) $*$ is commutative;
(3) $a * 1 = a$ for all $a \in [0, 1]$;
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0, 1]$.
A t-norm $*$ is continuous if it is continuous with both variables.

Lemma 2.3. *Let $\diamond$ be a t-norm. If $T_e \leq \diamond \leq T_m$, then there exists constant, $c \in [1, 2]$ such that $a \diamond b \geq \frac{(a \wedge b)^2}{c}$, $\forall a, b \in [0, 1]$.*

Proof. Let $a, b \in [0, 1]$ be arbitrary. Clearly,$T_m(a, b) = (a \wedge b) \geq (a \wedge b)^2$. For Eintein product (that is, Hamacher t- norm for $\lambda = 2$), we have $T_e(a, b) = \frac{ab}{2+ab-(a+b)} \geq \frac{(a \wedge b)^2}{2+(a \wedge b)-(a+b)} \geq \frac{(a \wedge b)^2}{2}$. Thus, for $T_e \leq \diamond$, it follows that

$\diamond(a,b) \geq T_e \geq \frac{(a\wedge b)^2}{2}$. Therefore, for each t-norm with $T_e \leq \diamond \leq T_m$, $a \diamond b \geq \frac{(a\wedge b)^2}{c}$ $\forall a, b \in [0,1]$ and for some $c \in [1,2]$. □

Definition 2.4 ([11]). A 3-tuple $(X, D, *)$ is said to be a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous $t-$ norm and D is a fuzzy set on $X \times X \times [0,\infty)$ satisfying the following conditions:

(D_1) $D(x,y,0) = 0$ for all $x, y \in X$;
(D_2) $D(x,y,t) = 1$ if and only if $x = y$;
(D_3) $D(x,y,t) = M(y,x,t)$ for all $x, y \in X$ and $t > 0$;
(D_4) $D(x,y,t) * M(y,z,s) \leq D(x,z,t+s)$ for all $x, y, z \in X$ and for all $s, t \geq 0$;
(D_5) $D(x,y,\_) : (0,\infty) \longrightarrow [0,1]$ is left continuous, $x, y \in X$ and $t, s > 0$.

In 1994 , George and Veeramni [4], slightly modified the notion of fuzzy metric introduced by Kramosil and Michalek by replacing (D_1) and (D_5) of Definition 2.4 respectively by

(D_1') $D(x,y,t) > 0$ for every fixed $x, y \in X$ and for all $t > 0$;
(D_5') $D(x,y,\_) : (0,\infty) \longrightarrow (0,1]$ is continuous for all $x, y \in X$.

Lemma 2.5. *If $(X, D, \diamond)$ is a fuzzy metric space and $*$ is a continuous t-norm such that $* \leq \diamond$, then $(X, D, *)$ is a fuzzy metric space.*

Proof. We need to verify only Definition 2.4(D_4). Since $(X, D, \diamond)$ is a fuzzy metric space, for all $x, y, z \in X$ and $t, s \in (0,\infty)$, $D(x,y,s+t) \geq D(x,z,s) \diamond D(y,z,t)$. Since $* \leq \diamond, D(x,z,s) \diamond D(y,z,t) \geq D(x,z,s) * D(y,z,t)$. Thus,
$D(x,y,s+t) \geq D(x,z,s) * D(y,z,t)$. Hence the result holds. □

Remark 2.6 (M. Grabiec, [5]). If $(X, D, \diamond)$ is a fuzzy metric space, then $D(x,y,\_)$ is a non-decreasing for all $x, y \in X$.

Definition 2.7 (V. George and P. Veeramani [4]). Let $(X, D, *)$ be a fuzzy metric space.

(i) A sequence $\{x_n\} \subset X$ is said to be D-Cauchy sequence, if for each $r \in (0,1)$ and $t > 0$,there exists $n(r) \in \mathbb{N}$ such that $D(x_n, x_m, t) > 1 - r$ for all $m, n \geq n(r)$.
(ii) A sequence $\{x_n\}$ is said to be D-convergent to $x \in X$, if for each $r \in (0,1)$ and $t > 0$,there exists $n(r) \in \mathbb{N}$ such that $D(x_n, x, t) > 1 - r$ for all $n \geq n(r)$.
(iii) $(X, D, *)$ is said be complete, if every D-Cauchy sequence is D-convergent.

3. Metric space induced by fuzzy metric space

In this section, we attempted to derive a metric from a given fuzzy metric in a sense of A. George and P. Veeramani [4]. We study the completeness and completion of some class of fuzzy metric spaces with respect to completeness and completion of their respective induced metric spaces.

Lemma 3.1. *If $(X, D, *)$ is a fuzzy metric space, then D is continuous.*

Proof. Let $\lim_n x_n = x_o$ and $\lim_n y_n = y_o$ in $(X, D, *)$. We need to show that $\lim_n D(x_n, y_n, t) = D(x_o, y_o, t)$. Since $\lim_n x_n = x_o$, $\lim_n y_n = y_o$, by Definition 2.7, it follows that $\lim_n D(x_n, x_o, t) = 1$, $\lim_n D(y_n, y_o, t) = 1$. From Definition 2.4, for $s > t$, we have

$$(3.1)\ (\forall n \in \mathbb{N})\ D(x_0, y_o, s+t) \geq D(x_o, x_n, t) * D(x_n, y_o, s) \geq D(x_o, x_n, t) * D(x_n, y_n, s-t) * D(y_n, y_o, t)$$

By letting $n \longrightarrow \infty$ in (3.1), for all $s > t$, we get

$$D(x_0, y_o, s+t) \geq \lim_n\{(D(x_o, x_n, t) * D(y_o, y_n, s)) * D(x_n, y_n, s-t)\} = (1 * 1) * \lim_n D(x_n, y_n, t) = \lim_n D(x_n, y_n, s-t).$$

Since $D(x, y, \_)$ is continuous,

$$D(x_0, y_o, s) = \lim_{t\to 0^+} D(x_0, y_o, s+t) \geq \lim_{t\to 0^+} \lim_n D(x_n, y_n, s-t) = \lim_n D(x_n, y_n, s).$$

Thus, $D(x_0, y_o, s) \geq \lim_n D(x_n, y_n, s)$. Similarly, $\lim_n D(x_n, y_n, s) \geq D(x_o, y_o, s)$. From the above discussions, we conclude that $D(x_0, y_o, s) = \lim_n D(x_n, y_n, s)$. □

Definition 3.2. Let X be a non-empty set and let $\gamma : X \times X \times [0, \infty) \to [0, \infty)$ be a mapping such that

(i) γ is continuous; $\gamma(x, y, t) = \gamma(y, x, t)$ for all $x, y \in X$ and $t \geq 0$;
(ii) $\gamma_{x,y} := \gamma(x, y, \_)$ is a non-decreasing function;
(iii) $0 < \gamma(x, y, t)$ for all $x, y \in X$ and for all $t > 0$.

We denote $\Gamma = \{\gamma : X \times X \times [0, \infty) \to [0, \infty) : \gamma \text{ satisfies (i) to (iii)}\}$.

Lemma 3.3. *Let $\gamma \in \Gamma$. If $\gamma_{x,y}$ is onto mapping, then $\gamma_{x,y}(0) = 0$ for all $x, y \in X$.*

Proof. Let $\gamma_{x,y}$ be onto. If $\gamma_{x,y}(0) \neq 0$ for some $x, y \in X$, then there exists $r > 0$ such that $0 < r < \gamma_{x,y}(0)$. Since $\gamma_{x,y}$ is onto, there exists $t_0 > 0$ such that $\gamma_{x,y}(t_0) = r$. Hence, $\gamma_{x,y}(t_0) < \gamma_{x,y}(0)$. This contradicts to Definition 3.2 (ii). Therefore, $\gamma_{x,y}(0) = 0$. □

Definition 3.4. Let $\phi : [0, \infty) \times (0, 1) \to (0, 1]$ be such that

(i) ϕ is continuous;
(ii) $\phi(t, \beta) \leq \phi(s, \alpha)$, $\forall\ \beta \leq \alpha$ and $\forall\ s \leq t$.
(iii) for each $t \in (0, \infty)$ and $\alpha \in (0, 1)$, $0 < \phi(t, \alpha) < 1$. We denote $\Phi = \{\phi : [0, \infty) \times (0, 1) \to (0, 1] : \phi \text{ satisfies (i) to (iii)}\}$.

Lemma 3.5. *Let $(\phi, \gamma, \alpha, t_0) \in (\Phi \times \Gamma \times (0, 1) \times (0, \infty)$ be fixed and let $\gamma \in \Gamma$. Let $\alpha \in (0, 1)$, $t_0 \in (0, \infty)$ and $x, y \in X$ be arbitrary.*

(i) $\phi_\alpha := \phi(\_, \alpha)$ is non-increasing; $\phi^{t_0} := \phi(t_0, \_)$ is a non-decreasing.
(ii) If ϕ_α is an onto mapping, then $\phi_\alpha(0) = 1$.
(iii) If ϕ_α and $\gamma_{x,y}$ are onto mappings, then $\phi_\alpha(\gamma_{x,y}(0)) = 1$.

Proof. Using Definition 3.4 (ii) and Definition 3.2, it is easy to verify the validity of the statements in Lemma 3.5(i). If $\phi_\alpha(0) \neq 1$, then there exists $r \in (0, 1)$ with $0 < \phi_\alpha(0) < r < 1$. Since ϕ_α is onto, there exists $k \in (0, \infty)$ such that $\phi_\alpha(k) = r$. By (i) of Lemma 3.5, ϕ_α is non -increasing. Hence $r = \phi_\alpha(k) \leq \phi_\alpha(0) < r$. This is a contradiction. Therefore, $\phi_\alpha(0) = 1$. Hence (ii) is proved. The proof of (iii) follows from Lemma 3.3 and (ii) of this lemma. □

Lemma 3.6. *Let $(X, D, *)$ be a fuzzy metric space, $(\phi, \gamma) \in \Phi \times \Gamma$. Assume that $a, x, y, u, v \in X$ and that α, $\beta \in (0, 1)$ with $\alpha \leq \beta$ are fixed.*

(i) If $D(x, y, t) \geq D(u, v, t)$, $\gamma_{x,y}(t) \geq \gamma_{u,v}(t)$ for all $t > 0$ and
$A = \{t > 0 : D(x, y, t) > \phi_\alpha(\gamma_{x,y}(t))\}$;

$B = \{t > 0 : D(u, v, t) > \phi_\alpha(\gamma_{u,v}(t))\}$, *then* $A \supseteq B$.

(*ii*) *If* $A = \{t > 0 : D(x, y, t) > \phi_\beta(\gamma_{x,y}(t))\}$;
$B = \{t > 0 : D(x, y, t) > \phi_\alpha(\gamma_{x,y}(t))\}$, *then* $A \subseteq B$.

(*iii*) *If* $A = \{z \in X : \inf\{t > 0 : D(a, z, t) > \phi_\alpha(\gamma_{a,z}(t))\} < t_0\}$ *and*
$B = \{z \in X : D(a, z, t_o) > \phi_\alpha(\gamma_{a,z}(t_0))\}$, *then* $B = A$.

(*iv*) *If* ϕ_α *and* $\gamma_{x,y}$ *are onto,* $A = \{t > 0 : D(x, y, t) > \phi_\alpha(\gamma_{x,y}(t))\}$ *and*
$B = \{t > 0 : D(u, v, t) > \phi_\alpha(\gamma_{u,v}(t))\}$, *then* $A \cap B \neq \varnothing$.

(*v*) *If* ϕ_α *and* $\gamma_{x,y}$ *are onto and* $D(x, y, t) > \phi_\alpha(\gamma_{x,y}(t))$ *for all* $t > 0$, *then* $x = y$.

Proof. It is easy to show the validity of (i) and (ii). So we verify only the validity of (iii) to (v). If $y \notin A$, then $\inf\{t > 0 : D(a, y, t) > \phi_\alpha(\gamma_{a,y}(t))\} \geq t_0$. That is, $D(a, y, t_o) \leq \phi_\alpha(\gamma_{y_0,y}(t_0)) \Longrightarrow y \notin B$. Thus, $B \subseteq A$. Conversely, if $y \notin B$, then $D(a, y, t_o) \leq \phi_\alpha(\gamma_{a,y}(t_0)) \Longrightarrow \inf\{t > 0 : D(a, y, t) > \phi_\alpha(\gamma_{a,y}(t))\} \geq t_o$. Hence $y \notin A \Longrightarrow A \subseteq B$. Therefore, $A = B$. Hence (iii) is proved.

Since $D(x, y, \_)$ is non-decreasing and ϕ_α is non-increasing, if $t_0 \in A$ for some $t_0 \in (0, \infty)$, then $t \in A$ for all $t \geq t_0$. Since ϕ_α is onto mapping, there exists $s > 0$ such that $\phi_\alpha(\gamma_{x,y}(s)) = \frac{\alpha}{\alpha+t_0}$. Now, Assume $A \cap B = \varnothing$. Then for $t_0 \in A$, we have

$$D(u, v, t_0) \leq \phi_\alpha(\gamma_{x,y}(t)) \leq \phi_\alpha(s) = \frac{\alpha}{\alpha + t_0} \text{ for all } t \geq \max(t_0, s). \tag{3.2}$$

Form (3.2), it follows $D(u, v, t_0) \neq 1$ for all $t_0 \in A$ and

$$\frac{t_0}{1 - D(u, v, t_0)} D(u, v, t_0) \leq \alpha, \text{for all } t_0 \in A. \tag{3.3}$$

Since we can take $t_0 \in A$ as large as we wish, Inequality (3.3) is valid only if $D(u, v, k) = 0$ for some $k \in A$. But, this contradicts to (D'_1). Thus, our assumption was wrong. Therefore, $A \cap B \neq \varnothing$. Thus, (*iv*) is proved.

Since ϕ_α and $\gamma_{x,y}$ are onto, by Lemma 3.5 and Continuity of ϕ_α, it follows that $\phi_\alpha(0) = 1 = \lim_{t\longrightarrow 0^+} \phi_\alpha(\gamma_{x,y}(t)) \leq \lim_{t\longrightarrow 0^+} D(x, y, t) \leq 1$. That is,

$$\lim_{t\longrightarrow 0^+} D(x, y, t) = 1. \tag{3.4}$$

Since $D(x, y, \_)$ is non-decreasing and (3.4) is valid, we get $D(x, y, t) = 1$ for all $t > 0$. Thus $x = y$. Thus, Lemma 3.5 is proved. Hence the result. □

Remark 3.7. Let $(X, D, *)$ be a fuzzy metric space. If $(\psi, \gamma, \alpha) \in \Psi \times \Gamma \times (0, 1)$ and $x, y \in X$ are fixed, then $\mathbb{A} = \{t > 0 : D(x, y, t) > \psi_\alpha(\gamma_{x,y}(t))\}$ is interval.

Definition 3.8. Let $(X, D, *)$ be a fuzzy metric space, $(\psi, \gamma, \alpha) \in \Psi \times \Gamma \times (0, 1)$ and $x, y \in X$ be fixed. A set $\mathbb{A} = \{t > 0 : D(x, y, t) > \psi_\alpha(\gamma_{x,y}(t))\}$ is said to be a D-induced interval.

Lemma 3.9. *If* $B, A \subset \mathbb{R}_+$ *and* $A \cap B \neq \varnothing$, *then*

(*i*) $\inf(A \cap B) \leq \inf(A) + \inf(B)$, *whenever* A *and* B *are intervals;*

(*ii*) $\inf\{A + B\} = \inf(A) + \inf(B)$.

Definition 3.10. Let $(X, D, *)$ be a fuzzy metric space. We say that a fuzzy metric D induces a metric on X if there exist an $\alpha \in (0, 1)$, $\phi \in \Phi$ and $\gamma \in \Gamma$ such that a function, $d : X^2 \to [0, \infty)$ defined by

$d(x,y) = \inf\{t > 0 : D(x,y,t) > \phi_\alpha(\gamma_{x,y}(t))\}$ is a metric on X. We call the corresponding metric space, (X,d), D-induced metric space of (ϕ,γ,α).

Notation: Hereafter, we refer (X,d) as D-induced metric space of (ϕ,t,α), if $\gamma_{x,y}(t) = t$ for all $x,y \in X$ and for all $t > 0$.

Theorem 3.11. *Let $(X,D,*)$ be a fuzzy metric space, $(\phi,\gamma,\alpha) \in \Phi \times \Gamma \times (0,1)$. Assume that both ϕ_α and $\gamma_{x,y}$ are onto mappings for every fixed $x,y,\in X$. Let*
$A = \{t > 0 : D(x,y,t+s) > \phi_\alpha(\gamma_{x,y}(t))\}$ and
$B = \{t > 0 : D(x,z,t) > \phi_\alpha(\gamma_{x,z}(t))\}$, $C = \{t > 0 : D(z,y,t) > \phi_\alpha(\gamma_{z,y}(t))\}$, $E = B \cap C$. If $E \subseteq A, \forall\ x,y,z \in X$, then D induces a metric on X.

Proof. Let $(X,D,*)$, (ϕ,γ), A, B, C and E be as stated in Theorem 3.11 . Define $d : X^2 \to [0,\infty)$ by $d(x,y) = \inf\{t > 0 : D(x,y,t) > \phi_\alpha(\gamma_{x,y}(t))\}$. Then
$0 = d(x,y) = \inf\{t > 0 : D(x,y,t) > \phi_\alpha(\gamma_{x,y}(t))\} \Longrightarrow D(x,y,t) > \phi_\alpha(\gamma_{x,y}(t))$ for all $t > 0$. Therefore, by Lemma 3.6, we have $x = y$. The converse is clear. Thus, $d_\alpha(x,y) = 0$ iff $x = y$. Since $D(x,y,t) = D(y,x,t)$ for all $t > 0$, it follows that $d(x,y) = d(x,y)$ for all $x,y \in X$. Let A and E be as stated in Theorem 3.11. By Lemma 3.6, it follows that $E \neq \varnothing$. If $E \subset A$, then $\inf(A) \leq \inf E$ and by Lemma 3.9, we get $\min(A) \leq \inf(E) \leq \inf(B) + \inf(C)$. That is,
$d(x,y) = \inf(A) \leq \inf(E) \leq \inf(B) + \inf(C) = d(x,z) + d(z,y)$. □

Theorem 3.12. *Let $(X,D,*)$ be a fuzzy metric space, $(\phi,\gamma,\alpha) \in \Phi \times \Gamma \times (0,1)$. Assume that both ϕ_α and $\gamma_{x,y}$ are onto mappings for every fixed $x,y \in X$. Let*
$A = \{t > 0 : D(x,y,t) > \phi_\alpha(\gamma_{x,y}(t))\}$ and
$B = \{t > 0 : D(x,z,t) > \phi_\alpha(\gamma_{x,y}(t))\}$, $C = \{s > 0 : D(z,y,s) > \phi_\alpha(\gamma_{z,y}(s))\}$, $E = B + C$. $E \subseteq A, \forall\ x,y,z \in X$ if and only if D induces a metric on X.

Proof. Let $(X,D,*)$, (ϕ,γ,α), A, B, C and E be as stated in Theorem 3.12. Define $d : X^2 \to [0,\infty)$ by $d(x,y) = \inf\{t > 0 : D(x,y,t) > \phi_\alpha(\gamma_{x,y}(t))\}$. If $0 = d(x,y) = \inf\{t > 0 : D(x,y,t) > \phi_\alpha(\gamma_{x,y}(t))\}$, then $D(x,y,t) > \phi_\alpha(\gamma_{x,y}(t))$ for all $t > 0$. Therefore, by Lemma 3.6, we have $x = y$. The converse is clear. Thus, $d(x,y) = 0$ iff $x = y$. Since $D(x,y,t) = D(y,x,t)$ for all $t > 0$, it follows that $d(x,y) = d(x,y)$ for all $x,y \in X$. Let A and E be as stated in the Theorem. By Lemma 3.6, it follows that $E \neq \varnothing$. If $E \subset A$, then $\inf(A) \leq \inf(E)$ and by Lemma 3.9, it follows that $\min(A) \leq \inf(E) = \inf(B) + \inf(C)$. That is,
$d(x,y) = \inf(A) \leq \inf(B) + \inf(C) = d(x,z) + d(z,y)$. Conversely, let d be a D-induced metric of (ϕ,γ,α) and both ϕ_α and $\gamma_{x,y}$ be onto mappings for every fixed $x,y \in X$. Since $\phi_\alpha(\gamma_{x,y})$ is onto for every fixed $x,y \in X$ and non-increasing; $D(x,y,\_)$ is a continuous for for every fixed $x,y \in X$ and non-decreasing, there exists $t_{xy} \in (0,\infty)$ such that $D(x,y,t_{xy}) = \phi_\alpha(\gamma(t_{xy}))$ for each $x \neq y$. Thus, if d is a D-induced metric on X, then $d(x,y) = t_{xy}$ and
$\{t > 0 : D(x,y,t) > \phi_\alpha(\gamma_{x,y}(t))\} = (t_{xy},\infty)$. If $x,y,z \in X$ is arbitrarily fixed, then
$B = \{t > 0 : D(x,z,t) > \phi_\alpha(\gamma_{x,y}(t))\} = (t_{xz},\infty)$,
$C = \{t > 0 : D(x,y,t) > \phi_\alpha(\gamma_{x,y}(t))\} = (t_{zy},\infty)$,
$A = \{t > 0 : D(x,y,t) > \phi_\alpha(\gamma_{x,y}(t))\} = (t_{xy},\infty)$ and $E = B + C = (t_{xz} + t_{zy},\infty)$.
Now if $E \not\subseteq A$, then $A \subset E$. Consequently, $\inf E < \inf A$. That is,
$d(x,z) + d(z,y) = \inf E < \inf A = d(x,y)$. This is a contradiction to assumption d is a D-induced metric on X. Therefore, $E \subseteq A$. □

Corollary 3.13. *If $(X, D, \wedge)$ be a fuzzy metric space, then D induces a metric on X, where $a \wedge b = \min(a, b)$ for all $x, y \in [0,1]$.*

Proof. Let $\alpha \in (0,1)$ be arbitrarily fixed. Consider
$\phi : [0, \infty) \times (0,1) \longrightarrow (0,1]$ and and $\gamma : X \times X \times [0, \infty)$ defined by
$\phi(t, \beta) = \frac{\beta}{\beta+t}$ and $\gamma(x, y, t) = t$ respectively. Clearly, $(\phi, t) \in \Phi \times \Gamma$ and both ϕ_α and $\gamma_{x,y}$ are onto mappings for every fixed $\alpha \in (0,1)$ and $x, y \in X$ respectively.
Let $A = \{t + s > 0 : D(x, z, t + s) > \phi_\alpha(t + s),\ s, t > 0\}$
$B = \{t > 0 : D(x, z, t) > \phi_\alpha(t)\} + \{s > 0 : D(z, y, s) > \phi_\alpha(s)\}$
$C = \{t + s > 0 : \min\{D(x, z, t), D(z, y, s)\} > \phi_\alpha(t + s)\ \ s, t > 0\}$
Now we need to show that $A \supseteq B$. Since $A \supseteq C$, it suffice to show $C \supseteq B$. Indeed, if $k_0 = t_o + s_0 \notin C$, then $\min\{D(x, z, t_o), D(z, y, s_0)\} \leq \phi_\alpha(t_0 + s_0)$. That is, either $D(x, z, t_0) \leq \phi_\alpha(t_o + s_0) \leq \phi_\alpha(t_0)$ or $D(z, y, s_0) \leq \phi_\alpha(t_0 + s_0) \leq \phi_\alpha(s_0)$. Hence $k_o = t_o + s_0 \notin B$. Thus, $B \subseteq C$. So, $B \subseteq A$. Therefore, all assumptions and conditions of Theorem 3.12 are satisfied and hence the conclusion of Corollary follows by Theorem 3.12. □

Theorem 3.14. *Let $(X, D, *)$ be a fuzzy metric space such that*
$\Omega = \{\alpha \in (0,1) : d_\alpha$ is a metric on X induced by $D\} \neq \varnothing$. If
$\alpha_0 = \inf(\Omega)$ and $\beta_0 = \sup(\Omega)$, and $\beta_0 \neq \alpha_0$, then $\{d_\alpha, \alpha \in (\alpha_0, \beta_0)\}$ is an increasing collection of metrics on X with respect to $\alpha \in (0,1)$.

Proof. Since $\Omega \neq \varnothing$, there exists $(\phi, \gamma, \alpha) \in \Phi \times \Gamma \times (0,1)$ such that
$d_\alpha(x, y) = \inf\{t > 0 : D(x, y, t) > \phi_\alpha(\gamma_{x,y}(t))\}$ defines a metric on X. Let α_0 and β_0 be as stated in the theorem and let $\alpha, \beta \in (\alpha_0, \beta_0)$ such that $\alpha \leq \beta$. Define $A = \{t > 0 : D(x.y, t) > \phi_\alpha(\gamma_{x,y}(t))\}$ and $B = \{t > 0 : D(x, y, t) > \phi_\beta(\gamma_{x,y}(t))\}$. By Lemma 3.6, $B \subseteq A$. Hence $\inf(A) \leq \inf(B)$. That is $d_\alpha(x, y) \leq d_\beta(x, y)$ for all $x, y \in X$. □

Theorem 3.15. *Let $(X, D, *)$ be a fuzzy metric space. If (X, d) is a D-induced metric space of (ϕ, t, α), then every D-Cauchy sequence is a d-Cauchy sequence.*

Proof. If $\{x_n\}$ is a D-Cauchy sequence, then for $1 > r > 0$ and $t > 0$, there exists a $k \in \mathbb{N}$ such that $D(x_n, x_m, t) > 1 - r$ for all $m > n \geq k$. Now let $0 < t_0$ be arbitrarily fixed. For $r_0 = 1 - \phi_\alpha(\frac{t_0}{2})$, there exists $n_0 \in \mathbb{N}$ such that
$D(x_n, x_m, \frac{t_0}{2}) > (1 - r_0) = \phi_\alpha(\frac{t_0}{2})$ for all $m > n \geq n_0 \geq k$. By definition of D-induced metric of (ϕ, t, α), we have $d_\alpha(x_n, x_m) = \inf\{t > 0 : D(x_n, x_m, t) > \phi_\alpha(t)\} \leq \frac{t_0}{2} < t_0$ for all $m > n \geq n_0$. Since $0 < t_0$, was arbitrary, we conclude that $\{x_n\}$ is a d-Cauchy sequence. □

Theorem 3.16. *Let $(X, D, *)$ be a fuzzy metric space. If (X, d_α) be a D-induced metric space of (ϕ, t, α). Then every D-convergent sequence is d-convergent sequence.*

Proof. The proof is similar to that of Theorem 3.15. □

Corollary 3.17. *Let $(X, D, *)$ be a fuzzy metric space and (X, d) be a D-induced metric space of (ϕ, t, α). If (X, d) is a complete metric space, then $(X, D, *)$ is a complete fuzzy metric space.*

Proof. Proof of corollary follows from Theorem 3.15 and Theorem 3.16. □

Corollary 3.18. *Let $(X, D, *)$ be a fuzzy metric space. If (X, d) is a D-induced metric space of (ϕ, t, α), then $(X, D, *)$ has a completion.*

Proof. Since (X, d) is a metric space, it has a completion (see, [19]), $(X^\star, d)$. Therefore, the result follows by Corollary 3.17. □

Theorem 3.19. *Let $(X, D, *)$ be a fuzzy metric space, (X, d) be a D-induced metric space of (ϕ, t, α) and ϕ_α is an onto mapping. A sequence $\{x_n\} \subset X$ is a d-Cauchy sequence if and only it is a D-Cauchy sequence.*

Proof. Let $(X, D, *), (X, d)$ and ϕ_α be as stated in the Theorem. If $\{x_n\}$ is a d-Cauchy sequence, then $\lim_{n,m} d_\alpha(x_n, x_m) = 0$. That is,

$$\begin{aligned} 0 &= \lim_{n,m}\{\inf\{t > 0 : D(x_n, x_m, t) > \phi_\alpha(t)\}\} \\ &= \inf\{t > 0 : \lim_{n,m} D(x_n, x_m, t) > \phi_\alpha(t)\} \end{aligned} \tag{3.5}$$

From (3.5), it follows that

$$\lim_{n,m} D(x_n, x_m, t) > \phi_\alpha(t) \text{ for all } t > 0. \tag{3.6}$$

By continuity of ϕ_α and Lemma 3.5, we have

$$\lim_{t \longrightarrow 0^+} \phi_\alpha(t) = 1. \tag{3.7}$$

From (3.6) and (3.7), it follows $\lim_{m,n} D(x_n, x_m, t) = 1$ for all $t > 0$.
The Converse follows from Theorem 3.15. □

Theorem 3.20. *Let $(X, D, *)$ be a fuzzy metric space, (X, d) be a D induced-metric space of onto pair (ϕ, t, α) and ϕ_α is an onto mapping. A sequence $\{x_n\} \subseteq X$ is a D- convergent if and only if it is a d-convergent .*

Proof. The proof of the theorem is similar to that of Theorem 3.19. □

Corollary 3.21. *Let $(X, D, *)$ be a fuzzy metric space, (X, d) be a D-induced metric space of onto pair (ϕ, t, α). $(X, D, *)$ is a complete fuzzy metric space if and only if (X, d) is a complete metric space.*

Proof. Proof of corollary follows from Theorem 3.19 and Theorem 3.20. □

Theorem 3.22. *Let $(X, D, *)$ and $(X, D, \diamond)$ be fuzzy metric spaces. Then $(X, D, \diamond)$ induces a metric space iff $(X, D, *)$ induces metric space.*

Proof. If $(X, D, \diamond)$ induces a fuzzy metric space, then there exists
$(\phi, \gamma, \alpha) \in \Phi \times \Gamma \times (0, 1)$ such that $d_\alpha : X^2 \to [0, \infty)$ defined by
$d_\alpha(x, y) = \{t > 0 : D(x, y, t) > \phi_\alpha(\gamma_{x,y}(t))\}$ defines a metric on X. Consequently, we have $D(x, y, t) > \phi_\alpha(\gamma_{x,y}(t))\ \forall\ t > 0 \Longrightarrow x = y$ and $d_\alpha(x, y) = d_\alpha(y, x)$ for all $x, y \in X$. If $A = \{t : D(x, y, s + t) > \phi_\alpha(t)\}$, $B = \{t : D(x, z, t) > \phi_\alpha(\gamma_{x,y}(t))\}$, $C = \{t : D(z, y, t) > \phi_\alpha(\gamma_{x,y}(t))\}$ and $E = B + C$, then by Theorem 3.12, $E \subseteq A$ for all $x, y, z \in X,\ t, s > 0$ irrespective of t-norm involved. Thus, $(X, D, *)$ induces a metric space if and only if $(X, D, \diamond)$ induces a metric space. □

4. s-metric and induced s-metric spaces

In this section, we introduce the concept of an s-metric and derive it from a fuzzy metric in a sense of A.George and P. Veeramani. We study the completeness and completion of some class of fuzzy metric spaces with respect to completeness and completion of their respective s-metric spaces.

Definition 4.1. Let X be nonempty set. A mapping $d : X^2 \longrightarrow [0, \infty)$ is said to be s- metric on X if there exists a continuous mapping (s-mapping)
$s : X^2 \longrightarrow [0, \infty)$ with $s(x, y) = s(y, x),\ \forall y, x \in X$ such that

(1) $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$;
(2) $d(x, y) = d(y, x)\ \forall x, y \in X$;
(3) $d(x, y) \leq s(x, z) + s(z, y)\ \forall x, y, z \in X$;

A pair (X, d) is said to be an s-metric space. If s is metric, we call d almost metric and (X, d) almost metric space

Example 4.2. Every metric is a s - metric.

Example 4.3. Let $X = [1, \infty)$ and let D be a fuzzy set in $X^2 \times (0, \infty)$ defined by $D(x, y, t) = \frac{x \wedge y}{x \vee y}$. A mapping, $d_{\alpha s}$, on X^2 defined by
$d_\alpha(x, y) = \inf\{t > 0 : D(x, y, t) > \frac{\alpha}{\alpha + t}\}$ is a s - metric on X, where $\alpha \in (0, 1)$ is fixed.

Proof. Let d_α and D be as stated in the Example. Then
$d_\alpha(x, y) = \inf\{t > 0 : D(x, y, t) > \frac{\alpha}{\alpha+t}\} = \frac{\alpha((x \vee y) - (x \wedge y))}{x \wedge y}$
$\Rightarrow d_\alpha(x, y) = 0$ iff $x = y$, $d_\alpha(x, y) = d_\alpha(y, x)$ and $d_\alpha(x, y) \leq \alpha|x - y|$. Define $s : X^2 \to [0, \infty)$ by $s(x, y) = \alpha|x - y|$. Thus, $d_\alpha(x, y) \leq s(x, y) \leq s(x, z) + s(z, y)$. Hence d is a s-metric on X. Moreover, d is almost metric on X. □

We define, a convergent sequence and a Cauchy sequence in an s-metric space as usual.

Definition 4.4. Let (X, d) be a s- metric space.

(1) A sequence $\{x_n\} \subset X$ is said to be d-convergent if for every given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, y) < \epsilon$ for some $y \in X$ and $\forall\ n \geq n_0$.
(2) A sequence $\{x_n\} \subset X$ is said to be d-Cauchy if for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon,\ \forall m > n \geq n_0$.

Proposition 4.5. *Let (X, d) be an s-metric space. (i) A sequence $\{x_n\}$ in X is convergent to $x \in X$ iff $\lim_n d(x_n, x) = 0$.*
(ii) A sequence $\{x_n\}$ in X is d-Cauchy iff $\lim_{n,m} d(x_n, x_m) = 0$.

Proof. (i) If $\lim_n x_n = x$, then by definition given $r \in (0, 1)$, the exists $n_0 \in \mathbb{N}$ such that $d(x, x_n) < r$ for all $n \geq n_0$. Since $r \in (0, 1)$ was arbitrary, we conclude that $\lim_n d(x, x_n) = 0$. The converse is clear. Similarly, (ii) holds. □

Definition 4.6. Let (X, d) be an s- metric space. We define

(1) s-open ball with center x_0 and radius r_0 by
$B_s(x_0, r_0) = \{y \in X : S(x_0, y) < r_0\}$ if $s(x, x) = 0$ for all $x \in X$
(2) d-open ball with center x_o and radius r_0 by
$B_d(x_0, r_0) = \{y \in X : d(x_0, y) < r_0\}$.

Proposition 4.7. *If (X, d) is an s-metric space with $s(x, x) = 0$ for all $x \in X$, then for each $x \in X$ and each $r > 0$ (i) $B_s(x, r) \subseteq B_d(x, r)$; (ii) $\exists\ t > 0$ such that $B_d(x, r) \subseteq B_s(x, t)$.*

Proof. Let (X, d) be an s-metric space and let $x \in X,\ r > 0$ be arbitrary.

(i) Since (X, d) a s- metric space, for every $x, y \in X$, we have $d(x, y) \leq s(x, y)$. So, if $z \notin B_d(x, r)$, then $s(x, z) \geq d(x, z) \geq r$. That is,
$z \notin B_s(x, r) \Longrightarrow B_s(x, r) \subseteq B_d(x, r)$.

(ii) Suppose to the contrary $B_d(x, r) \nsubseteq B_s(x, t)$ for each $t > 0$. Assume $r \leq t$. So, if $B_d(x, r) \nsubseteq B_s(x, t)$, then there exists $z \in B_d(x, r)$ such that
$z \notin B_s(x, t) \Longrightarrow r > s(x, z) \geq t$ (by (i)) $\Longrightarrow r > t$. This is the contradiction to $r \leq t$. Thus, there exists $t > 0$ such that $B_d(x, r) \subseteq B_s(x, t)$. □

Definition 4.8. Let (X, d) be an s-metric space such that $s(x, x) = 0$ for all $x \in X$.

(1) A non-empty set $A \subset X$ is said to be s-open if for each $z \in A$, there exists $r_o > 0$ such that $B_s(z, r_o) \subset A$.
(2) A non-empty set $A \subset X$ is said be d-open if for each $z \in A$, there exists $r_o > 0$ such that $B_d(z, r_o) \subset A$.

Proposition 4.9. *Let (X, d) be an s-metric space such that $s(x, x) = 0$ for all $x \in X$. If $\tau_s = \{A \subset X : A \text{ is } s\text{-open set } \}$ and $\tau_d = \{A \subset X : A \text{ is } d \text{ open set } \}$, then $\tau_s = \tau_d$.*

Proof. If $A \in \tau_s$, then for every $x \in A$, there exists $r > 0$ such that
$B_s(x, r) \subset A$. By Proposition 4.7, there exists $r_0 > 0$ such that
$B_d(x, r_0) \subset B_s(x, r) \subset A$. Therefore, A is a d-open set in X. Thus, $A \in \tau_d$. That is $\tau_s \subset \tau_d$. Conversely, if $A \in \tau_d$, then for every $x \in A$, there exists $r > 0$ such that $B_d(x, r) \subset A$. By Proposition 4.7, $B_s(x, r) \subset B_d(x, r) \subset A$. Therefore, A is a s-open set in X. Thus, $A \in \tau_s$. That is, $\tau_d \subset \tau_s$. □

It is not difficult to show τ_d (also τ_s if $s(x, x) = 0$ for all $x \in X$) is a topology on X. We call, this topology d-derived topology on X.

Definition 4.10. Let $(X, D, *)$ be a fuzzy metric space. We say that a fuzzy metric D induces an s-metric on X if there exists $(\phi, \gamma, \alpha) \in \Phi \times \Gamma \times (0, 1)$ such that a function, $d : X^2 \longrightarrow [0, \infty)$ defined by
$d(x, y) = \inf\{t > 0 : D(x, y, t) > \phi_\alpha(\gamma_{x,y}(t))\}$ is an s-metric on X. We call the corresponding induced s-metric space (X, d) a D-induced s-metric space of(ϕ, γ, α).

Theorem 4.11. *Every D-induced s-metric of (ϕ, γ, α) is continuous.*

Proof. Let (X, d) be a D-induced s-metric space of (ϕ, γ, α). If $\lim_n x_n = x$, $\lim_n y_n = y$, then by Lemma 3.1, Definition 3.2, Definition 3.4 and Definition 4.10, it follows that

$$\begin{aligned} \lim_n d(x_n, y_n) &= \inf\{t > 0 : \lim_n D(x_n, y_n, t) > lim_n \phi_\alpha(\gamma_{x_n,y_n}(t))\} \\ &= \inf\{t > 0 : D(x, y, t) > \phi_\alpha(\gamma_{x,y}(t))\} \\ &= d(x, y). \end{aligned}$$

Therefore, d-is continuous. □

Corollary 4.12. *If (X, d) is a D-induced s-metric space of (ϕ, γ, α), then a limit of d-convergent sequence is unique.*

Proof. Let (X, d) be D-induced s-metric space. Let $\{x_n\}$ be d-convergent sequence. If $\lim_n x_n = x$ and $\lim_n x_n = y$, then by Theorem 4.11, we get
$d(x, y) = \lim_n d(x, x_n) = d(x, x) = 0$. Therefore, by Definition 4.1 it follows that $x = y$. ☐

Theorem 4.13. *Let $(X, D, *)$ be a fuzzy metric space. If $T_e \leq * \leq T_m$, then D induces an s-metric.*

Proof. Suppose$(X, D, *)$ is a fuzzy metric space such that $T_e \leq * \leq T_m$. Let $(\phi, t, \alpha) \in \Phi \times \Gamma \times (0, 1)$ and ϕ_α be an onto mapping. Define
$d : X^2 \to [0, \infty)$ by $d_{\alpha_s}(x, y) = \inf\{t > 0 : D(x, y, t) > \phi_\alpha(\gamma_{x,y}(t))\}$. By lemma 3.6, $\inf\{t > 0 : D(x, y, t) > \phi_\alpha(t)\} = 0$ if and only if $x = y$. That is, $d(x, y) = 0$ if and only if $x = y$. Since $D(x, y, t) = D(y, x, t)$ for all $x, y \in X$, $t > 0$, it follows that $d(x, y) = d(y, x)$ for all $x, y \in X$. Let $x, y, z \in X$ and $s, t \in (0, \infty)$ be arbitrary. By Lemma 2.3 and Definition 2.4, we have
$D(x, y, k) \geq D(x, z, t) * D(z, y, s) \geq \frac{(D(x,z,t) \wedge D(z,y,s))^2}{c}$ for some
$c \in [1, 2]$, where $k = s + t$. Define

$$\begin{aligned} A = &\ \{k > 0 : D(x, y, k) > \phi_\alpha(k)\}, \\ B = &\ \{k > 0 : D(x, z, s) * D(z, y, t) > \phi_\alpha(k)\}, \\ C = &\ \{k > 0 : (D(x, z, t) \wedge D(z, y, s))^2 > c\phi_\alpha(k)\}. \end{aligned}$$

Lemma 3.6, it follows that $A \supseteq B \supseteq C$. So, $\inf(A) \leq \inf(B) \leq \inf(C)$. That is, $d(x, y) = \inf(A) \leq \inf(C)$. But

$$\begin{aligned} \inf(C) &\leq \inf\{t > 0 : (D(x, z, t)^2 > c\phi_\alpha(\gamma_{x,y}(t))\} + \{s > 0 D(z, y, s)^2 > c\phi_\alpha(s)\} \\ &= s(x, z) + s(z, y), \text{where} \end{aligned}$$

s is a mapping $s : X^2 \to [0, \infty)$ defined by
$s(x, y) = \inf\{t > 0 : D(x, y, t) > \sqrt{c\phi_\alpha(\gamma_{x,y}(t))}\}$. Clearly, $s(x, y) = s(y, x)$ for all $x, y \in X$. Also, $d(x, y) \leq s(x, z) + s(z, y)$ for all $x, y \in X$. ☐

Theorem 4.14. *Let $(X, D, *)$ be a fuzzy metric space such that $T_e \leq * \leq T_m$. If (X, d) is a D-induced s metric space of (ϕ, t, α) and ϕ_α is an onto maping, then a sequence $\{x_n\} \subset X$ is D-Cauchy iff d-Cauchy.*

Proof. The proof the theorem is similar to proof Theorem 3.19. ☐

Theorem 4.15. *Let $(X, D, *)$ be a fuzzy metric space such that $T_e \leq * \leq T_m$. If (X, d) is a D-induced s metric space of (ϕ, t, α) and ϕ_α is onto mapping, then a sequence $\{x_n\} \subset X$ is D-convergent if and only if d-convergent.*

Proof. The proof is similar to that of Theorem 4.14. ☐

Corollary 4.16. *Let $(X, D, *)$ be a fuzzy metric space such that $T_e \leq * \leq T_m$. If (X, d) is a D-induced s metric space of onto pair (ϕ, t, α) and ϕ_α is onto mapping, then $(X, D, *)$ is a complete fuzzy metric space if and only if (X, d) is a complete s-metric space.*

Proof. Proof of corollary follows from Theorem 4.14 and Theorem 4.15. ☐

5. On topological spaces (X, τ_D) and (X, τ_d)

In this section, we investigate the relationship between fuzzy metric D-induced topology T_D and that of D-induced metric (or s-metric) induced topology τ_d.

Definition 5.1. Let $(X, D, *)$ be a fuzzy metric space and let (X, τ_D) be a corresponding topological space induced by a fuzzy metric D. A set $A \subseteq X$ is said to be a τ_D-open if and only if given $t_0 > 0$, there exists $0 < r < 1$ such that for each $a \in A$, $B(a, r, t_0) \subset A$.

Theorem 5.2. *Let $(X, D, *)$ be a fuzzy metric space. If (X, d) is a D-induced metric space of (ϕ, γ, α), then $\tau_d \subseteq \tau_D$.*

Proof. Let (X, d) be a D-induced metric space of (ϕ, γ, α). If $A \in \tau_d$, then for each $a \in A$, there exists $t_o \in (0, 1)$ such that $B_d(a, t_o) \subseteq A$. That is,
$B = \{b \in X : d(a, b) < t_0\} \subseteq A$ or equivalently,
$B = \{b \in X : \inf\{t > 0 : D(a, b, t) > \phi_\alpha(\gamma_{a,b}(t))\} < t_0\} \subseteq A$. By Lemma 3.6(iii), it follows that $B = \{b \in X : D(a, b, t_o) > \phi_\alpha(\gamma_{a,b}(t_0))\} \subseteq A$.

Since $\overline{B} = \{b \in X : d(a, b) \leq t_0\}$ is closed and bounded subset of a metric space, it is compact set in (X, τ_d) and by continuity of $\phi_\alpha(\gamma_{a,.}(t_0))$, there exists $b' \in \overline{B}$ such that $t'_0 = \phi_\alpha(\gamma_{a,b'}(t_0)) = \max_{b\in\overline{B}} \phi_\alpha(\gamma_{a,b}(t_0))$. By Definition 3.2 and Definition 3.4, $0 < t'_0 < 1$. Put $t'_0 = 1 - r$ for some $r \in (0, 1)$, and consider $C = \{b \in \overline{B} : D(a, b, t_o) > 1 - r\}$. We claim $C \subseteq B$. For if $C \nsubseteq B$, then there exists $z \in C$ with $z \notin B$. But $z \notin B \Longrightarrow D(a, z, t_0) \leq \phi_\alpha(\gamma_{a,z}(t_0)) \leq t'_0 = 1 - r$ (since $z \in \overline{B}$) i.e $z \notin C$. This contradicts to assumption $z \in C$. Thus, $C \subseteq B \subseteq A$ i.e $A \in \tau_D$. □

Corollary 5.3. *Let $(X, D, *)$ be a fuzzy metric space, (X, d) be a D-induced metric space of (ϕ, γ, α). If $A \subset X$ is compact in (X, τ_d), then it is compact in (X, τ_D).*

Proof. Proof of the corollary follows by Theorem 5.2. □

Corollary 5.4. *Let $(X, D, *)$ be a fuzzy metric space and let (X, d) be a D-induced metric space of (ϕ, γ, α). If a mapping $G : (X, \tau_D) \longrightarrow (X, \tau_D)$ is continuous, then $G : (X, \tau_D) \longrightarrow (X, \tau_d)$ is continuous.*

Proof. Let $G : (X, \tau_D) \longrightarrow (X, \tau_D)$ be continuous mapping and let V be open in (X, τ_d), then by Theorem 5.2, V is open in (X, τ_D). Since
$G : (X, \tau_D) \longrightarrow (X, \tau_D)$ is continuous mapping, $G^{-1}(V)$ is open in (X, τ_D). Thus, $G : (X, \tau_D) \longrightarrow (X, \tau_d)$ is continuous. □

Theorem 5.5. *Let $(X, D, *)$ be a fuzzy metric space. If (X, d) is a D-induced metric space of (ϕ, γ, α) and both ϕ_α and $\gamma_{x,y}$ are onto mappings for each fixed $(\alpha, (x, y)) \in (0, 1) \times X^2$, then $\tau_D = \tau_d$.*

Proof. Let $(X, D, *)$ and (X, d) be as stated in the theorem. If $\varnothing \neq A \in \tau_D$, then for each $x_0 \in A$ and $t_0 > 0$, there exists $r_0 \in (0, 1)$ such that $B(x_0, r_0, t_0) \subset A$, i.e $B = \{b \in X : D(x_0, b, \frac{t_0}{2}) > 1 - r_0\} \subset A$. Now fix such $r_0 \in (0, 1)$. Since both ϕ_α and $\gamma_{x,y}$ are onto mappings for each fixed $(\alpha, (x, y) \in (0, 1) \times X^2$), there exists $t_b \in (0, \infty)$ such that $\phi_\alpha(\gamma_{x_0,b}(t_b)) = 1 - r_0$ for each $b \in B$. If we put $t'_0 = \inf\{t_b : \phi_\alpha(\gamma_{x_0,b}(t_b)) = 1-r_0\}$ and $s_0 = \min\{t'_0, \frac{t_0}{2}\}$, then $\psi_\alpha(\gamma_{x_o,b}(t'_0)) \geq 1-r_0$ for all $b \in B$ and $\mathbb{B} = \{b \in X : D(x_0, b, \frac{t_0}{2}) > \psi_\alpha(\gamma_{x_o,b}(t'_0))\} \subseteq B$, i.e $\mathbb{B} \subset A$.

Observe that $t'_0 > 0$. For if $t'_0 = 0$, then there exist a subset
$C = \{b_\lambda : \lambda \in I = \text{some index set}\}$ of B and sequence
$\{t_{b_\lambda} : \lambda \in I = \text{ some index set}\} \subseteq \{t_b : \phi_\alpha(\gamma_{x_0,b}(t_b)) = 1-r_0\}$ such that $\lim_\lambda t_{b_\lambda} = 0$.

By continuity of $\phi_\alpha(\gamma_{x,y})$ and Lemma 3.5, we have
$1 = \lim_\lambda \phi_\alpha(\gamma_{x_0,b_\lambda}(t_{b_\lambda})) > 1 - r_0 \Longrightarrow \phi_\alpha(\gamma_{x_0,b_\lambda}(t_{b_\lambda})) > 1 - r_0$ for some b_λ and its corresponding t_{b_λ}. But, this contradicts to assumption
$\phi_\alpha(\gamma_{x_0,b_\lambda}(t_{b_\lambda})) = 1 - r_0$. Therefore, $t_0' \neq 0$.
Using Remark 2.6, Definition 3.2, and Lemma 3.5, we have the following:

$$(5.1) \quad \left\{ \begin{array}{l} D(x_0, b, s_0) \leq \min\{D(x_0, b, \frac{t_0}{2}), D(x_0, b, t_0')\}, \\ \max\{\phi_\alpha(\gamma_{x_0,b}(\frac{t_0}{2})), \phi_\alpha(\gamma_{x_0,b}(t_0'))\} \leq \phi_\alpha(\gamma_{x_0,b}(s_0)), \ \forall \alpha \in (0,1) \end{array} \right. \ \forall\, b \in X.$$

Define $E = \{b \in X : D(x_0, b, s_0) > \phi_\alpha(\gamma_{x_0,b}(s_0))\}$ We claim that $E \subseteq \mathbb{B}$. Indeed, if $y \notin \mathbb{B}$, then by (5.1), it follows that
$D(x_0, y, s_0) \leq D(x_0, y, \frac{t_0}{2}) \leq \phi_\alpha(\gamma_{x_0,b}(t_0')) \leq \phi_\alpha(\gamma_{x_0,b}(s_0))$. Hence $y \notin E$. Thus, $E \subseteq \mathbb{B}$. By definition of induced d, for each $b \in E$, it follows that
$d(x_0, b) = \inf\{t > 0 : D(x_0, b, t) > \phi_\alpha(\gamma_{x_0,b}(t))\} \leq s_0$. Therefore,
$\{b \in X : \inf\{t > 0 : D(x_0, b, t) > \phi_\alpha(\gamma_{x_0,b}(t))\} \leq s_0 < t_0\} \subseteq E \subseteq B \subseteq A$ or equivalently, $\{b \in X : d(x_0, b) \leq s_0 < t_0\} \subseteq E \subseteq A$. Since $x_0 \in A$ was arbitrary, for each $x \in A$, there exists a $t_0 > 0$ such that $\{b \in X : d(x_0, b) < t_0\} \subseteq A$. Thus, $A \in \tau_d$. The converse follows by Theorem 5.2. □

Corollary 5.6. *Let $(X, D, *)$ be a fuzzy metric space . If (X, d) is a D-induced metric space of (ϕ, γ, α) and both ϕ_α and $\gamma_{x,y}$ are onto mappings for every fixed $(\alpha, (x, y)) \in (0,1) \times X^2$, then X is τ_D compact if and only if X is τ_d compact.*

Proof. Since both ϕ_α and $\gamma_{x,y}$ are onto for every fixed $x, y \in X$, by Theorem 5.5, $\tau_D = \tau_d$. Therefore, X is τ_D compact iff X is τ_d compact. □

Corollary 5.7. *Let $(X, D, *)$ be a fuzzy metric space. If (X, d) is a D- induced metric space of (ϕ, γ, α), both ϕ_α and $\gamma_{x,y}$ are onto mappings for every fixed $(\alpha, (x, y)) \in (0,1) \times X^2$ and (X, τ) is a fuzzy metrizable with respect to fuzzy metric D, then (X, τ) is metrizable topological space.*

Proof. Since both ϕ_α and $\gamma_{x,y}$ are onto mappings, by Theorem 5.5 $\tau_D = \tau_d$. Thus, (X, τ) is fuzzy metrizable with respect to D implies that $\tau = \tau_D = \tau_d$. □

Corollary 5.8. *Let $(X, D, \wedge)$ is a fuzzy metric space, Where $a \wedge b = \min(a, b)$ for all $a, b \in [0, 1]$. If (X, τ) is a fuzzy metrizable topological space with respect to fuzzy metric D, then (X, τ) is a metrizable topological space.*

Proof. The proof Corollary follows by Corollary 3.13 and Corollary 5.7. □

Theorem 5.9. *Let $(X, D, *)$ be a fuzzy metric space. If (X, d) is a D-induced s-metric space of (ϕ, t, α), then $\tau_d \subseteq \tau_D$.*

Proof. Let $(X, D, *)$ and (X, d) be as stated in the theorem. If $\varnothing \neq A \in \tau_d$, then for each $a \in A$, there exists $t_0 \in (0,1)$ such that $B_{d_s}(a, t_0) \subseteq A$. That is,
$\{b \in X : d(a, b) < t_0\} \subseteq A$ or equivalently,
$\{b \in X : \inf\{t > 0 : D(a, b, t) > \phi_\alpha(t)\} < t_0\} \subseteq A$. By Lemma 3.6, it follows that $\{b \in X : D(a, b, t_o) > \phi_\alpha(t_0)\} \subseteq A$. By putting $\phi_\alpha(t_0) = 1 - r$ for some $0 < r < 1$, we get $\{b \in X : D(a, b, t_o) > 1 - r\} \subseteq A$. Hence $A \in \tau_D$. □

Theorem 5.10. *Let $(X, D, *)$ be a fuzzy metric space such that $T_e \leq * \leq T_m$. If (X, d) is a D-induced s-metric space of (ϕ, t, α) and ϕ_α is an onto mapping, then $\tau_d = \tau_D$.*

Proof. The proof is similar to that of Theorem 5.5. Hence omitted. □

6. Application of induced metric and s- metric

In this section, the idea of induced metric and induced s- metric are applied to investigate the completeness, completion and (or metrizability) of some well know fuzzy metric spaces (or topological space).

Theorem 6.1. *Let $X = (0, \infty)$,$a * b = ab$ for all $a, b \in [a, b]$ and D be fuzzy set on $X^2 \times (0, \infty)$ defined by $D(x, y, t) = \frac{t+(x\wedge y)}{t+(x\vee y)}$. Then*
(i) $(X, D, *)$ is not complete fuzzy metric space;
(ii) $(X, D, *)$ has a completion ;
(iii) (X, τ_D) is metrizable topological space.

Proof. $(X, D, *)$ is fuzzy metric space (V. Gregori et al. [6]). Define $\gamma : X \times X \times [0, \infty) \longrightarrow [0, \infty)$ and $\phi : [0, \infty) \times (0, 1) \longrightarrow (0, 1]$ by $\gamma(x, y, t) = t$ for all $x, y \in X,\ t \geq 0$ and $\phi(t, \beta) = \frac{\beta}{\beta+t}$ for all $\beta \in (0, 1),\ t \geq 0$ respectively. Clearly, $(\phi, \gamma) \in \Phi \times \Gamma$ and $\phi_\alpha(\gamma_{x,y})$ is an onto mapping for every fixed $\alpha \in (0, 1)$ and every $x, y \in X$ with $\phi_\alpha(\gamma_{x,y}(t)) = \frac{\alpha}{\alpha+t}$ for all $x, y \in X$. Since $T_e \leq * \leq T_m$, by Theorem 4.13, a mapping $d : X^2 \to [0, \infty)$ defined by

$$d(x, y) = \inf\{t > 0 : D(x, y, t) > \frac{\alpha}{\alpha + t}\} = \frac{-(x \wedge y) + \sqrt{(x \wedge y)^2 + 4\alpha|x - y|}}{2}$$

is an s-metric on X. Moreover, $d(x, y) \leq \sqrt{\alpha|x - y|} = s(x, y)$ for all $x, y \in X$, where s is a mapping $s : X^2 \to [0, \infty)$ defined by $s(x, y) = \sqrt{\alpha|x - y|}$, i.e. (X, d) is a D-induced s-metric space of (ϕ, t, α) and ϕ_α is an onto mapping. By Corollary 4.16, $(X, D, *)$ is complete if and only if (X, d) is complete. Therefore, it suffice to show that (X, d) is not complete. But $\{x_n\}_{n\in\mathbb{N}}$ is d-Cauchy sequence
$\Longleftrightarrow \lim_{n,m} d(x_n, x_m) = 0$
$\Longleftrightarrow \lim_{n,m} \frac{-(x_n\wedge x_m)+\sqrt{(x_n\wedge x_m)^2+4\alpha|x_n-x_m|}}{2} = 0$
$\Longleftrightarrow \lim_{n,m} \alpha|x_n - x_m| = 0 \Longleftrightarrow \{x_n\}_{n\in N}$ is s-Cauchy sequence. Since (X, s) is not complete metric space, it follows that (X, d) is not complete s-metric space. Thus $(X, D, *)$ is not complete fuzzy metric space. Hence (i) of the theorem is proved.

To prove (ii), we observe that a sequence $\{x_n\}$ is D-Cauchy iff it is s-Cauchy. Since s is metric on X, it has a completion. By Corollary 3.18, D has a completion. Hence By Proposition 4.9, $d = \tau_s$. Since (X, d) is a D-induced s-metric space of (ϕ, t, α) and ϕ_α is an onto mapping for every fixed $\alpha \in (0, 1)$, by Theorem 5.10, we have $\tau_D = \tau_d$. Therefore, (X, τ_D) is metrizable. Thus, (iii) of the theorem is proved. □

Theorem 6.2. *Let $X = (0, \infty)$, $a * b = ab,\ a, b \in [0, 1]$, and $D : X^2 \times (0, \infty) \longrightarrow (0, 1]$ defined by $D(x, y, t) = \frac{x\wedge y}{x\vee y}$. Then*
(i) (D, τ_D) is metrizable topological space;
(ii) $(X, D, *)$ is a complete fuzzy metric space.

Proof. One can easily verify that $(X, D, *)$ is a fuzzy metric space.
(i): Define $\gamma : X \times X \times [0, \infty) \longrightarrow [0, \infty)$ and $\phi : [0, \infty) \times (0, 1) \longrightarrow (0, 1]$ by $\gamma(x, y, t) = \frac{t}{\min(x,y)}$ and $\phi(t, \alpha) = \frac{\alpha}{\alpha+t}$ respectively. Clearly, $(\phi, \gamma) \in \Phi \times \Gamma$ and both ϕ_α and $\gamma_{x,y}$ are onto mappings for every fixed $(x, y, \alpha) \in X \times X \times (0, 1)$ with $\phi_\alpha(\gamma_{x,y}(t)) = \frac{\min(x,y)\alpha}{\min(x,y)\alpha+t}$ for all $x, y \in X$ and $t \geq 0$. A mapping
$d : X \times X \longrightarrow [0, \infty)$ defined by

$$\begin{aligned} d(x, y) &= \inf\{t > 0 : D(x, y, t) > \phi_\alpha(\gamma_{x,y}(t))\} \\ &= \inf\{t > 0 : \frac{\min(x, y)}{\max(x, y)} > \frac{\min(x, y)\alpha}{\min(x, y)\alpha + t}\} \\ &= \alpha(\max(x, y) - \min(x, y)) = \alpha|x - y| \end{aligned}$$

is metric on X. Since (X, d) is an D-induced metric of (ϕ, γ, α) and both $\phi_\alpha(\gamma_{x,y})$ is an onto mappings for every fixed $(x, y, \alpha) \in X \times X \times (0, 1)$, by Theorem 5.5 $\tau_D = \tau_d$. That is, (X, τ_D) is metrizable topological space.

(ii): Define $\gamma : X \times X \times [0, \infty) \longrightarrow [0, \infty)$ and $\phi : [0, \infty) \times (0, 1) \longrightarrow (0, 1]$ by $\gamma(x, y, t) = t$ and $\phi(t, \alpha) = \frac{\alpha}{\alpha+t}$ respectively. Clearly, $(\phi, \gamma) \in \Phi \times \Gamma$ and $\phi_\alpha(\gamma_{x,y})$ is an onto mapping for every fixed $(x, y, \alpha) \in X \times X \times (0, 1)$ with $\phi_\alpha(\gamma_{x,y}(t)) = \phi_\alpha(t) = \frac{\alpha}{\alpha+t}$ for all $x, y \in X$ and $t \geq 0$. since $T_e \leq * \leq T_D$, by Theorem 4.13, a mapping $d : X \times X \longrightarrow [0, \infty)$ defined by

$$\begin{aligned} d(x, y) &= \inf\{t > 0 : D(x, y, t) > \phi_\alpha(\gamma_{x,y}(t))\} \\ &= \inf\{t > 0 : \frac{\min(x, y)}{\max(x, y)} > \frac{\alpha}{\alpha + t}\} \\ &= \frac{\alpha(\max(x, y) - \min(x, y))}{\min(x, y)} = \frac{\alpha|x - y|}{\min(x, y)} \end{aligned}$$

is an s-metric on X i.e. (X, d) is a D-induced s-metric space of (ϕ, t, α). Since ϕ_α is onto mapping for evert fixed $\alpha \in (0, 1)$, by Corollary 4.16, $(X, D, *)$ is complete if and only if (X, d) is complete. Therefore, it suffice to show that (X, d) is complete.

Now, we observe that, any null sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ with respect to usual metric on $\mathbb{R}$ is not d-Cauchy. For if a null sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ with respect to usual metric on $\mathbb{R}$ is a d-Cauchy, then given $\epsilon > 0$, there exists $n_0 \in N$ such that

$$d(x_n, x_m) < \epsilon,\ \forall\ n, m \geq n_0 \tag{6.1}$$

Since $\{x_n\}_{n\in N} \subset X$ and $\lim_n x_n = 0$, there exists a decreasing sub-sequence $\{x_{n_j}\}_{n_j\in N}$ of $\{x_n\}_{n\in N}$ such that $\lim_{n_j} x_{n_j} = 0$. Hence for n_0 as in (6.1), we have

$$d(x_j, x_i) < \epsilon,\ \forall\ n_j \geq n_i \geq n_0 \tag{6.2}$$

By fixing n_{i_0} in (6.2), we have, $\forall\ n_j \geq n_{i_0} \geq n_0$
$d(x_{n_j}, x_{n_{i_0}}) < \epsilon \Longrightarrow \frac{\alpha|x_{n_j} - x_{n_{i_0}}|}{x_{n_j} \wedge x_{n_{i_0}}} \Longrightarrow \frac{\alpha|x_{n_j} - x_{n_{i_0}}|}{x_{n_j}} \Longrightarrow \alpha|1 - \frac{x_{n_{i_0}}}{x_{n_j}}|$
$\Longrightarrow \lim_{n_j} \alpha|1 - \frac{x_{n_{i_0}}}{x_{n_j}}| = \infty$ (since $x_{n_{i_0}} \neq 0$ is fixed and $\lim_{n_j} x_{n_j} = 0$). This contradicts to both (6.1) and (6.2). Hence, given $\epsilon > 0$, there is no $n_0 \in N$ satisfying (6.1). Therefore, every null sequence in X with respect to usual metric on $\mathbb{R}$ is not d-Cauchy sequence. If $\{x_n\}_{n\in N} \subset X$ be a d-Cauchy sequence, then given $\epsilon > 0$,

there exists $n_0 \in N$ such that $d(x_n, x_m) < \epsilon \Longrightarrow \frac{\alpha|x_n - x_m|}{x_n \wedge x_m} < \epsilon,\ \forall\ n, m \geq n_0$, i.e.

$$\alpha|x_n - x_m| < \epsilon(x_n \wedge x_m),\ \forall\ n, m \geq n_0. \tag{6.3}$$

Since $\{x_n\}_{n\in N} \subset X$ is a d-Cauchy sequence, form first part we have, $x_m \wedge x_n \neq 0\ \forall\ n, m \geq n_0$. Thus,

$$\frac{\alpha\epsilon}{x_n \wedge x_m} > 0,\ \forall\ n, m \geq n_0 \tag{6.4}$$

By replacing ϵ in (6.3) by $\frac{\alpha\epsilon}{x_n \wedge x_m}$ in (6.4), we obtain that, $\forall\ n, m \geq n_o$, $\alpha|x_n - x_m| < \frac{\alpha\epsilon}{x_n \wedge x_m}(x_n \wedge x_m) \Longrightarrow |x_n - x_m| < \epsilon,\ \forall\ n, m \geq n_0$. That is $\{x_n\}_{n\in N} \subset X$ is a Cauchy sequence in $(\mathbb{R}, |\ |)$. Since $(\mathbb{R}, |\ |)$ is complete, $\lim_n x_n = x \in \mathbb{R}$. Since $\lim_n x_n \neq 0$ (by first part of the proof) and $\{x_n\}_{n\in N} \subset X, \lim_n x_n = x > 0$, it follows that $\lim_n x_n = x \in X$. Therefore, (X, d) is a complete s-metric space. Thus, $(X, D, *)$ is a complete fuzzy metric space. □

Note: Theorem 6.2 (ii) has been proved by V.Gregori et al. [6] using an other method.

7. Conclusion

In this paper, we derived a metric (or s-metric) form a fuzzy metric in a sense of A. George and P. Veeramani [4], where the infimum of an induced interval is used to evaluate distance between two points. It is proved that every fuzzy-metric space $(X, D, *)$ with its continuous t-norm $*$ between τ_e(= Einstein product) and τ_m(= minimum) induces at least an s-metric space (X, d).

The topological properties such as the compactness and metrizabilty (fuzzy metrizabilty) of a fuzzy metric induced topological space (X, τ_D) are investigated using its D-induced metric induced topological space (X, τ_d) or D-induced s-metric induced topological space (X, τ_d). Similarly, we investigated the completeness and completion of a fuzzy metric space $(X, D, *)$ using the completion and completeness of its corresponding D-induced metric space (X, d) or D-induced s-metric space (X, d).

Since every fuzzy-metric space $(X, D, *)$ with continuous t-norm $*$ between T_e and T_m induces at least an an s-metric space (X, d), the completeness and completion(respectively,compactness and metrizablity) of $(X, D, *)$(respectively, of (X, τ_D)) can be treated form the perspective of its D-induced metric (or s-metric). we believe that this result will be helpful to study and characterize fuzzy-metric using its corresponding D-induced metric or D-induced s-metric and to apply it to fixed point theory in fuzzy settings.

Similar results can be obtained for fuzzy metric space in a sense of I. Kramosil and J. Michalek [11].

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References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.

[2] Z. Deng. Fuzzy pseudo-metric spaces, J. Math. Anal. Appl. 86 (1) (1982) 74–95.

[3] J. X. Fang, On fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 46 (1992) 107–113.

[4] A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994) 395–399.

[5] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems 27 (1988) 385–389.

[6] V. Gregori, J-J. Minana and S. Morillas, Some questions in fuzzy metric spaces, Fuzzy Sets and Systems 204 (2012) 71–85.

[7] V. Gregori and S. Romaguera. Fuzzy quasi-metric spaces. Appl. Gen. Topol. 5 (1) (2004) 129–136.

[8] V. Gregori and A. Sapena, On fixed-point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 25 (2002) 245–252.

[9] M. Imdad and J. Ali, Some common fixed point theorems in fuzzy metric spaces, Math. Commun. 11 (2006) 153–163.

[10] O. Kaleva and S. Siekkala, On fuzzy metric spaces, Fuzzy Sets and Systems 12 (1984) 215–229.

[11] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetika 11 (1975) 336–334.

[12] S. Kumar and D. Mihet, G-completeness and M-completeness in fuzzy metric spaces: a note on a common fixed point theorem, Acta Math. Hungar. 126 (2010) 253–257.

[13] D. Mihet, A Banach contraction theorem in fuzzy metric spaces, Fuzzy Sets and Systems 144 (2004) 431–439.

[14] D. Mihet, On fuzzy contractive mappings in fuzzy metric spaces, Fuzzy Sets and Systems 158 (2007) 915–921.

[15] D. Mihet, Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces, Fuzzy Sets and Systems 159 (2008) 739–744.

[16] D. Mihet, A class of contractions in fuzzy metric spaces, Fuzzy Sets and Systems 161 (2010) 1131–1137.

[17] S. N. Mishra, N. Sharma and S. L. Singh, Common fixed points of maps on fuzzy metric spaces, Int. J. Math. Math. Sci. 17 (1994) 253–258.

[18] J. N. Mordeson, D. S. Malik and N. Kuroki, Fuzzy Semigroups, Studies in Fuzziness and Soft Computing, vol. 131, Springer-Verlag 2003.

[19] J. Muscat, Functional Analysis:An Introduction to Metric Spaces, Hilbert Spaces, and Banach Algebras, Springer 2014.

[20] P. Pao-Ming and L. Ying-Ming. Fuzzy topology. I. neighborhood structure of a fuzzy point and moore-smith convergence, J. Math. Anal. Appl. 76 (1980) 571–599.

[21] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512–517.

[22] B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960) 314–334.

[23] S. Sedghi, I. Altun and N. Shobe, Coupled fixed point theorems for contractions in fuzzy metric spaces, Nonlinear Anal. 72 (2010) 1298–1304.

[24] S. Sharma, Common fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 127 (2002) 345–352.

[25] H. Sherwood, On the completion of probabilistic metric spaces, Probab. Theory Related Fields 6 (1) (1966) 62–64.

[26] F. Shi, Pointwise pseudo-metrics in l-fuzzy set theory, Fuzzy Sets and Systems 121 (2001) 209–216.

[27] T. Simsekler and S. Yuksel, Fuzzy soft topological spaces, Ann. Fuzzy Math. Inform. 5 (1) (2013) 87–96.

[28] B. Singh and M. S. Chauhan, Common fixed points of compatible maps in fuzzy metric spaces, Fuzzy Sets and Systems 115 (2000) 471–475.

[29] B. Tanay and M. B. Kandemir, Topological structure of fuzzy soft sets, Comput. Math. Appl. 61 (2011) 2952–2957.

[30] J. Tang and Y. Luo, A new type of generalized fuzzy quasi-ideals of ordered semigroups, Ann. Fuzzy Math. Inform. 8 (6) (2014) 941–963.
[31] M. Vasuki, A common fixed point theorem in an fuzzy metric space, Fuzzy Sets and Systems 97 (1998) 395–397.
[32] M. Vasuki and P. Veeramani, Fixed point theorems and Cauchy sequences in fuzzy metric spaces, Fuzzy Sets and Systems 135 (2003) 415–417.
[33] C. Vetro, Fixed points in weak non-Archimedean fuzzy metric spaces, fuzzy sets and Systems 162 (2011) 84–90.
[34] G. Yun, S. Hwang and J. Chang, Fuzzy Lipschitz maps and fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 161 (2010) 1117–1130.
[35] L. A. Zadeh, Fuzzy sets, Iformation and Control 8 (1965) 338–353.
[36] J. Zhan and Y. B. Jun, Soft BL-algebras based on fuzzy sets, Comput. Math. Appl. 59 (2010) 2037–2046.

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