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Numerical solution of fuzzy differential equations by milne's fifth order predictor-corrector method

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Abstract. In this paper three numerical methods to solve for fuzzy differential equations are discussed. These methods are Milne's explicit five-step method, implicit four-step and predictor-corrector is obtained by combining Milne's explicit five-step method and implicit four-step methods. Convergence and stability of the proposed methods are also proved in detail. In addition, these methods are illustrated by solving two fuzzy Cauchy problems.

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1. INTRODUCTION

Fuzzy differential equations (FDEs) are utilized for the purpose of the modelling problems in science and engineering. Most of the problems in sciense and engineering require the solutions of a Fuzzy differential equation which are satisfied in fuzzy initial conditions, therefore a fuzzy initial value problem is occurs, and should be solved. It is too complicated to obtain the exact solutions of FDE which models the numerical problems. The concept of a fuzzy derivative was first introduced by Chahg and Zadeh [6] it was followed by Dubois and Prade [7], who defined and used the extension principle. The fuzzy differential equation and the fuzzy initial value problem were regulary treated by Nieto, Rodriguez-Lopez [21] and Seikkala [24] and by many other researchers [3, 4, 8, 9, 12, 13, 19]. The numerical methods for solving fuzzy differential equations are introduced in [1, 2, 20, 22, 23]. Allahviranloo, Ahmady and E.Ahmady [5] are also discussed in numerical solution of FDEs by predictor-corrector method. In the last few years many works have been discussed for solving FDE in particular,Jayakumar and Kanagarajan [11, 14, 15, 16, 17] have discussed numerical solutions of hybrid fuzzy differental equations.

In this article, we develope numerical methods for solving FDEs. In section 2 we list some basic definition for fuzzy valued functions. In section 3 we reviews the fuzzy cauchy problem. In section 4 Milne's fifth order predictor-corrector method for solving fuzzy differential equations are proposed. Predictor-corrector five step algorithm is discussed in section 5. Convergence and stability of the mentioned methods are proved in section 6. section 7 contains some numerical examples to illustrated the theory. Finaly conclution is drawn.

2. Preliminaries

2.1. **Multistep method.**

Definition 2.1. An m-step method for solving the initial-value problem is one whose difference equation for finding the approximation $y(t_{i+1})$ at the mesh point t_{i+1} can be represented by the following equation:

$$\begin{aligned} y(t_{i+1}) &= a_{m-1}y(t_i) + a_{m-2}y(t_{i-1}) + ... + a_0 y(t_{i+1-m}) \\ &+h\{b_m f(t_{i+1}, y_{i+1}) + b_{m-1}f(t_i, y_i) + ... + b_0 f(t_{i+1-m}, y_{i+1-m})\}, \end{aligned} \tag{2.1}$$

for $i = m-1, m, ..., N-1$, such that $a = t_0 \leq t_1 \leq ... \leq t_N = b$, $h = \frac{(b-a)}{N} = t_{i+1} - t_i$, and $a_0, a_1, ..., a_{m-1}, b_0, b_1, ..., b_m$ are constants with the starting values

$$y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2, ..., y_{m-1} = \alpha_{m-1}.$$

When $b_m = 0$, the method is known as explicit, since (2.1) gives y_{i+1} explicit in terms of previously determined values. When $b_m \neq 0$, the method is know as implicit , since y_{i+1} occurs on both sides of (2.1) and is specified only implicitly.

With consideration Definition 2.1, the multistep method is

Milne's explicit five-step method:

$$y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2, \quad y_3 = \alpha_3, \quad y_4 = \alpha_4,$$

$$\begin{aligned} y_{i+1} = y_{i-4} + \tfrac{h}{144}[&95f(t_{i-4}, y_{i-4}) - 50f(t_{i-3}, y_{i-3}) + 600f(t_{i-2}, y_{i-2} - 350 \\ &f(t_{i-1}, y_{i-1}) + 425f(t_i, y_i)], \end{aligned}$$

where $i = 4, 5, \ldots, N-1$.

Milne's implicit four-step method :

$$y_1 = \alpha_1, \quad y_2 = \alpha_2, \quad y_3 = \alpha_3, \quad y_4 = \alpha_4,$$

$$\begin{aligned} y_{i+4} = y_i + \frac{h}{90}[&29f(t_{i+1}, y_{i+1}) + 124f(t_i, y_i) + 24f(t_{i-1}, y_{i-1}) + 4f(t_{i-2}, y_{i-2}) \\ &-4f(t_{i-1}, y_{i-1})], \end{aligned}$$

where $i = 4, 5, \cdots, N-1$.

Definition 2.2. Associated with the difference equation

$$\left.\begin{aligned} &y_{i+1} = a_{m-1}y_i + a_{m-2}y_{i-1} + \ldots + a_0 y_{i+1-m} + hF(t_i, h, y_{i+1}, y_i, \ldots, y_{i+1-m}), \\ &y_0 = \alpha, \quad y_1 = \alpha_1, \ldots, y_{m-1} = \alpha_{m-1}, \end{aligned}\right\} \tag{2.2}$$

is a polynomiyal, called the characteristic polynomial of the method given by

$$p(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \ldots - a_1\lambda - a_0.$$

If $|\lambda_i| \leq 1$ for each $i = 1, 2, \ldots, m$, and all roots with absolute value 1 are simple roots, then the difference method is said to satisfy the root condition.

Theorem 2.3. *A multistep method of the form (2.2) is stable if and only if it satisfies the root condition.*

Proof. See [10]. □

Notations used in this chapter is as follows:
A tilde is placed over a symbol to denote a fuzzy set so $\widetilde{\alpha_1}, \widetilde{f}(t), \ldots$
An arbitrary fuzzy number with an ordered pair of functions $(\underline{u}(\alpha), \overline{u}(\alpha))$, $0 \leq \alpha \leq 1$, which satisfy the following requirements is represented.
1. $\underline{u}(\alpha)$ is a bounded left continuous nondecreasing function over [0,1],
2. $\overline{u}(\alpha)$ is a bounded left continuous nonincreasing function over [0,1],
3. $\underline{u}(\alpha) \leq \overline{u}(\alpha), 0 \leq \alpha \leq 1$.

Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded α-level intervals. It means if $v \in E$ then the α-level set

$$[v]^\alpha = \{s | v(s) \geq \alpha\}, \quad 0 < \alpha \leq 1,$$

is a closed bounded intervel which is denoted by

$$[v]^\alpha = [\underline{v}^\alpha, \overline{v}^\alpha].$$

Let I be a real interval. A mapping $y : I \to E$ is called a fuzzy process and its α-leval set is denoted by

$$[y(t)]^\alpha = [\underline{y}^\alpha(t), \overline{y}^\alpha(t)], \quad t \in I, \quad \alpha \in (0, 1].$$

Triangular fuzzy numbers are those fuzzy sets in E which are characterized by an ordered triple $(x^l, x^c, x^r) \in R^3$ with $x^l \leq x^c \leq x^r$ such that $[U]^0 = [x^l, x^r]$ and $[U]^1 = \{x^c\}$ then

$$[U]^\alpha = [x^c - (1-\alpha)(x^c - x^l), x^c + (1-\alpha)(x^r - x^c)], \tag{2.3}$$

for any $\alpha \in I$.

Definition 2.4. The supremum metric d_∞ on E is defined by

$$d_\infty(U, V) = sup\{d_H([U]^\alpha, [V]^\alpha) : \alpha \in I\},$$

and (E, d_∞) is a complete metric space.

Definition 2.5. A mapping $F : I \to E$ is Hukuhara differentiable at $t_0 \in T \subseteq R$ if for some $h_0 > 0$ the Hukuhara difference

$$F(t_0 + \Delta t) \sim_h F(t_0), \quad F(t_0) \sim_h F(t_0 - \Delta t),$$

exist in E for all $0 < \Delta t < h_0$ and if there exists an $F'(t_0) \in E$ such that

$$\lim_{\Delta t \to 0+} d_\infty \frac{(F(t_0 + \Delta t) \sim_h F(t_0))}{\Delta t} - F'(t_0) = 0$$

and

$$\lim_{\Delta t \to 0+} d_\infty \frac{(F(t_0) \sim_h F(t_0 - \Delta t))}{\Delta t} - F'(t_0) = 0,$$

the fuzzy set $F'(t_0)$ is called the Hukuhara derivative of F at t_0.

Recall that $U \sim_h V = W \in E$ are defined on level sets, where $[U]^\alpha \sim_h [V]^\alpha = [W]^\alpha$ for all $\alpha \in I$. By consideration of definition of the metric d_∞, all the level set mappings $[F(t_0)]^\alpha$ are Hukuhara differentiable at t_0 with Hukuhara derivatives $[F'(t_0)]^\alpha$ for each $\alpha \in I$, when $F : I \to E$ is Hukuhara differentiable at t_0 with Hukuhara derivative $F'(t_0)$.

Definition 2.6. The fuzzy integral

$$\int_a^b y(t)dt, \quad 0 \le a \le b \le 1,$$

is defined by

$$\left[\int_a^b y(t)dt\right]^\alpha = \left[\int_a^b \underline{y}^\alpha(t)dt, \int_a^b \overline{y}^\alpha(t)dt\right],$$

provided the Lebesgue integrals on the right exist.

Remark 2.7. If $F : I \to E$ is Hukuhara differentiable and its Hukuhara derivative F' is integrable over [0,1], then

$$F(t) = F(t_0) + \int_{t_0}^t F'(s)ds.$$

for all values of t_0, t where $0 \le t_0 \le t \le 1$.

Definition 2.8. A mapping $y : I \to E$ is called a fuzzy process. We denote

$$[y(t)]^\alpha = [\underline{y}^\alpha(t), \overline{y}^\alpha(t)], \quad t \in I, \quad 0 < \alpha \le 1.$$

The Seikkala derivative $y'(t)$ of a fuzzy process y is defined by

$$[y'(t)]^\alpha = [(\underline{y}^\alpha)'(t), (\overline{y}^\alpha)'(t)], \qquad 0 < \alpha \le 1.$$

provided that this equation defines a fuzzy number $y'(t) \in E$.

Remark 2.9. If $y : I \to E$ is Seikkala differentiable and its Seikkala derivative y' is integrable over [0,1], then

$$y(t) = y(t_0) + \int_{t_0}^t y'(s)ds,$$

for all values of t_0, t where $t_0, t \in I$.

3. A fuzzy Cauchy problem

Consider the first-order fuzzy differential equation $y' = f(t, y)$ where y is a fuzzy function of $t, f(t, y)$ is a fuzzy function of crisp variable t and fuzzy variable y, and y' is Hukuhara or Seikkala fuzzy derivative of y. If an initial value $\widetilde{y}(t_0) = \widetilde{\alpha}_0$ is given, a fuzzy Cauchy problem of first order will be obtained as follows:

$$\left.\begin{array}{l} y'(t) = f(t, y(t)), \quad t_0 \leq t \leq T, \\ \widetilde{y}(t_0) = \widetilde{\alpha}_0, \end{array}\right\} \tag{3.1}$$

Sufficient conditions for the existence of a unique solution to equation (2.4)are

(i) Continuity of f,

(ii) Lipschitz condition $d_\infty(f(t, x), (f(t, y)) \leq L d_\infty(x, y), \quad L > 0$.

3.1. Interpolation of fuzzy number

The problem of interpolation for fuzzy sets is as follows:
Suppose that at various time instant t information $f(t)$ is presented as fuzzy set. The aim is to approximate the function $f(t)$, for all t in the domine of f.
Let $t_0 < t_1 < \ldots < t_n$ be $n + 1$ distinct points in R and let $\widetilde{u_0}, \widetilde{u_1}, \ldots, \widetilde{u_n}$ be $n + 1$ fuzzy sets in E.
A fuzzy polynomial interpolation of the data is a fuzzy value continuous function $f : R \to E$ satisfying:
(i) $f(t_i) = \widetilde{u}_i, \quad i = 1, \ldots, n$.
(ii) If the data is cricp, then the interpolation f is a crisp polynomial.
A function f which fulfilling these condition may be constructed as follows.
Let $C^i_\alpha = [\widetilde{u}_i]^\alpha$ for any $\alpha \in [0, 1], \quad i = 0, 1, 2, \ldots, n$. For each $x = (x_0, x_1, \ldots, x_n) \in R^{n+1}$, the unique polynomial of degree $\leq n$ denoted by P_X such that

$$P_X(t_i) = x_i, \qquad i = 0, 1, 2, \ldots, n,$$

$$P_X(t) = \sum_{i=0}^{n} x_i \left(\prod_{i \neq j} \frac{t - t_j}{t_i - t_j} \right).$$

Finally, for each $t \in R$ and all $\xi \in R$ is defined by $f(t) \in E$ by

$$(f(t))(\xi) = sup\{\alpha \in [0, 1] : \exists X \in C^0_\alpha \times \ldots \times C^n_\alpha \quad \text{such that} \quad P_X(t) = \xi\}.$$

The interpolation polynomial can be written level set wise as

$$[f(t)]^\alpha = \{y \in R : y = P_X(t), \;\; x \in [\widetilde{u}_i]^\alpha, \; i = 1, 2, ..., n\}, \qquad \text{for} \;\; 0 \leq \alpha \leq 1.$$

When the data $\widetilde{u}_i$ presents as triangular fuzzy numbers, values of the interpolation polynomial are also triangular fuzzy numbers. Then $f(t)$ has a particular simple form that is well situated to computation.

Theorem 3.1. *Let $(t_i, \widetilde{u}_i), \quad i = 0, 1, 2, \ldots, n$ be the observed data and suppose that each of the $\widetilde{u}_i = (u^l_i, u^c_i, u^r_i)$ is an element of E. Then for each $t \in [t_0, t_n]$,*

$\widetilde{f}(t) = (f^l(t), f^c(t), f^r(t)) \in E,$

$$\begin{aligned} f^l(t) &= \sum_{l_i(t)\geq 0} l_i(t)u_i^l + \sum_{l_i(t)<0} l_i(t)u_i^r, \\ f^c(t) &= \sum_{l=0}^{n} l_i(t)u_i^c, \\ f^r(t) &= \sum_{l_i(t)\geq 0} l_i(t)u_i^r + \sum_{l_i(t)<0} l_i(t)u_i^l, \end{aligned}$$

such that $l_i(t) = \prod_{j\neq i}^{n} \frac{t-t_j}{t_i-t_j}$.

Proof. See [18]. □

4. Milne's fifth order predictor- corrector method

4.1. Milne's explicit five step method. Now we are going to solve fuzzy initial value problem $y'(t) = f(t, y(t))$ by Milne's five step method.
Let the fuzzy initial values be $\widetilde{y}(t_{i-1}),\ \widetilde{y}(t_i),\ \widetilde{y}(t_{i+1}),\ \widetilde{y}(t_{i+2}),\ \widetilde{y}(t_{i+2}),\ \widetilde{y}(t_{i+3})$

i.e. $\widetilde{f}(t_{i-1}, y(t_{i-1})),\ \widetilde{f}(t_i, y(t_i)),\ \widetilde{f}(t_{i+1}, y(t_{i+1}))\ ,\ \widetilde{f}(t_{i+2}, y(t_{i+2})),\ \widetilde{f}(t_{i+3}, y(t_{i+3})),$

which are triangular fuzzy numbers and are shown by

$$\begin{aligned} &\left\{f^l(t_{i-1}, y(t_{i-1})), f^c(t_{i-1}, y(t_{i-1})), f^r(t_{i-1}, y(t_{i-1}))\right\}, \\ &\left\{f^l(t_i, y(t_i)), f^c(t_i, y(t_i)), f^r(t_i, y(t_i))\right\}, \\ &\left\{f^l(t_{i+1}, y(t_{i+1})), f^c(t_{i+1}, y(t_{i+1})), f^r(t_{i+1}, y(t_{i+1}))\right\}, \\ &\left\{f^l(t_{i+2}, y(t_{i+2})), f^c(t_{i+2}, y(t_{i+2})), f^r(t_{i+2}, y(t_{i+2}))\right\}, \\ &\left\{f^l(t_{i+3}, y(t_{i+3})), f^c(t_{i+3}, y(t_{i+3})), f^r(t_{i+3}, y(t_{i+3}))\right\}, \end{aligned}$$

also

$$\widetilde{y}(t_{i+4}) = \widetilde{y}(t_{i-1}) + \int_{t_{i-1}}^{t_{i+4}} \widetilde{f}(t, y(t))dt. \tag{4.1}$$

By fuzzy interpolation for $\widetilde{f}(t_{i-1}, y(t_{i-1}),\ \widetilde{f}(t_i, y(t_i)),\ \widetilde{f}(t_{i+1}, y(t_{i+1})),\ \widetilde{f}(t_{i+2}, y(t_{i+2})),$ $\widetilde{f}(t_{i+3}, y(t_{i+3}))$, we have;

$$\begin{aligned} f^l(t, y_k(t)) &= \sum_{j=i-1}^{i+3} l_j(t) f^l(t_{k,j}, y(t_{k,j})) + \sum_{j=i-1}^{i+3} l_j(t) f^r(t_{k,j}, y(t_{k,j})), \\ f^c(t, y_k(t)) &= \sum_{j=i-1}^{i+3} l_j(t) f^c(t_{k,j}, y(t_{k,j}), \\ f^r(t, y_k(t)) &= \sum_{j=i-1}^{i+3} l_j(t) f^r(t_{k,j}, y(t_{k,j})) + \sum_{j=i-1}^{i+3} l_j(t) f^l(t_{k,j}, y(t_{k,j})). \end{aligned}$$

For $t_{i-1} \leq t \leq t_{i+4}$:

$$
\begin{aligned}
l_{i-1}(t) &= \frac{(t-t_i)(t-t_{i+1})(t-t_{i+2})(t-t_{i+3})}{(t_{i-1}-t_i)(t_{i-1}-t_{i+1})(t_{i-1}-t_{i+2})(t_{i-1}-t_{i+3})} \geq 0,\\
l_{i}(t) &= \frac{(t-t_{i-1})(t-t_{i+1})(t-t_{i+2})(t-t_{i+3})}{(t_{i}-t_{i-1})(t_{i}-t_{i+1})(t_{i}-t_{i+2})(t_{i}-t_{i+3})} \leq 0,\\
l_{i+1}(t) &= \frac{(t-t_{i-1})(t-t_{i})(t-t_{i+2})(t-t_{i+3})}{(t_{i+1}-t_{i-1})(t_{i+1}-t_{i})(t_{i+1}-t_{i+2})(t_{i+1}-t_{i+3})} \geq 0,\\
l_{i+2}(t) &= \frac{(t-t_{i-1})(t-t_{i})(t-t_{i+1})(t-t_{i+3})}{(t_{i+2}-t_{i-1})(t_{i+2}-t_{i})(t_{i+2}-t_{i+1})(t_{i+2}-t_{i+3})} \leq 0,\\
l_{i+3}(t) &= \frac{(t-t_{i-1})(t-t_{i})(t-t_{i+1})(t-t_{i+2})}{(t_{i+3}-t_{i-1})(t_{i+3}-t_{i})(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})} \geq 0,
\end{aligned}
$$

therefore the following results will be obtained:

$$
\begin{aligned}
f^l(t,y(t)) &= l_{i-1}(t)f^r(t_{i-1},y(t_{i-1})) + l_i(t)f^l(t_i,y(t_i)) + l_{i+1}(t)f^r(t_{i+1},y(t_{i+1}))\\
&\quad + l_{i+2}(t)f^l(t_{i+2},y(t_{i+2})) + l_{i+3}(t)f^r(t_{i+3},y(t_{i+3})) \qquad (4.2)\\
f^c(t,y(t)) &= l_{i-1}(t)f^c(t_{i-1},y(t_{i-1})) + l_i(t)f^c(t_i,y(t_i)) + l_{i+1}(t)f^c(t_{i+1},y(t_{i+1}))\\
&\quad + l_{i+2}(t)f^c(t_{i+2},y(t_{i+2})) + l_{i+3}(t)f^c(t_{i+3},y(t_{i+3})) \qquad (4.3)\\
f^r(t,y(t)) &= l_{i-1}(t)f^l(t_{i-1},y(t_{i-1})) + l_i(t)f^r(t_i,y(t_i)) + l_{i+1}(t)f^l(t_{i+1},y(t_{i+1}))\\
&\quad + l_{i+2}(t)f^r(t_{i+2},y(t_{i+2})) + l_{i+3}(t)f^l(t_{i+3},y(t_{i+3})). \qquad (4.4)
\end{aligned}
$$

From (2.2) and (4.1) it follows that:

$$
\widetilde{y}^{\alpha}(t_{i+4}) = [\underline{y}^{\alpha}(t_{i+4}), \overline{y}^{\alpha}(t_{i+4})],
$$

where

$$
(4.5) \quad \underline{y}^{\alpha}(t_{i+4}) = \underline{y}^{\alpha}(t_{i-1}) + \int_{t_{i-1}}^{t_{i+4}} \left\{\alpha f^c(t,y(t)) + (1-\alpha)f^l(t,y(t))\right\} dt,
$$

$$
(4.6) \quad \overline{y}^{\alpha}(t_{i+4}) = \overline{y}^{\alpha}(t_{i-1}) + \int_{t_{i-1}}^{t_{i+4}} \left\{\alpha f^c(t,y(t)) + (1-\alpha)f^r(t,y(t))\right\} dt.
$$

If (4.2) and (4.3) are used in (4.5) and (4.3),(4.4) in (4.6), then

$$\begin{aligned}\underline{y}^{\alpha}(t_{i+4}) &= \underline{y}^{\alpha}(t_{i-1})\\ &+\int_{t_{i-1}}^{t_{i+4}}\Big\{\alpha[l_{i-1}(t)f^{c}(t_{i-1},y(t_{i-1}))+l_{i}(t)f^{c}(t_{i},y(t_{i}))+l_{i+1}(t)f^{c}(t_{i+1},y(t_{i+1}))\\ &+l_{i+2}(t)f^{c}(t_{i+2},y(t_{i+2}))+l_{i+3}(t)f^{c}(t_{i+3},y(t_{i+3}))]\\ &+(1-\alpha)[l_{i-1}(t)f^{r}(t_{i-1},y(t_{i-1}))+l_{i}(t)f^{l}(t_{i},y(t_{i}))+l_{i+1}(t)f^{r}(t_{i+1},y(t_{i+1}))\\ &+l_{i+2}(t)f^{l}(t_{i+2},y(t_{i+2})+l_{i+3}(t)f^{r}(t_{i+3},y(t_{i+3}))]\Big\}dt.\end{aligned}$$

$$\begin{aligned}\overline{y}^{\alpha}(t_{i+4}) &= \overline{y}^{\alpha}(t_{i-1})\\ &+\int_{t_{i-1}}^{t_{i+4}}\Big\{\alpha[l_{i-1}(t)f^{c}(t_{i-1},y(t_{i-1}))+l_{i}(t)f^{c}(t_{i},y(t_{i}))+l_{i+1}(t)f^{c}(t_{i+1},y(t_{i+1}))\\ &+l_{i+2}(t)f^{c}(t_{i+2},y(t_{i+2}))+l_{i+3}(t)f^{c}(t_{i+3},y(t_{i+3}))]\\ &+(1-\alpha)[l_{i-1}(t)f^{l}(t_{i-1},y(t_{i-1}))+l_{i}(t)f^{r}(t_{i},y(t_{i}))+l_{i+1}(t)f^{l}(t_{i+1},y(t_{i+1}))\\ &+l_{i+2}(t)f^{r}(t_{i+2},y(t_{i+2}))+l_{i+3}(t)f^{l}(t_{i+3},y(t_{i+3}))]\Big\}dt.\end{aligned}$$

The following results will be obtained by integration:

$$\begin{aligned}\underline{y}^{\alpha}(t_{i+4}) &= \underline{y}^{\alpha}(t_{i-1})+\frac{95h}{144}[\alpha f^{c}(t_{i-1},y(t_{i-1}))+(1-\alpha)f^{l}(t_{i-1},y(t_{i-1}))]-\frac{50h}{144}[\alpha f^{c}(t_{i},y(t_{i}))\\ &+(1-\alpha)f^{r}(t_{i},y(t_{i}))]+\frac{600h}{144}[\alpha f^{c}(t_{i+1},y(t_{i+1}))+(1-\alpha)f^{l}(t_{i+1},y(t_{i+1})]\\ &-\frac{350h}{144}[\alpha f^{c}(t_{i+2},y(t_{i+2}))+(1-\alpha)f^{r}(t_{i+2},y(t_{i+2}))]+\frac{425h}{144}[\alpha f^{c}(t_{i+3},y(t_{i+3}))\\ &+(1-\alpha)f^{l}(t_{i+3},y(t_{i+3}))],\end{aligned}$$

$$\begin{aligned}\overline{y}^{\alpha}(t_{i+4}) &= \overline{y}^{\alpha}(t_{i-1})+\frac{95h}{144}[\alpha f^{c}(t_{i-1},y(t_{i-1}))+(1-\alpha)f^{r}(t_{i-1},y(t_{i-1}))]-\frac{50h}{144}[\alpha f^{c}(t_{i},y(t_{i}))\\ &+(1-\alpha)f^{l}(t_{i},y(t_{i}))]+\frac{600h}{144}[\alpha f^{c}(t_{i+1},y(t_{i+1}))+(1-\alpha)f^{r}(t_{i+1},y(t_{i+1}))]\\ &-\frac{350h}{144}[\alpha f^{c}(t_{i+2},y(t_{i+2}))+(1-\alpha)f^{l}(t_{i+2},y(t_{i+2}))]+\frac{425h}{144}[\alpha f^{c}(t_{i+3},y(t_{i+3}))\\ &+(1-\alpha)f^{r}(t_{i+3},y(t_{i+3}))].\end{aligned}$$

Thus

$$(4.7)\quad \begin{aligned} \underline{y}^{\alpha}(t_{i+4}) &= \underline{y}^{\alpha}(t_{i-1}) + \tfrac{h}{144}[95\underline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) - 50\overline{f}^{\alpha}(t_i, y(t_i)) + 600\underline{f}^{\alpha}(t_{i+1}, y(t_{i+1})) \\ &\quad -350\overline{f}^{\alpha}(t_{i+2}, y(t_{i+2})) + 425\underline{f}^{\alpha}(t_{i+3}, y(t_{i+3}))], \\ \overline{y}^{\alpha}(t_{i+4}) &= \overline{y}^{\alpha}(t_{i-1}) + \tfrac{h}{144}[95\overline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) - 50\underline{f}^{\alpha}(t_i, y(t_i)) + 600\overline{f}^{\alpha}(t_{i+1}, y(t_{i+1})) \\ &\quad -350\underline{f}^{\alpha}(t_{i+2}, y(t_{i+2})) + 425\overline{f}^{\alpha}(t_{i+3}, y(t_{i+3}))] \end{aligned}$$

Therefore Explicit five step method is obtained as follows:

$$(4.8)\quad \begin{cases} \underline{y}^{\alpha}(t_{i+4}) = \underline{y}^{\alpha}(t_{i-1}) + \dfrac{h}{144}[95\underline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) - 50\overline{f}^{\alpha}(t_i, y(t_i)) + 600\underline{f}^{\alpha}(t_{i+1}, y(t_{i+1})) \\ \qquad -350\overline{f}^{\alpha}(t_{i+2}, y(t_{i+2})) + 425\underline{f}^{\alpha}(t_{i+3}, y(t_{i+3})) \\ \overline{y}^{\alpha}(t_{i+4}) = \overline{y}^{\alpha}(t_{i-1}) + \dfrac{h}{144}[95\overline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) - 50\underline{f}^{\alpha}(t_i, y(t_i)) + 600\overline{f}^{\alpha}(t_{i+1}, y(t_{i+1})) \\ \qquad -350\underline{f}^{\alpha}(t_{i+2}, y(t_{i+2})) + 425\overline{f}^{\alpha}(t_{i+3}, y(t_{i+3})), \\ \underline{y}^{\alpha}(t_{i-1}) = \alpha_0,\ \underline{y}^{\alpha}(t_i) = \alpha_1,\ \underline{y}^{\alpha}(t_{i+1}) = \alpha_2,\ \underline{y}^{\alpha}(t_{i+2}) = \alpha_3, \\ \overline{y}^{\alpha}(t_{i-1}) = \alpha_4,\ \overline{y}^{\alpha}(t_i) = \alpha_5,\ \overline{y}^{\alpha}(t_{i+1}) = \alpha_6\ ,\overline{y}^{\alpha}(t_{i+2}) = \alpha_7. \end{cases}$$

4.2. **Milne's implicit four step method.** Similar way the fuzzy initial value problem $y'(t) = f(t, y(t))$ can be solved by Adams-Moulton four-step method. The Implicit four step method is obtained as follows:

$$(4.9)\quad \begin{cases} \underline{y}^{\alpha}(t_{i+4}) = \underline{y}^{\alpha}(t_{i+3}) + \frac{h}{90}[29\underline{f}^{\alpha}(t_{i+4}, y(t_{i+4})) + 124\underline{f}^{\alpha}(t_{i+3}, y(t_{i+3})) + 24\underline{f}^{\alpha}(t_{i+2}, y(t_{i+2})) \\ \qquad +4\underline{f}^{\alpha}(t_{i+1}, y(t_{i+1})) - \overline{f}^{\alpha}(t_i, y(t_i))] \\ \overline{y}^{\alpha}(t_{i+4}) = \overline{y}^{\alpha}(t_{i+3}) + \frac{h}{90}[29\overline{f}^{\alpha}(t_{i+4}, y(t_{i+4})) + 124\overline{f}^{\alpha}(t_{i+3}, y(t_{i+3})) + 24\overline{f}^{\alpha}(t_{i+2}, y(t_{i+2})) \\ \qquad +4\overline{f}^{\alpha}(t_{i+1}, y(t_{i+1})) - \underline{f}^{\alpha}(t_i, y(t_i))] \\ \underline{y}^{\alpha}(t_i) = \alpha_0,\ \underline{y}^{\alpha}(t_{i+1}) = \alpha_1,\ \underline{y}^{\alpha}(t_{i+2}) = \alpha_2,\ \underline{y}^{\alpha}(t_{i+3}) = \alpha_3 \\ \overline{y}^{\alpha}(t_i) = \alpha_4,\ \overline{y}^{\alpha}(t_{i+1}) = \alpha_5\ ,\overline{y}^{\alpha}(t_{i+2}) = \alpha_6,\ \overline{y}^{\alpha}(t_{i+3}) = \alpha_7 \end{cases}$$

5. Predictor-corrector five step method

The following algorithm is based on Explicit five-step method as a predictor and also an iteration of implicit four-step method as a corrector.

ALGORITHM:

Fix $k \in Z^+$. To approximate the solution of following fuzzy initial value problem.

$$x_k'(t) = f(t_{k,i}, x(t_{k,i}), \lambda_k(x_k))$$

$$\underline{y}^\alpha(t_{k,i-1}) = \underline{\alpha}_0,\ \underline{y}^\alpha(t_{k,i}) = \underline{\alpha}_1,\ \underline{y}^\alpha(t_{k,i+1}) = \underline{\alpha}_2,\ \underline{y}^\alpha(t_{k,i+2}) = \underline{\alpha}_3,\ \underline{y}^\alpha(t_{k,i+3}) = \underline{\alpha}_4.$$

$$\overline{y}^\alpha(t_{k,i-1}) = \overline{\alpha}_0,\ \overline{y}^\alpha(t_{k,i}) = \overline{\alpha}_1,\ \overline{y}^\alpha(t_{k,i+1}) = \overline{\alpha}_2,\ \overline{y}^\alpha(t_{k,i+2}) = \overline{\alpha}_3,\ \overline{y}^\alpha(t_{k,i+3}) = \overline{\alpha}_4.$$

positive integer N_k is chosen.

Step 1. Let $h = \dfrac{t_{k+1} - t_k}{N_k}$,

$$\underline{w}^\alpha(t_{k,0}) = \underline{\alpha}_0, \underline{w}^\alpha(t_{k,1}) = \underline{\alpha}_1, \underline{w}^\alpha(t_{k,2}) = \underline{\alpha}_2, \underline{w}^\alpha(t_{k,3}) = \underline{\alpha}_3, \underline{w}^\alpha(t_{k,4}) = \underline{\alpha}_4,$$

$$\overline{w}^\alpha(t_{k,0}) = \overline{\alpha}_0, \overline{w}^\alpha(t_{k,1}) = \overline{\alpha}_1, \overline{w}^\alpha(t_{k,2}) = \overline{\alpha}_2, \overline{w}^\alpha(t_{k,3}) = \overline{\alpha}_3, \overline{w}^\alpha(t_{k,4}) = \overline{\alpha}_4,$$

Step 2. Let $i = 1$,

Step 3. Let

$$\begin{aligned}\underline{w}^{(0)\alpha}(t_{i+4}) = \underline{w}^\alpha(t_{i-1}) + \tfrac{h}{144}[&95\underline{f}^\alpha(t_{i-1}, w(t_{i-1})) - 50\overline{f}^\alpha(t_i, w(t_i)) \\ &+600\underline{f}^\alpha(t_{i+1}, w(t_{i+1})) - 350\overline{f}^\alpha(t_{i+2}, w(t_{i+2})) + 425\underline{f}^\alpha(t_{i+3}, w(t_{i+3}))],\end{aligned}$$

$$\begin{aligned}\overline{w}^{(0)\alpha}(t_{i+4}) = \overline{w}^\alpha(t_{i-1}) + \tfrac{h}{144}[&95\underline{f}^\alpha(t_{i-1}, w(t_{i-1})) - 50\overline{f}^\alpha(t_i, w(t_i)) \\ &+600\overline{f}^\alpha(t_{i+1}, w(t_{i+1})) - 350\underline{f}^\alpha(t_{i+2}, w(t_{i+2})) + 425\overline{f}^\alpha(t_{i+3}, w(t_{i+3}))].\end{aligned}$$

Step 4. Let $t_{i+4} = t_0 + (i+4)h$.

Step 5. Let

$$\begin{cases}\underline{w}^\alpha(t_{i+3}) = \underline{y}^\alpha(t_{i+2}) + \frac{h}{90}[29\underline{f}^\alpha(t_{i+3}, w(t_{i+3})) + 124\underline{f}^\alpha(t_{i+2}, w(t_{i+2})) \\ \quad +24\overline{f}^\alpha(t_{i+1}, w(t_{i+1})) + 4\underline{f}^\alpha(t_i, w(t_i)) - \overline{f}^\alpha(t_{i-1}, w(t_{i-1}))], \\ \\ \overline{w}^\alpha(t_{i+3}) = \overline{w}^\alpha(t_{i+2}) + \frac{h}{90}[29\overline{f}^\alpha(t_{i+3}, w(t_{i+3})) + 124\overline{f}^\alpha(t_{i+2}, w(t_{i+2})) \\ \quad +24\underline{f}^\alpha(t_{i+1}, y(t_{i+1})) + 4\overline{f}^\alpha(t_i, w(t_i)) - \underline{f}^\alpha(t_{i-1}, y(t_{i-1}))].\end{cases}$$

Step 6. $i = i + 1$.

Step 7. If $i \leq N - 4$ go to step 3.

Step 8. Algorithm will be completed and $(\underline{w}^\alpha(t_{k+1}), \overline{w}^\alpha(t_{k+1})$ approximates real value of $(\underline{x}^\alpha(t_{k+1}), \overline{x}^\alpha(t_{k+1})$.

6. Convergence and Stability

To integrate the system given in equation (4.9) from t_0 a prefixed $T > t_0$ the interval $[t_0, T]$ will be replaced by a set of discrete equally spaced grid point $t_0 < t_1 < t_2 < ... < t_N = T$ which the exact solution $(\underline{Y}(t,\alpha), \overline{Y}(t,\alpha))$ is approximated by some $(\underline{y}(t,\alpha), \overline{y}(t,\alpha))$. The exact and approximate solutions at $t_n,\ \ 0 \leq n \leq N$ are denoded by $Y_n(t,\alpha) = (\underline{Y}_n(t,\alpha), \overline{Y}_n(t,\alpha))$, and $y_n(t,\alpha) = (\underline{y}_n(t,\alpha), \overline{y}_n(t,\alpha))$, respectively. The grid points which the solution is calculated are $t_n = t_0 + nh, h = \frac{(T-t_0)}{N}, 1 \leq n \leq N$.

From (13), the polygon curves

$$\underline{y}(t,h,\alpha) = \{[t_0, \underline{y_0}(\alpha)], [t_1, \underline{y_1}(\alpha)], \ldots, [t_N, \underline{y}_N(\alpha)]\},$$

$$\overline{y}(t,h,\alpha) = \{[t_0, \overline{y_0}(\alpha)], [t_1, \overline{y_1}(\alpha)], \ldots, [t_N, \overline{y}_N(\alpha)]\}$$

are the Milne's implicit approximates to $\underline{Y}(t,\alpha)$ and $\overline{Y}(t,\alpha)$, respectively, over the interval $t_0 \leq t \leq t_N$. The following lemmas will be applied to show convergence of these apptoximates, i.e.

$$\lim_{h\to 0} \underline{y}(t,h,\alpha) = \underline{Y}(t,\alpha), \quad \lim_{h\to 0} \overline{y}(t,h,\alpha) = \overline{Y}(t,\alpha).$$

Lemma 6.1. *Let a sequence of numbers* $\{W_n\}_{n=0}^{N}$ *satisfy*

$$|w_{n+1}| \leq A|w_n| + B|w_{n-1}| + C|w_{n-2}| + D|w_{n-3}| + E, \qquad 0 \leq n \leq N-1$$

for some given positive constants A,B,C D and E. Then

$$\begin{aligned}
|w_{n+1}| \leq\ & (A^{n-2} + \alpha_1 A^{n-4}B + \alpha_2 A^{n-5}C + ... + \alpha_w AD^{[\frac{n-5}{2}]})|w_3| \\
& +(A^{n-3}B + \beta_1 A^{n-5}B^2 + ... + \beta_s B^{[\frac{n-5}{2}]}D^{[\frac{n-5}{2}]})|w_2| \\
& +(A^{n-3}C + \gamma_1 A^{n-5}CB^2 + ... + \gamma_t C^{[\frac{n-5}{2}]}D^{[\frac{n-5}{2}]})|w_1| \\
& +(A^{n-3}D + \eta_1 A^{n-5}CD^2 + ... + \eta_q BD^{[\frac{n-3}{2}]})|w_0| \\
& +(A^{n-3} + A^{n-4} + .. + 1)E + (\delta_1 A^{n-5} + \delta_2 A^{n-6} + ... + \delta_m A + 1)BE \\
& +(\varsigma_1 A^{n-6} + \varsigma_2 A^{n-7} + \varsigma_2 A^{n-8} + ... + \varsigma_l A + 1)CE \\
& +(\lambda_1 A^{n-7} + \lambda_2 A^{n-8} + ... + \lambda_p A + 1)DE \\
& +(\mu_1 A^{n-8} + \mu_2 A^{n-9} + ... + \mu_r A + 1)B^2E + ..., n \textit{ odd} \quad \textit{and}
\end{aligned}$$

$$\begin{aligned}
|w_{n+1}| \leq\ & (A^{n-2} + \alpha_1 A^{n-4}B + \alpha_2 A^{n-5}C + ... + \alpha_w BD^{[\frac{n-4}{2}]})|w_3| \\
& +(A^{n-3}B + \beta_1 A^{n-5}B^2 + \beta_2 A^{n-7}CB^3 + ... + \beta_s AD^{[\frac{n-4}{2}]})|w_2| \\
& +(A^{n-3}C + \gamma_1 A^{n-5}CB^2 + ... + \gamma_t B^{[\frac{n-4}{2}]}D^{[\frac{n-4}{2}]})|w_1| \\
& +(A^{n-3}D + \eta_1 A^{n-5}CD^2 + ... + \eta_q C^{[\frac{n-4}{2}]-}D^{[\frac{n-4}{2}]})|w_0| \\
& +(A^{n-3} + A^{n-4} + .. + 1)E + (\delta_1 A^{n-5} + \delta_2 A^{n-6} + ... + \delta_m A + 1)BE \\
& +(\varsigma_1 A^{n-6} + \varsigma_2 A^{n-7} + \varsigma_2 A^{n-8} + ... + \varsigma_l A + 1)CE \\
& +(\lambda_1 A^{n-7} + \lambda_2 A^{n-8} + ... + \lambda_p A + 1)DE \\
& +(\mu_1 A^{n-8} + \mu_2 A^{n-9} + ... + \mu_r A + 1)B^2E + ..., n \textit{ even}
\end{aligned}$$

where $\alpha_w, \beta_s, \gamma_t, \delta_m, \varsigma_l, \lambda_p, \eta_q$ *are constants for all* w, s, t, m, l, q *and* r.
The proof, by using mathematical induction is straightforward.

Theorem 6.2. *For arbitrary fixed* $\alpha : 0 \leq \alpha \leq 1$ *, the Milne's expilicit four step approximates of (4.9) converges to the exact solution* $\underline{Y}(t,\alpha), \overline{Y}(t,\alpha)$ *for* $\underline{y}, \overline{y} \in c^5[t_0, T]$.

Proof. It is sufficient to show

$$\lim_{h\to 0} \underline{y}_N(\alpha) = \underline{Y}(T,\alpha), \lim_{h\to 0} \overline{y}_N(\alpha) = \overline{Y}(T,\alpha).$$

By using exact value the following results will be obtained:

$$\begin{aligned}\underline{Y}_{n+1}(t;\alpha) = \underline{Y}_n(t;\alpha) + \frac{h}{90}[29f(t_{n+1}, \underline{Y}_{n+1}(t;\alpha)) + 124f(t_n, \underline{Y}_n(t;\alpha)) \\ -24f(t_{n-1}, \overline{Y}_{n-1}(t;\alpha)) + 4f(t_{n-2}, \underline{Y}_{n-2}(t;\alpha)) - 3f(t_{n-3}, \underline{Y}_{n-3}(t;\alpha))] \\ -\frac{19h^5}{1440} Y^{('''''')}(\underline{\xi}_n),\end{aligned}$$

$$\begin{aligned}\overline{Y}_{n+1}(t;\alpha) = \overline{Y}_n(t;\alpha) + \frac{h}{90}[29f(t_{n+1}, \overline{Y}_{n+1}(t;\alpha)) + 124f(t_n, \overline{Y}_n(t;\alpha)) \\ -24f(t_{n-1}, \underline{Y}_{n-1}(t;\alpha)) + 4f(t_{n-2}, \overline{Y}_{n-2}(t;\alpha)) - f(t_{n-3}, \overline{Y}_{n-3}(t;\alpha))] \\ -\frac{19h^6}{1440} Y^{('''''')}(\overline{\xi}_n),\end{aligned}$$

where $t_n < \underline{\xi}_n, \overline{\xi}_n < t_{n+1}$, Consequently

$$\begin{aligned}\underline{Y}_{n+1}(t;\alpha) - \underline{y}_{n+1}(t;\alpha) = \underline{Y}_n(t;\alpha) - \underline{y}_n(t;\alpha) + \frac{h}{90}[29(f(t_{n+1}, \underline{Y}_{n+1}(t;\alpha) - f(t_{n+1}, \underline{y}_{n+1})(t;\alpha))) \\ +124(f(t_n, \underline{Y}_n(t;\alpha)) - f(t_n, \underline{y}_n(t;\alpha))) - 24(f(t_{n-1}, \overline{Y}_{n-1}(t;\alpha)) \\ -f(t_{n-1}, \overline{y}_{n-1}(t;\alpha))) + 4(f(t_{n-2}, \underline{Y}_{n-2}(t;\alpha)) - f(t_{n-2}, \underline{y}_{n-2}(t;\alpha))) \\ -(f(t_{n-3}, \underline{Y}_{n-3}(t;\alpha)) - f(t_{n-3}, \underline{y}_{n-3}(t;\alpha)))] - \frac{19h^6}{1440} Y^{('''''')}(\underline{\xi}_n),\end{aligned}$$

$$\begin{aligned}\overline{Y}_{n+1}(t;\alpha) - \overline{y}_{n+1}(t;\alpha) = \overline{Y}_n(t;\alpha) - \overline{y}_n(t;\alpha) + \frac{h}{90}[29(f(t_{n+1}, \overline{Y}_{n+1}(t;\alpha) - f(t_{n+1}, \overline{y}_{n+1}(t;\alpha)))) \\ +124(f(t_n, \overline{Y}_n(t;\alpha)) - f(t_n, \overline{y}_n(t;\alpha))) - 24(f(t_{n-1}, \underline{Y}_{n-1}(t;\alpha)) \\ -f(t_{n-1}, \underline{y}_{n-1}(t;\alpha))) + 4(f(t_{n-2}, \overline{Y}_{n-2}(t;\alpha)) - f(t_{n-2}, \overline{y}_{n-2}(t;\alpha))) \\ -(f(t_{n-3}, \overline{Y}_{n-3}(t;\alpha)) - f(t_{n-3}, \overline{y}_{n-3}(t;\alpha)))] - \frac{19h^6}{1440} Y^{('''''')}(\overline{\xi}_n).\end{aligned}$$

Denote $w_n = \underline{Y}_n(t;\alpha) - \underline{y}_n(t;\alpha), v_n = \overline{Y}_n - \overline{y}_n(t;\alpha)$.
Then

$$|w_{n+1}| \leq \left(1 + \frac{124hL_1}{90}\right)|w_n| + \left(\frac{24hL_2}{90}\right)|w_{n-1}| + \left(\frac{29hL_3}{90}\right)|w_{n+1}| + \left(\frac{4hL_4}{90}\right)|w_{n-2}|$$
$$+ \left(\frac{hL_5}{90}\right)|v_{n-3}| + \frac{19}{1440}h^6\underline{M},$$

$$|v_{n+1}| \leq \left(1 + \frac{124hL_6}{90}\right)|v_n| + \left(\frac{24hL_7}{90}\right)|v_{n-1}| + \left(\frac{29hL_8}{90}\right)|v_{n+1}| + \left(\frac{4hL_9}{90}\right)|v_{n-2}|$$
$$+ \left(\frac{hL_{10}}{90}\right)|w_{n-3}| + \frac{19}{1440}h^6\overline{M},$$

where $\underline{M} = \max_{t_0 \leq t \leq T} |\underline{Y}^{('''''')}(t,r)| and \;\; \overline{M} = \max_{t_0 \leq t \leq T} |\overline{Y}^{('''''')}(t,r)|$ and is put

$L = max\{L_1, L_2, L_3, L_4 L_5, L_6, L_7, L_8, L_9, L_{10}\} \leq \frac{720}{251h}$,
Then

$$|w_{n+1}| \leq \left(1 + \frac{153hL}{90-29hL}\right)|w_n| + \left(\frac{24hL}{90-29hL}\right)|w_{n-1}| + \left(\frac{4hL}{90-29hL}\right)|v_{n-2}|$$
$$+ \left(\frac{hL}{90-29hL}\right)|w_{n-3}| + \left(\frac{19}{1440-464hL}h^6\overline{M}\right),$$

$$|v_{n+1}| \leq \left(1 + \frac{153hL}{90-29hL}\right)|v_n| + \left(\frac{24hL}{90-29hL}\right)|v_{n-1}| + \left(\frac{4hL}{90-29hL}\right)|w_{n-2}|$$
$$+ \left(\frac{hL}{90-29hL}\right)|v_{n-3}| + \left(\frac{19}{1440-464hL}h^6\overline{M}\right),$$

are resulted,where $|u_n| = |w_n| + |v_n|$,then by Lemma 6.1 and $w_0 = v_0 = 0$(also with $w_1 = v_1 = 0$):

$$|u_n| \leq \frac{\left(1 + \frac{153hL}{90-29hL}\right)^{n-1} - 1}{\frac{153hL}{90-29hL}} \times \frac{19}{1440-464hL}h^6(\underline{M} + \overline{M})$$
$$+ \left\{\delta_1\left(1 + \frac{153hL}{90-29hL}\right)^{n-5} + \delta_2\left(1 + \frac{153hL}{90-29hL}\right)^{n-6} + \ldots\ldots + \delta_m\left(1 + \frac{153hL}{90-29hL}\right) + 1\right\}$$
$$\times \left(\frac{24hL}{90-29hL}\right)\left(\frac{19}{1440-464hL}h^6(\underline{M} + \overline{M})\right)$$

$$+\left\{\zeta_1\left(1+\frac{153hL}{90-29hL}\right)^{n-6}+\zeta_2\left(1+\frac{153hL}{90-29hL}\right)^{n-7}+.......+\zeta_p\left(1+\frac{153hL}{90-29hL}\right)+1\right\}$$

$$\times\left(\frac{4hL}{90-29hL}\left(\frac{19}{1440-464hL}h^6\right)(\underline{M}+\overline{M})\right)$$

$$+\left\{\lambda_1\left(1+\frac{153hL}{90-29hL}\right)^{n-7}+\lambda_2\left(1+\frac{153hL}{90-29hL}\right)^{n-8}+........+\lambda_q\left(1+\frac{10hL}{24-9hL}\right)+1\right\}$$

$$\times\left(\frac{hL}{90-29hL}\right)\left(\frac{19}{1440-464hL}h^6(\underline{M}+\overline{M})\right)+$$

$$+\left\{\xi_1\left(1+\frac{395hL}{720-251hL}\right)^{n-8}+\xi_2\left(1+\frac{395hL}{720-251hL}\right)^{n-9}+......+\xi_q\left(1+\frac{10hL}{24-9hL}\right)+1\right\}$$

$$\times\left(\frac{24hL}{90-29hL}\right)^2\left(\frac{19}{1440-464hL}h^6(\underline{M}+\overline{M})\right)+........$$

are obtained. If $h \to 0$, then $w_n \to 0, v_n \to 0$ which concludes the proof. □

Remark 6.3. Above theorem results that convergence order is $O(h^5)$

Theorem 6.4. *For arbitrary fixed $r : 0 \le r \le 1$ the Milne's explicit four step method approximates of Equation (4.8) converge to the exact solutions $\underline{Y}(t,\alpha), \overline{Y}(t,\alpha)$ for $\underline{Y}, \overline{Y} \in C^5[t_0, T]$.*

Proof. .Similar to Theorem 6.2. □

Remark 6.5. It is easy to show that convergence order of Milne's explicit five-step method is $O(h^5)$.

Theorem 6.6. .
Milne's expilicit four-step and five-step methods are stable.

Proof. For Milne's explicit four-step method, exist only one characteristic polynomial $p(\lambda) = \lambda^4 - \lambda^3$ and it is clear that satisfies the root condition by Theorem 6.2; Milne's explicit four-step method is stable. Also, for Milne's explicit five-step method, there only one characteristic polynomial $p(\lambda) = \lambda^4 - \lambda^3$ and it satisfies the root condition, therefore it is a stable. □

Theorem 6.7. .
Milne's implicit three-step and four-step methods are stable.

Proof. Similar to Theorem 6.6. □

7. Numerical Examples

7.1. **Example.** Consider the fuzzy initial value problem,

$$\begin{aligned}
&y'(t) = y(t), \quad t \in I = [0,1],\\
&y(0) = [0.75 + 0.25\alpha, 1.125 - 0.125\alpha], \quad 0 < \alpha \leq 1\\
&y(0.1) = [(0.75 + 0.25\alpha)e^{0.1}, (1.125 - 0.125\alpha)e^{0.1}],\\
&y(0.2) = \left[(0.75 + 0.25\alpha)e^{0.2}, (1.125 - 0.125\alpha)e^{0.2}\right],\\
&y(0.3) = [(0.75 + 0.25\alpha)e^{0.3}, (1.125 - 0.125\alpha)e^{0.3}],\\
&y(0.4) = [(0.75 + 0.25\alpha)e^{0.4}, (1.125 - 0.125\alpha)e^{0.4}],
\end{aligned}$$

The exact solution at $t = 1$ is given by

$$Y(1;\alpha) = [(0.75 + 0.25\alpha)e, \quad (1.125 - 0.125\alpha)e], \quad 0 < \alpha \leq 1.$$

By using the Milne's-fifth order predictor-corrector method the following results are obtained:

Table 7.1

α	Adams-3		Milne's-5		Exact Solution	
	$\underline{y}(t_i;\alpha)$	$\overline{y}(t_i;\alpha)$	$\underline{y}(t_i;\alpha)$	$\overline{y}(t_i;\alpha)$	$\underline{Y}(t_i;\alpha)$	$\overline{Y}_2(t_i;\alpha)$
0.1	0.856507464	1.229502650	0.856507462	1.229502646	0.856507462	1.229502646
0.2	0.884136737	1.215688014	0.884136735	1.215688010	0.884136735	1.215688010
0.3	0.911766010	1.201873382	0.911766007	1.201873373	0.911766007	1.201873373
0.4	0.939395283	1.188058741	0.939395280	1.188058737	0.939395280	1.188058740
0.5	0.967024556	1.174244104	0.967024553	1.174244100	0.967024553	1.174244100
0.6	0.994653829	1.160429468	0.994653826	1.160429464	0.994653826	1.160429464
0.7	1.022283102	1.146614831	1.022283099	1.146614828	1.022283099	1.146614828
0.8	1.049912375	1.132800195	1.049912372	1.132800191	1.049912372	1.132800191
0.9	1.077541649	1.118985558	1.077541645	1.118985555	1.077541645	1.118985555
0.10	1.105170922	1.105170922	1.105170918	1.105170918	1.105170918	1.105170918

Table 7.2.

Error analysis in Adams third order and Milne's fifth order predictor-corrector methods

α	Error in Adams-3		Error in Milne's-5	
	$\underline{y}(t_i;\alpha)$	$\overline{y}(t_i;\alpha)$	$\underline{Y}(t_i;\alpha)$	$\overline{Y}(t_i;\alpha)$
0.1	-2.7126e-9	-2.7126e-9	-2.5646e-14	-2.5646e-14
0.2	-2.8001e-9	-2.8001e-9	-2.6423e-14	-2.6423e-14
0.3	-2.8876e-9	-2.8876e-9	-2.7311e-14	-2.7311e-14
0.4	-2.9751e-9	-2.9751e-9	-2.8089e-14	-2.8089e-14
0.5	-3.0625e-9	-3.0625e-9	-2.8977e-14	-2.8977e-14
0.6	-3.1501e-9	-3.1501e-9	-2.9754e-14	-2.9754e-14
0.7	-3.2376e-9	-3.2376e-9	-3.0420e-14	-3.0420e-14
0.8	-3.3251e-9	-3.3251e-9	-3.1530e-14	-3.1530e-14
0.9	-3.1426e-9	-3.1426e-9	-3.2196e-14	-3.2196e-14
1.0	-3.5001e-9	-3.5001e-9	-3.3085e-14	-3.3085e-14

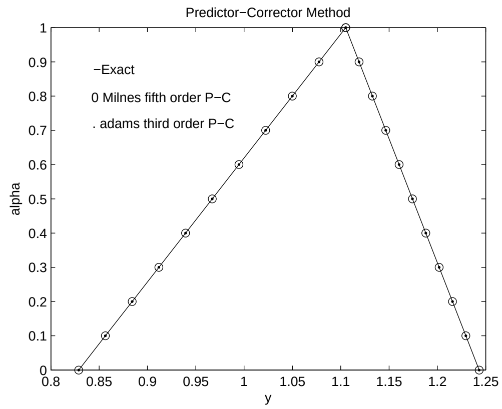

FIGURE 1. h=0.01

7.2. **Example.** Consider the fuzzy initial value problem

$$y'(t) = -y(t), \quad t \in I = [0,1],$$
$$y(0) = [0.75 + 0.25\alpha, 1.125 - 0.125\alpha], \quad 0 < \alpha \leq 1$$

$$\underline{y}(0.01) = (0.9375 + 0.0625\alpha)e^{-0.01} - (1-\alpha)(0.1875)e^{0.01},$$

$$\overline{y}(0.01) = (0.9375 + 0.0625\alpha)e^{-0.01} + (1-\alpha)(0.1875)e^{0.01},$$

$$\underline{y}(0.02) = (0.9375 + 0.0625\alpha)e^{-0.02} - (1-\alpha)(0.1875)e^{0.02},$$

$$\overline{y}(0.02) = (0.9375 + 0.0625\alpha)e^{-0.02} + (1-\alpha)(0.1875)e^{0.02},$$

$$\underline{y}(0.03) = (0.9375 + 0.0625\alpha)e^{-0.03} - (1-\alpha)(0.1875)e^{0.03},$$

$$\overline{y}(0.03) = (0.9375 + 0.0625\alpha)e^{-0.03} + (1-\alpha)(0.1875)e^{0.03},$$

$$\underline{y}(0.04) = (0.9375 + 0.0625\alpha)e^{-0.04} - (1-\alpha)(0.1875)e^{0.04},$$

$$\overline{y}(0.04) = (0.9375 + 0.0625\alpha)e^{-0.04} + (1-\alpha)(0.1875)e^{0.04}.$$

The exact solution at $t = 0.1$ is given by

$$Y(0.1;\alpha) = [(0.9375 + 0.0625\alpha)e^{-0.1} - (1-\alpha)(0.1875)e^{0.1}, (0.9375 + 0.0625\alpha)e^{-0.1} + (1-\alpha)(0.1875)e^{0.1}]$$

By using the Milne's fifth order predictor-corrector method the following results are obtained.

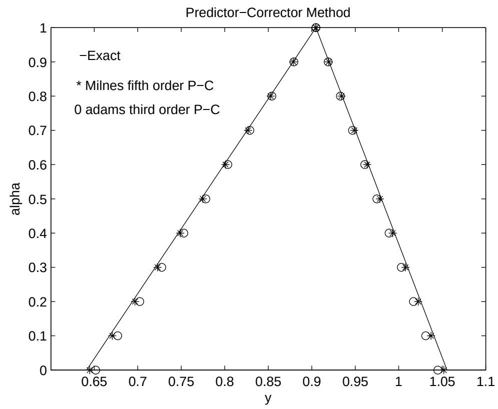

FIGURE 2. h=0.01

Table 7.3.

α	Adam's-3		Milne's-5		Exact Solution	
	$\underline{y}(t_i;\alpha)$	$\overline{y}(t_i;\alpha)$	$\underline{y}(t_i;\alpha)$	$\overline{y}(t_i;\alpha)$	$\underline{Y}(t_i;\alpha)$	$\overline{Y}(t_i;\alpha)$
0.1	0.676896473	1.030984160	0.667442721	1.040437906	0.670686728	1.037193898
0.2	0.702223245	1.016967855	0.693819909	1.025371185	0.696703471	1.022487623
0.3	0.727550017	1.002951551	0.720197098	1.010304464	0.722720215	1.007781347
0.4	0.752876789	0.988935247	0.746574287	0.995237743	0.748736958	0.993075072
0.5	0.778203561	0.974918943	0.772951475	0.980171022	0.774753701	0.978368796
0.6	0.803530333	0.960902638	0.799328664	0.965104301	0.800770444	0.963662520
0.7	0.828857105	0.946886334	0.825705852	0.950037581	0.826787188	0.948956245
0.8	0.854183877	0.932870030	0.852083041	0.934970860	0.852803931	0.934249969
0.9	0.879510649	0.918853726	0.878460229	0.919904139	0.878820675	0.919543694
1.0	0.904837421	0.904837421	0.904837418	0.904837418	0.904837418	0.904837418

Table 7.4.
Error analysis in Adams third order and Milne's fifth order predictor-corrector methods

α	Error in Adams-3		Error in Milne's-5	
	$\underline{y}(t_i;\alpha)$	$\overline{y}(t_i;\alpha)$	$\underline{Y}(t_i;\alpha)$	$\overline{Y}(t_i;\alpha)$
0.1	-9.4538e-3	9.4538e-3	-3.2440e-3	3.2440e-3
0.2	-8.4033e-3	8.4033e-3	-2.8836e-3	2.8836e-3
0.3	-7.3529e-3	7.3529e-3	-2.5231e-3	2.5231e-3
0.4	-6.3025e-3	6.3025e-3	-2.1627e-3	2.1627e-3
0.5	-5.2521e-3	5.2521e-3	-1.8022e-3	1.8022e-3
0.6	-4.2017e-3	4.2017e-3	-1.4418e-3	1.4418e-3
0.7	-3.1513e-3	3.1513e-3	-1.0813e-3	1.0813e-3
0.8	-2.1008e-3	2.1008e-3	-7.2089e-4	7.2089e-4
0.9	-1.0504e-3	1.0504e-3	-3.6045e-4	3.6045e-4
1.0	-3.1640e-9	3.1640e-9	-3.3196e-14	-3.3196e-14

8. Conclution

In this paper, we have applied itrative solution of Milne's fifth order predictor-corrector method for finding the numerical solution of fuzzy differential equations. Comparision of solution of Example 7.1 and 7.2 shows that our proposed method gives better solution then Adam's third order predictor-corrector method.

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