Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 10, No. 5, (November 2015), pp. 789–798
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

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Fuzzy fractional integro-differential equations under generalized Caputo differentiability

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Received 22 November 2014 ; Revised 22 January 2015; Accepted 24 April 2015

Abstract. In this paper, existence and uniqueness theorems for nonlinear fuzzy fractional Fredholm integro-differential equations are investigated. We interpret Fredholm integro-differential equations under fractional generalized Hukuhara derivatives in the Caputo sense, and for this interpretation, we prove existence results in two type of differentiability. Moreover, two examples are given to illustrate the usefulness of our main results.

2010 AMS Classification: 34A07, 34A08

Keywords: Generalized Hukuhara differentiability; Caputo generalized Hukuhara derivative; Nonlinear fuzzy fractional Fredholm integro-differential equation; Fixed point theorem; Method of successive iteration.

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1. Introduction

In this work, we study the existence and uniqueness of solutions for the following nonlinear fuzzy fractional integro-differential equations with fuzzy initial condition

$$\begin{cases} ({}_{gH}D^q_* u)(t) = f(t, u(t), Ku(t)) \quad , \quad t \in J = [a, b] \\ u(t_0) = u_0 \in \mathbb{R}_\mathcal{F} \end{cases} \tag{1.1}$$

where $0 < q \leq 1$ is a real number and the operator ${}_{gH}D^q_*$ denote the Caputo fractional generalized derivative of order q , $f : J \times \mathbb{R}_\mathcal{F} \times \mathbb{R}_\mathcal{F} \times \to \mathbb{R}_\mathcal{F}$ is bounded continuous function which satisfies some assumptions that will be specified later, and

$$Ku(t) = \int_a^b k(t, s, u(s))ds \tag{1.2}$$

with $K : J \times J \to R$.

The study of fuzzy Riemann- Liouville fractional differential equations has been initiated as an independent subject in conjunction with fuzzy valued analysis [1, 4, 16] and fuzzy Caputo fractional differential equations [3] . Sadia Arshad et al in [7] prove some results on the existence and uniqueness of solutions of Riemann-Liouville fuzzy fractional differential equations and in [8] study the existence and uniqueness of the solution for a class of fractional differential equation with fuzzy initial value. Two new uniqueness results for fuzzy fractional differential equations involving Riemann-Liouville generalized Hukuhara differentiability have been investigated in [5] with the Nagumo-type condition and the Krasnoselskii-Krein-type condition. In [3] the existence and uniqueness of the solution for a class of fuzzy Caputo fractional differential equation with initial value have been studied.

Alikhani et al in [2] prove the existence and uniqueness results for fuzzy fractional integral and integro-differential equations involving Riemann-Liouville differential operators .

In this paper the nonlinear fuzzy fractional Fredholm integro-differential equation under generalized fuzzy Caputo derivative is introduced. Also we study the problems of existence and uniqueness of the solutions of this set of equations.

2. Preliminaries

In this section, we recall some basic concepts which are used throughout the paper.

Definition 2.1. A fuzzy number is a function such as $u : \mathbb{R} \longrightarrow [0, 1]$ satisfying the following properties:

1. u is normal, i. e. $\exists t_0 \in \mathbb{R}$ with $u(t_0) = 1$,
2. u is a convex fuzzy set i. e. $u((1-\lambda)t_1 + \lambda t_2) \geq \min\{u(t_1), u(t_2)\}, \forall t_1, t_2 \in \mathbb{R}, \lambda \in [0, 1]$,
3. u is upper semi-continuous on $\mathbb{R}$,
4. $\overline{\{t \in \mathbb{R} : u(t) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

The set of all fuzzy real numbers is denoted by $\mathbb{R}_{\mathcal{F}}$.

For $0 < \alpha \leq 1$ the α-level set $u(\alpha)$ of a fuzzy number u is the subset of points $t \in \mathbb{R}$ with membership grade $u(t)$ of at least α, that is $u(\alpha) = \{t \in \mathbb{R} | u(t) \geq \alpha\} = [u_\alpha^-,\ u_\alpha^+]$ and $u(0) = \overline{\{t \in \mathbb{R} | u(t) > 0\}}$. Then from (1) to (4), it follows that if u belongs to $\mathbb{R}_{\mathcal{F}}$ then the α-level set $u(\alpha)$ is a closed interval for all $\alpha \in [0,\ 1]$. For arbitrary $u,\ v \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$, $k > 0$ the addition and scalar multiplication are defined by $u \oplus v = u(\alpha) + v(\alpha)$, $k \odot u = [ku_\alpha^-, ku_\alpha^+]$ respectively.

Definition 2.2. The Hausdorff distance between fuzzy numbers is given by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}^+ \cup \{0\}$ as in [9, 11, 15]

$$d(u,\ v) = \sup_{\alpha \in [0,\ 1]} d_H\Big(u(\alpha),\ v(\alpha)\Big) = \sup_{\alpha \in [0,\ 1]} \max\Big\{|u_\alpha^- - v_\alpha^-|,\ |u_\alpha^+ - v_\alpha^+|\Big\}$$

where d_H is the Hausdorff metric. The metric space $(\mathbb{R}_{\mathcal{F}},\ d)$ is complete, separable and locally compact and the following properties from [13] for metric d are valid:

1. $d(u \oplus w,\ v \oplus w) = d(u,\ v), \qquad \forall u,\ v,\ w \in \mathbb{R}_{\mathcal{F}};$
2. $d(\lambda u,\ \lambda v) = |\lambda| d(u,\ v), \qquad \forall \lambda \in \mathbb{R},\ u,\ v \in \mathbb{R}_{\mathcal{F}};$
3. $d(u \oplus v,\ w \oplus z) \leq d(u,\ w) + d(v,\ z) \qquad \forall u,\ v,\ w,\ z \in \mathbb{R}_{\mathcal{F}};$
4. $d(u \ominus v,\ w \ominus z) \leq d(u,\ w) + d(v,\ z)$ as long as $u \ominus v$ and $w \ominus z$ exist, where $u,\ v,\ w,\ z \in \mathbb{R}_{\mathcal{F}}$.

Definition 2.3 ([9])**.** Let $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v + w$, then w is called the Hukuhara difference of u and v, and it is denoted by $u \ominus v$.

Definition 2.4 ([10])**.** The generalized Hukuhara difference of two fuzzy number $u, v \in \mathbb{R}_{\mathcal{F}}$ is defined as follows

$$u \ominus_{gH} v = w \Longleftrightarrow \begin{cases} \quad (i\ ) \quad \text{u = v + w} \ , \\ or \quad (ii) \quad \text{v = u + (-1) w} \ . \end{cases}$$

Please note that α-level representation of fuzzy-valued function $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ expressed by $f_\alpha(t) = [f_\alpha^-(t), f_\alpha^+(t)]$, $t \in [a, b], 0 \leq \alpha \leq 1$.

Definition 2.5 ([10])**.** The generalized Hukuhara derivative of a fuzzy-valued function $f : (a, b) \longrightarrow \mathbb{R}_{\mathcal{F}}$ at t_0 is defined as

$$f_{gH}^{'}(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) \ \ominus_{gH} \ f(t_0)}{h},$$

if $(f)_{gH}^{'}(t_0) \in \mathbb{R}_{\mathcal{F}}$, we say that f is generalized Hukuhara differentiable (gH-differentiable) at t_0.
Also we say that f is [(i) -gH]-differentiable at t_0 if

$$(f_{gH}^{'})_\alpha(t_0) = [(f_\alpha^-)^{'}(t_0), (f_\alpha^+)^{'}(t_0)], \qquad 0 \leq \alpha \leq 1, \tag{2.1}$$

and that f is [(ii) -gH]-differentiable at t_0 if

$$(f_{gH}^{'})_\alpha(t_0) = [(f_\alpha^+)^{'}(t_0), (f_\alpha^-)^{'}(t_0)], \qquad 0 \leq \alpha \leq 1. \tag{2.2}$$

Definition 2.6 ([17])**.** We say that a point $t_0 \in (a, b)$, is a switching point for the differentiability of f, if in any neighborhood V of t_0 there exist points $t_1 < t_0 < t_2$ such that
type (I) at t_1 (2.1) holds while (2.2) does not hold and at t_2 (2.2) holds and (2.1) does not hold, or
type (II) at t_1 (2.2) holds while (2.1) does not hold and at t_2 (2.1) holds and (2.2) does not hold.

Definition 2.7 ([6])**.** A fuzzy-valued function $f : [a,\ b] \to \mathbb{R}_{\mathcal{F}}$ is said to be continuous at $t_0 \in [a,\ b]$ if for each $\epsilon > 0$ there is $\delta > 0$ such that $d(f(t), f(t_0)) < \epsilon$, whenever $t \in [a,\ b]$ and $|t - t_0| < \delta$. We say that f is fuzzy continuous on $[a, b]$ if f is continuous at each $t_0 \in [a,\ b]$.

Definition 2.8 ([12]))**.** Let $f : [a,\ b] \to \mathbb{R}_{\mathcal{F}}$, for each partition $P = \{t_0, t_1, ..., t_n\}$ of $[a, b]$ and for arbitrary $\xi_i \in [t_{i-1}, t_i]$, $1 \leq i \leq n$, suppose $R_p = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})$, and $\Delta := max\{|t_i - t_{i-1}|, 1 \leq i \leq n\}$. The definite Riemann integral of $f(t)$ over [a, b] is

$$\int_a^b f(t)dt = \lim_{\Delta \to 0} R_p,$$

provided that this limit exists in the metric d.
Note that if the fuzzy function $f(t)$ is continuous in the metric d, Lebesgue integral and Riemann integral yield the same value, and also

$$\left(\int_a^b f(t)dt\right)_\alpha = \left[\int_a^b f_\alpha^-(t)dt, \int_a^b f_\alpha^+(t)dt\right], \qquad 0 \le \alpha \le 1,$$

Definition 2.9 ([14]). Consider $f : [a, b] \to \mathbb{R}$, fractional derivative of $f(t)$ in the Caputo sense is defined as
(2.3)

$$(D_*^q f)(t) = (I^{m-q} D^m f)(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{(q-m-1)} f^{(m)}(s) ds \qquad q-1 < m \le q\,,\ m \in \mathbb{N}\,,\ s > a$$

where D stand for classic derivative .

Throughout this paper, we consider the notations $A^{\mathbb{F}}[a, b]$ for the space of fuzzy-valued functions from $[a, b]$ into $\mathbb{R}_{\mathcal{F}}$ that are absolutely continuous on $[a, b]$. Also, $C^{\mathbb{F}}[a, b]$ denote the set of fuzzy-valued function which are fuzzy continuous on all of $[a,\ b]$ such that the continuity is one-sided at endpoints $a,\ b$. Also, we denote the space of all Lebesgue integrable fuzzy-valued functions on the bounded interval $[a, b] \subset \mathbb{R}$ by $L^{\mathbb{F}}[a, b]$.

Definition 2.10 ([16]). Let $f \in L^{\mathbb{F}}[a, b]$. The fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as following:

$$(I_a^q f)(t) = \frac{1}{\Gamma(q)} \int_a^t \frac{f(s)ds}{(t-s)^{1-q}}, \qquad a < s < t, \qquad 0 < q \le 1. \tag{2.4}$$

Definition 2.11 ([3]). Let $f \in A^{\mathbb{F}}[a, b]$. The fractional generalized Hukuhara Caputo derivative of fuzzy-valued function f is defined as following:
(2.5)

$$(\ {}_{gH}D_*^q f)(t) = I_a^{1-q}(f'_{gH})(t) = \frac{1}{\Gamma(1-q)} \int_a^t \frac{(f'_{gH})(s)ds}{(t-s)^q}, \qquad a < s < t, \qquad 0 < q \le 1$$

Also we say that f is ${}^{cf}[(i) - gH]$-differentiable at t_0 if

$$({}_{gH}D_*^q f)_\alpha(t_0) = [(D_*^q f_\alpha^-)(t_0), (D_*^q f_\alpha^+)(t_0)], \qquad 0 \le \alpha \le 1 \tag{2.6}$$

and that f is ${}^{cf}[(ii) - gH]$-differentiable at t_0 if

$$({}_{gH}D_*^q f)_\alpha(t_0) = [(D_*^q f_\alpha^+)(t_0), (D_*^q f_\alpha^-)(t_0)], \qquad 0 \le \alpha \le 1 \tag{2.7}$$

Definition 2.12 ([3]). We say that a point $t_0 \in (a, b)$ is a switching point for the differentiability of f, if in any neighborhood V of t_0 there exist points $t_1 < t_0 < t_2$ such that
type (I) at t_1 (2.6) hold while (2.7) does not hold and at t_2 (2.7) holds and (2.6) does not hold, or
type (II) at t_1 (2.7) hold while (2.6) does not hold and at t_2 (2.6) holds and (2.7) does not hold.

3. Fuzzy fractional integro-differential equations of Fredholm type

In this section, we give the main results on the existence of solution of Eq.(1.1). For this purpose, we need the following Lemma and assumptions:

Lemma 3.1 ([3]). *Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy-valued function such that $f \in A^{\mathbb{F}}[a,b]$,*

$$I_a^q({}_{gH}D_*^q f)(t) = f(t) \ominus_{gH} f(a)$$

Lemma 3.2. *The initial value problem (1.1) is equivalent to one of the following integral equations:*

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, u(s), Ku(s))ds$$

if $u(t)$ be ${}^{cf}[(i)-gH]$-differentiable,

$$u(t) = u_0 \ominus \frac{-1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s, u(s), Ku(s))ds$$

if $u(t)$ be ${}^{cf}[(ii)-gH]$-differentiable, and

$$u(t) = \begin{cases} u_0 + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s, u(s), Ku(s))ds, & t \in [a,\ c]; \\ u_0 \ominus \frac{-1}{\Gamma(q)} \int_c^t (t-s)^{q-1} f(s, u(s), Ku(s))ds, & t \in [c,\ b]. \end{cases}$$

if there exists a point $c \in (a,\ b)$ such that $u(t)$ is ${}^{cf}[(i)-gH]$-differentiable on $[a,\ c]$ and ${}^{cf}[(ii)-gH]$-differentiable on $[c,\ b]$ and $f(c, u(c), Ku(c)) \in \mathbb{R}$.

Proof. *By using of Definition 2.10 and Eq.(1.1) , we have*

$$I_{t_0}^q(({}_{gH}D_*^q u)(t)) = I_{t_0}^q(f(t, u(t), Ku(t)))$$

So Lemma 3.1 concludes that

$$u(t) \ominus_{gH} u_0 \quad = \quad \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, u(s), Ku(s))ds$$

Let $u(t)$ be ${}^{cf}[(i)-gH]$-differentiable, then

$$u(t) \quad = \quad u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, u(s), Ku(s))ds \tag{3.1}$$

and, if $u(t)$ be ${}^{cf}[(ii)-gH]$-differentiable,

$$u(t) \quad = \quad u_0 \ominus \frac{-1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, u(s), Ku(s))ds \tag{3.2}$$

Now consider the type of ${}^{cf}[gH]$-differentiability switches at $c \in (a,b)$ of type (I). Then on interval $[a,c)$ the solution of (1.1), $u(t)$, is as Eq.(3.1) and on $[c,b]$ as Eq. (3.2). □

Now we make the following assumptions:
$(\mathbf{A_1})$. $f : J \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \times \to \mathbb{R}_{\mathcal{F}}$ is continuous and there exist real positive functions L_1, L_2 such that

$$d(f(t, x_1, y_1), f(t, x_2, y_2)) \le L_1(t)d(x_1, x_2) + L_2(t)d(y_1, y_2)$$

for all $x_1, x_2, y_1, y_2 \in C^{\mathbb{F}}[a,b]$.
($\mathbf{A_2}$). $k : J \times J \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is continuous and there exist real positive functions K^* such that

$$d(k(t,s,x_1), k(t,s,x_2)) \leq K^* d(x_1, x_2)$$

for all $x_1, x_2 \in C^{\mathbb{F}}[a,b]$.
($\mathbf{A_3}$). There exist a number v such that $\lambda \leq v < 1$, $t \in J$

$$\lambda = I_a^q L(1 + K^*)$$

and
$I_a^q L = \sup_{t\in J}\{I_a^q L_1, I_a^q L_2\}$

Theorem 3.3. *Assume that the hypotheses $(A_1), (A_2)$ holds. Then the initial value problem (1.1) has a unique solution which is ${}^{cf}[(i)-gH]$-differentiable on J, provided that $\lambda < 1$, where λ is given in (A_3).*

Proof. *Suppose that $u(t)$ is ${}^{cf}[(i)-gH]$-differentiable and $u_0 \in \mathbb{R}_{\mathcal{F}}$ be fixed. We want to prove that the mapping $F : C^{\mathbb{F}}[a,b] \to C^{\mathbb{F}}[a,b]$ defined by*

$$(Fu)(t) = u_0 + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s, u(s), Ku(s))\, ds \quad , \quad t \in [a,b]$$

is a contraction. For this, let $u, v \in C^{\mathbb{F}}[a,b]$ by means of (A_1) and the properties of distance (2.2), we show that

$$\begin{aligned}
d(Fu(t), Fv(t)) &\leq \frac{1}{\Gamma(q)} \int_a^t \mid (t-s)^{q-1} \mid d(f(s,u(s),Ku(s)), f(s,v(s),Kv(s)))\, ds \\
&\leq \frac{1}{\Gamma(q)} \int_a^t \mid (t-s)^{q-1} \mid [L_1 d(u,v) + L_2 d(Ku,Kv)]\, ds \\
&\leq \frac{1}{\Gamma(q)} \int_a^t \mid (t-s)^{q-1} \mid .L_1.d(u,v)\, ds + \frac{1}{\Gamma(q)} \int_a^t \mid (t-s)^{q-1} \mid .L_2.d(Ku,Kv)\, ds \\
&\leq I_a^q L_1.d(u,v) + I_a^q L_2 K^*.d(u,v) \\
&\leq I_a^q L(1+K^*).d(u,v) \\
&< d(u,v)
\end{aligned}$$

There fore, F is a contraction mapping on $C^{\mathbb{F}}[a,b]$ and has a fixed point $Fu(t) = u(t)$. Hence the initial value problem (1.1) has unique ${}^{cf}[(i)-gH]$-differentiable solution.

Theorem 3.4. *Assume that the Eq.(1.1) satisfies the conditions $(A_1), (A_2)$. Then there exists a unique solution $u(t)$ of (1.1) on $[a,b]$ and the successive iterations*

$$\begin{aligned}
u_0(t) &= u_0 \\
u_{n+1}(t) &= u_0 \ominus \frac{-1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s, u_n(s), Ku_n(s))\, ds
\end{aligned} \tag{3.3}$$

are convergence to $u(t)$ which is ${}^{cf}[(ii)-gH]$-differentiable on J, provided that $\lambda < 1$, where λ is given in (A_3).

Proof. *First we prove that the sequence* $\{u_n\}$ *(3.3) is a cauchy sequence in* $C^{\mathbb{F}}[a,b]$. *For this purpose, we have*

$$\begin{aligned} d(u_1,u_0) &= d\big(u_0 \ominus \frac{-1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(s,u_0(s),Ku_0(s))\,ds, u_0\big) \\ &\leq \frac{1}{\Gamma(q)}\int_a^t \mid (t-s)^{q-1} \mid d(f(s,u_0(s),Ku_0(s)),\tilde{0})\,ds \\ &\leq I_a^q M \end{aligned}$$

where $M = \sup_{t\in J} d(f(t,u,Ku),\tilde{0})$.
Using properties of Definition 2.2 and Lipschitz continuity of f, *gives*

$$\begin{aligned} d(u_{n+1},u_n) &\leq \frac{1}{\Gamma(q)}\int_a^t \mid (t-s)^{q-1} \mid d(f(s,u_n(s),Ku_n(s)), f(s,u_{n-1}(s),Ku_{n-1}(s)))\,ds \\ &\leq \frac{1}{\Gamma(q)}\int_a^t \mid (t-s)^{q-1} \mid L_1.d(u_n,u_{n-1})\,ds + \frac{1}{\Gamma(q)}\int_a^t \mid (t-s)^{q-1} \mid L_2.d(Ku_n,Ku_{n-1})\,ds \\ &\leq I_a^q L(1+K^*).d(u_n,u_{n-1}) \leq \lambda d(u_n,u_{n-1}) \leq \lambda^n d(u_1,u_0) \leq \lambda^n I_a^q M \end{aligned}$$

Since $\lambda < 1$ *the sequence* $\{u_n\}$ *is a cauchy sequence in* $C^{\mathbb{F}}[a,b]$. *Therefore there exist* $u \in C^{\mathbb{F}}[a,b]$ *such that* $\{u_n\}$ *converges to* u. *So we must show that* u *is a solution of the problem (1.1).*

$$\begin{aligned} & d(u(t) + \frac{-1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(s,u(s),Ku(s))\,ds, u_0) \\ = \; & d(u(t) + \frac{-1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(s,u(s),Ku(s))\,ds, u_{n+1}(t) + \frac{-1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(s,u_n(s),Ku_n(s))\,ds) \\ \leq \; & d(u(t),u_{n+1}) + \frac{1}{\Gamma(q)}\int_a^t \mid (t-s)^{q-1} \mid L_1.d(u(s),u_n)\,ds + \frac{1}{\Gamma(q)}\int_a^t \mid (t-s)^{q-1} \mid L_2.d(Ku(s),Ku_n)\,ds \\ \leq \; & d(u(t),u_{n+1}) + I_a^q L(1+K^*).d(u(t),u_n) \end{aligned}$$

The right-hand side tends to zero as $n \to \infty$. *Therefore, we conclude that*

$$u(t) + \frac{-1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(s,u(s),Ku(s))\,ds = u_0$$

Now Lemma 3.2 implies that u *is a solution of the problem (1.1) .*

To prove the uniqueness, assume that $v(t)$ *is a solution of problem (1.1) on* J, *i.e.*

$$v(t) = u_0 \ominus \frac{-1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(s,v(s),Kv(s))\,ds$$

Using the recurrence formula for $u_n(t)$, *we obtain*

$$d(v(t),u_n(t)) \leq \frac{1}{\Gamma(q)}\int_a^t \mid (t-s)^{q-1} \mid d(f(s,v(s),Kv(s)),f(s,u_{n-1}(s),Ku_{n-1}(s)))ds$$
$$(3.4) \qquad \leq I_a^q L(1+K^*).d(v,u_{n-1}) \leq \lambda d(v,u_{n-1})$$

Hence we have

$$d(v(t),u_n(t)) < \lambda d(v,u_{n-1})$$

By setting $\delta_n = d(v(t),u_n(t))$, the Eq.(3.4) leads to

$$\delta_n \leq \lambda\delta_{n-1} \qquad n \geq 1$$

and finally

$$\delta_n \leq \lambda^n\delta_0 \qquad n \geq 1$$

condition (A_3) guarantees that $\delta_n \to 0$ as $n \to \infty$, that is

$$v(t) = \lim_{n\to\infty} u_n(t) = u(t), \qquad t \in J$$

4. Examples

In this section, two examples are given to illustrate the efficiency of our main results.

Example 4.1. Consider the fractional integro-differential equation with fuzzy initial value

$$(4.1) \qquad \begin{cases} ({}_{gH}D_*^q u)(t) = f(t) + \frac{1}{2}u(t) + \frac{1}{3}\int\limits_0^1 stu(s)ds \quad , \quad t \in [0,1] \\ u(0) = (0.52, 1, 1.23) \in \mathbb{R}_{\mathcal{F}} \end{cases}$$

with the exact solution $u^* : [0,1] \to \mathbb{R}_{\mathcal{F}}$ that given by

$$u_\alpha^*(t) = [u_\alpha^{*-}(t),\ u_\alpha^{*+}(t)] = [(0.52+0.48\alpha)e^t, (1.23-0.23\alpha))e^t], \qquad \alpha \in [0,1]$$

where $f(t)$ is chosen accordingly. Since the solution of problem (4.1) is ${}^{cf}[(i)-gH]$-differentiable on $[0,1]$, by Theorem 3.4 we have

$$d(Fu(t),Fv(t)) \leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} d(\frac{1}{2}u(t) + \frac{1}{3}\int\limits_0^1 stu(s)ds, \frac{1}{2}v(t) + \frac{1}{3}\int\limits_0^1 stv(s)ds)ds$$
$$\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[\frac{1}{2}d(u,v) + \frac{1}{3}d(\int\limits_0^1 stu(s)ds, \int\limits_0^1 stv(s)ds)]ds$$
$$\leq \frac{1}{\Gamma(q)}\int_0^t \frac{1}{2}(t-s)^{q-1}.d(u,v)ds + \frac{1}{\Gamma(q)}\int_a^t \frac{1}{3}(t-s)^{q-1}.d(\int\limits_0^1 stu(s)ds, \int\limits_0^1 stv(s)ds)ds$$
$$\leq I_0^q(\frac{1}{2}).H(u,v) + I_0^q(\frac{1}{3}).H(u,v) \leq I_0^q(\frac{5}{6}).H(u,v)$$

So, Theorem 3.4 implies that $\Gamma(q+1) > 0.833$, hence for all $0 < q < 1$, problem (4.1) has unique solution on $[0,1]$.

Example 4.2. Consider the following fractional integro-differential equation

$$(4.2)\quad \begin{cases} ({}_{gH}D_*^q u)(t) = f(t) + (0.1t)u(t) + 0.01 \int\limits_1^2 e^{st}u(s)ds \quad , \quad t \in [1,2] \\ u(0) = (0.23, 1, 1.84) \in \mathbb{R}_{\mathcal{F}} \end{cases}$$

The exact solution is

$$u^*(t) = -(0.23, 1, 1.84)t^2$$

that can be expressed by

$$u^*_\alpha(t) = [(-1.84 + 0.84\alpha))t^2, (-0.23 - 0.77\alpha)t^2], \quad \alpha \in [0,1] \quad t \in [1,2]$$

where $f(t)$ is chosen accordingly. As regards the solution of problem (4.2) is ${}^{cf}[(ii)-gH]$-differentiable on$[1,2]$, by Theorem 3.4 we must be have $2^{-q}\Gamma(q+1) > 0.74$. Therefore problem (4.2) for fractional derivative of order $0 < q < 0.2735$, has a unique solution on the interval $[1,2]$.

5. Conclusions

Following the ideas recently developed in [3],in this paper, we presented the existence and uniqueness theorems for fractional Fredholm integro-differential equations under generalized fuzzy Caputo Hukuhara differentiability with fuzzy initial conditions.

For future works, we will study the numerical methods for solving nonlinear fuzzy fractional integro-differential equations.

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