Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 10, No. 5, (November 2015), pp. 775–788
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI

On the sequence spaces of interval valued fuzzy numbers

Mehmet Şengönül

Received 20 March 2015; Revised 6 May 2015; Accepted 18 May 2015

Abstract. The purpose of this paper is to introduce the null, convergent, p-absolute convergent series and bounded sequence spaces of interval valued fuzzy numbers $c_0(E^2)$, $c(E^2)$, $\ell_p(E^2)$ and $\ell_\infty(E^2)$, respectively, consisting of all sequences $u = (u_k)$ such that (u_k) is a sequence of interval valued fuzzy numbers. Also, we have shown that these spaces are complete module spaces. The $\alpha-$, $\beta-$ and $\gamma-$duals of the module spaces of IVFNs have been computed. In the final section, we have defined the interval valued fuzzy matrix transformation as the way of traditional matrix transformations.

2010 AMS Classification: 03E72, 46A45, 40C05

Keywords: Fuzzy number, Interval valued fuzzy number, Cofinal, Dual, Matrix transformations.

Corresponding Author: Mehmet Şengönül (msengonul@yahoo.com)

1. Introduction

As it is well known, the idea of fuzzy sets and fuzzy operations was first introduced by A. L. Zadeh [19]. Since then many authors have discussed various aspects of the theory and its applications such as fuzzy sequence spaces of fuzzy numbers. But also, Nanda [14], Talo and Başar [15], Altınok et al. [1, 2, 3] and Hong et al.[8] studied sequences space of fuzzy numbers and some properties of the sequence of fuzzy numbers.

Well-known generalization of an ordinary fuzzy set is interval-valued fuzzy set which is attributed to Gorzalczany [7] and Turksen [16]. They applied to the fields of approximate inference, signal transmission and control, etc. Yao and Lin [18] studied fuzzy shortest-path network problems, S. Jay Chen, S. Ming Chen [4, 5] studied fuzzy risk analysis based on measures of similarity between interval-valued fuzzy numbers and handling information filtering problems based on interval-valued fuzzy numbers.

In literature, one can reach many documents about interval valued fuzzy set theory and its applications.

Recently, Chen [9] has introduced the distance between interval valued fuzzy sets and defined interval valued fuzzy numbers. Also, Hong and Lee [8], Meenakshi and Kaliraja [12] and Li [9] have studied different properties of interval valued fuzzy numbers. In this article, the sequence spaces of the interval valued fuzzy numbers were defined and some properties were researched.

2. Preliminaries

Throughout the paper, we denote the set of real numbers by $\mathbb{R}$, unit closed and bounded interval by I, that is, $I = [0, 1]$ and $[I] = \{x = [x_\ell, x_r] : 0 \leq x_\ell \leq x_r \leq 1\}$. An interval number is a closed subset of real numbers [6]. Let's denote the set of all real valued interval numbers by E_i. Any element of E_i is denoted by $\overline{x}$. That is $\overline{x} = \{x \in \mathbb{R} : a \leq x \leq b\}$.

The set of all interval numbers E_i is a metric space with d [13] defined by

$$d(\overline{x}, \overline{y}) = \max\{|x_\ell - y_\ell|, |x_r - y_r|\}. \tag{2.1}$$

Moreover, it is known that E_i is a complete metric space. In the special case $\overline{x} = [a, a]$ and $\overline{y} = [b, b]$, we obtain usual metric of the $\mathbb{R}$ with $d(\overline{x}, \overline{y}) = |a - b|$.

Let A be a fuzzy set in $\mathbb{R}$ which is characterized by a membership function $u_A : \mathbb{R} \to [0, 1]$. A fuzzy number (FN) is a function u from $\mathbb{R}$ to $[0, 1]$, which satisfies the following properties:

FN1. u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,

FN2. u is fuzzy convex, i.e., for any $x, y \in \mathbb{R}$ and $\mu \in [0, 1]$, $u\,[\mu x + (1 - \mu)y] \geq \min\{u(x), u(y)\}$,

FN3. u is upper semi-continuous,

FN4. The closure of $\{x \in \mathbb{R} : u(x) \geq 0\}$, denoted by u^0, is compact.

FN1, FN2, FN3 and FN4 imply that for each $\alpha \in [0, 1]$, the α-level set defined by $[u]^\alpha = \{x \in \mathbb{R} : u(x) > \alpha\}$ is in E_i, as well as the support u^0, i.e., $[u]^\alpha = [u_\ell(\alpha), u_r(\alpha)]$ for each $\alpha \in [0, 1]$. We denote the set of all fuzzy numbers by E^1.

Define a map $\overline{d} : E^1 \times E^1 \longrightarrow \mathbb{R}$ by $\overline{d}(u, v) = \sup\limits_{0 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha)$. It is known that E^1 is a complete metric space with the metric $\overline{d}$, [11].

Let $u, v \in E^1$ and $\lambda \in \mathbb{R}$. Then the operations addition and scalar multiplication are defined on E^1 in terms of $\alpha-$level sets by

$$u + v = w \Leftrightarrow [w]^\alpha = [u]^\alpha + [v]^\alpha \ \text{ and } \ [\lambda u]^\alpha = \lambda [u]^\alpha \quad \text{for all} \ \ \alpha \in [0, 1].$$

A sequence $u = (u_k)$ of fuzzy numbers is a function u from the set $\mathbb{N}$, the set of all positive integers, into E^1, and fuzzy number u_k denotes the value of the function at k and is called the k^{th} term of the sequence. Let $c_0(E^1)$, $c(E^1)$ and $\ell_\infty(E^1)$ denote of all null, convergent and bounded sequences of fuzzy numbers, respectively.

In [14], it is shown that $c_0(E^1)$, $c(E^1)$ and $\ell_\infty(E^1)$ are complete metric spaces with the metric $D_\infty(u, v) = \sup_{k \in \mathbb{N}} \overline{d}(u_k, v_k)$, where $u = (u_k), v = (v_k) \in \ell_\infty(E^1)$ $(or \ \ c_0(E^1), c(E^1))$.

Let X be an ordinary set and $[I] = \{[\alpha_1, \alpha_2] : \alpha_1, \alpha_2 \in [0, 1] \text{ and } \alpha_1 \leq \alpha_2\}$. The the mapping $u : X \to [I], \ \ x \to u(x)$ is called an interval valued fuzzy set on X, [17].

The membership function of u is $u(x) = [u^-(x), u^+(x)]$ for all $x \in X$. The functions $u^- : X \to [0,1]$ and $u^+ : X \to [0,1]$ are called as upper fuzzy set and lower fuzzy set, the functions $u^-(x)$ and $u^+(x)$ are the membership functions of the fuzzy sets u^- and u^+, respectively.

Suppose that $u(x) = [u^-(x), u^+(x)]$ is an interval valued fuzzy set on the set $\mathbb{R}$. If u^- and u^+ are fuzzy numbers on $\mathbb{R}$ and the inequality $u^-(x) \leq u^+(x)$ holds for all $x \in \mathbb{R}$ then u is called as interval valued fuzzy number and we may write it for brief (`IVFN`).

Let E^2 denote all interval valued fuzzy numbers on $\mathbb{R}$. For brief, here after, we shall write $u = [u^-, u^+]$ instead of $u(x) = [u^-(x), u^+(x)]$.

Lemma 2.1 ([17]). *$u \in E^2$ if and only if u^- and u^+ are ordinary fuzzy numbers with $u^- \leq u^+$ for all $x \in \mathbb{R}$.*

Let's suppose that $u_1, u_2 \in E^2$ and $\lambda \in \mathbb{R}$. Then the partial ordering relation and some algebraic operations on E^2 are defined as follows:

Ordering: $u_1 \leq u_2 \Leftrightarrow [u_1^-, u_1^+] \leq [u_2^-, u_2^+] \Leftrightarrow u_1^- \leq u_2^-$ and $u_1^+ \leq u_2^+$, Addition: $u_1 + u_2 = \{v \in E^1 : u_1^- + u_2^- \leq v \leq u_1^+ + u_2^+\}$, Scalar multiplication: if $\lambda \geq 0$ then $\lambda u = \{v \in E^1 : \lambda u^- \leq v \leq \lambda u^+\}$ and if $\lambda < 0$ then $\lambda u = \{v \in E^1 : \lambda u^+ \leq v \leq \lambda u^-\}$, Multiplication : $u_1 u_2 = \{v \in E^1 : \min\{u_1^- u_2^-, u_1^- u_2^+, u_1^+ u_2^-, u_1^+ u_2^+\} \leq v \leq \max\{u_1^- u_2^-, u_1^- u_2^+, u_1^+ u_2^-, u_1^+ u_2^+\}\}$.

Definition 2.2. Suppose that $u = [u^-, u^+] \in E^2$. If $u^- = u^+$, then u is called degenerate interval valued fuzzy number.

It can be easily seen that a degenerate interval valued fuzzy number is an ordinary fuzzy number. Thus, we have the following proposition:

Proposition 2.3. *The set of all fuzzy numbers E^1 can be embedded in E^2.*

Definition 2.4. Let $\tau^2 \subset E^2$ and let us consider function $||.|| : \tau^2 \to \mathbb{R}$. The function $||.||$ is called module on the set τ^2 if it has the following properties:

`M1`. $||u|| = \theta \Leftrightarrow u = \theta$, where θ is zero element of the set E^2,

`M2`. $||\lambda u|| = |\lambda| ||u||$,

`M3`. $||u + v|| \leq ||u|| + ||v||$.

If the function $||.|| : \tau^2 \to \mathbb{R}$ satisfy `M1`, `M2` and `M3` then τ^2 is called module sequence space of the `IVFN`s. And if τ^2 is complete with respect to the module $||.||$ then τ^2 is called complete module sequence space of the `IVFN`s.

Let $u, v \in E^2$ and we define

$$D(u, v) = \max\left\{\overline{d}(u^-, v^-), \overline{d}(u^+, v^+)\right\} \tag{2.2}$$

The module of the `IVFN` u is defined as the non negative real number $D(u, \theta)$ which corresponds to the distance from u to θ.

In [9], it is shown that E^2 is metric space metric spaces with the metric defined by (2.2).

Lemma 2.5. *Define the module*

$$D(u, \theta) = ||u||_{E^2} = \max\left\{\overline{d}(u^-, \overline{0}), \overline{d}(u^+, \overline{0})\right\}, \tag{2.3}$$

Then E^2 is complete module space of the `IVFN`s with the module defined by (2.3).

Proof. The conditions M1, M2, and M3 are clearly satisfied. To show that E^2 is complete in this module, suppose that

$(u_n) = (u_0, u_1, u_2, ..., u_n, ...) = ([u_0^-, u_0^+], [u_1^-, u_1^+], ..., [u_n^-, u_n^+], ...)$ be a fundamental(Cauchy) sequence in E^2(see, [9]) for each n. Then we have

$$||u_n - u_m||_{E^2} = \max\left\{\overline{d}(u_n^-, u_m^-), \overline{d}(u_n^+, u_m^+)\right\} < \epsilon.$$

Hence we obtain $\overline{d}(u_n^-, u_m^-) < \epsilon$ and $\overline{d}(u_n^+, u_m^+) < \epsilon$. This shows that (u_n^-) and (u_n^+) are Cauchy sequence of fuzzy numbers in E^1. However, E^1 is complete and so (u_n^-) and (u_n^+) are convergent in E^1 for all $n \in \mathbb{N}$. Let us suppose that $\lim_n u_n^- = u_0^-$ and $\lim_n u_n^+ = u_0^+$ for each $k \in \mathbb{N}$. Since $\overline{d}(u_n^-, u_m^-) < \epsilon$ and $\overline{d}(u_n^+, u_m^+) < \epsilon$ for all $n, m \geq k$, $\lim_{m\to\infty} \overline{d}(u_n^-, u_m^-) = \overline{d}(u_n^-, \lim_m u_m^-) = \overline{d}(u_n^-, u_0^-) < \epsilon$ and $\lim_{m\to\infty} \overline{d}(u_n^+, u_m^+) = \overline{d}(u_n^+, \lim_m u_m^+) = \overline{d}(u_n^+, u_0^+) < \epsilon$. This means that $u_n^- \to u_0^-$ as $n \to \infty$ and $u_n^+ \to u_0^+$ as $n \to \infty$ in E^2, i.e., $(u_n) \to u_0$ as $n \to \infty$. On the other hand, since

$$\begin{aligned} ||u_0||_{E^2} &= \max\left\{\overline{d}(u_0^-, u_n^- - u_n^-), \overline{d}(u_0^+, u_n^+ - u_n^+)\right\} \\ &= \max\left\{\overline{d}(u_0^- - \overline{0}, u_n^- - u_n^-), \overline{d}(u_0^+ - \overline{0}, u_n^+ - u_n^+)\right\} \\ &\leq \max\left\{\overline{d}(u_0^-, u_n^-), \overline{d}(u_0^+, u_n^+)\right\} + \max\left\{\overline{d}(\overline{0}, u_n^-), \overline{d}(\overline{0}, u_n^+)\right\} < \infty \end{aligned}$$

this shows that $u_0 \in E^2$. □

Now, let us give the following new definitions:

Definition 2.6. A sequence space of IVFNs is subspace of $w(E^2)$, where $w(E^2) = \{(u_k) = ([u_k^-, u_k^+])_{k\in\mathbb{N}} : u : \mathbb{N} \to E^2, k \to u(k) = [u_k^-, u_k^+] \text{ and } u_k^-, u_k^+ \in E^1\}$. If $(u_k) \in w(E^2)$ then (u_k) is called a sequence of IVFNs.

Definition 2.7. A sequence $(u_k) \in w(E^2)$ is said to be bounded if and only if there exists two IVFNs m and M such that $m \leq u_k \leq M$ for all $k \in \mathbb{N}$.

Definition 2.8. A sequence $u = (u_k)$ of IVFNs is said to be convergent to the IVFN u_0, written as $\lim_k u_k = u_0$, if for every $\epsilon > 0$ there exists a positive integer m such that $D(u_k, u_0) < \epsilon$ for $k \geq m$ that is a sequence $u = (u_k)$ of IVFNs is said to be convergent to the IVFN u_0 if for each $\epsilon > 0$ there exists a positive integer m such that $D(u_k, u_0) = \sup_k \max\left\{\overline{d}(u_k^-, u_0^-), \overline{d}(u_k^+, u_0^+)\right\} < \epsilon$ for all $k \geq m$.

Definition 2.9. A sequence $u = (u_k)$ of IVFNs is said to be fundamental sequence if for every $\epsilon > 0$ there exists a positive integers n, m such that $D(u_n, u_m) < \epsilon$ for $n, m \geq k$.

Definition 2.10. Let $\lambda(E^2)$ be a sequence space of the IVFNs. If the function $||.|| : \lambda(E^2) \to \mathbb{R}$ is satisfies M1, M2 and M3 then $\lambda(E^2)$ is called module sequence space of the IVFNs. And if $\lambda(E^2)$ is complete with respect to a module then $\lambda(E^2)$ is called complete module sequence space of the IVFNs.

Since the set of all close intervals on $\mathbb{R}$ is a quasivector space [10], the set $w(E^2)$ can be regarded as a quasivector space. For $u = (u_k), v = (v_k)$ and $\alpha, \beta \in \mathbb{R}$, the following rules are clearly satisfied: $(u_k) + (v_k) = (v_k) + (u_k)$; $(u_k) + [(v_k) + (z_k)] = [(u_k) + (v_k)] + (z_k)$; $(u_k) + (v_k) = (u_k) + (z_k)$ implies $(v_k) = (z_k)$; $\alpha[(u_k) + (v_k)] = \alpha(u_k) + \alpha(v_k)$; $(\alpha + \beta)(u_k) = \alpha(u_k) + \beta(u_k)$, (where $\alpha\beta \geq 0$); $\alpha(\beta(u_k)) =$

$(\alpha\beta)(u_k)$; $(u_k) = [\overline{1}, \overline{1}](u_k)$. The zero element of $w(E^2)$ is the sequence $\theta = (\theta_k) = ([\theta_k^-, \theta_k^+])$ all terms of which are zero interval valued fuzzy number, where $\theta_k^- = \overline{0}$ and $\theta_k^+ = \overline{0}$.

The rest of this paper proceeds as follows:

In section 3, we have introduced the sequence spaces of IVFNs. In section 4, we have stated and proved the theorems determining the $\alpha-$, $\beta-$ and $\gamma-$duals of the sequence space IVFNs. Finally, in section 5, the classes $(\ell_\infty(E^2) : \ell_\infty(E^2))$ and $(c_0(E^2) : c_0(E^2))$ of infinite matrix of IVFNs are characterized. Now, we can introduce sequence spaces of IVFNs.

3. The sequence spaces of Ivfns

Now, let us define the sequence spaces $c_0(E^2)$, $c(E^2)$, $cs(E^2)$, $bs(E^2)$, $\ell_p(E^2)$ and $\ell_\infty(E^2)$ as the set of all null sequences of IVFNs, the set of all convergent sequences of IVFNs, the sets of all convergent series of IVFNs, the sets of all bounded series of IVFNs, p-absolutely convergent series of the IVFNs and the set of all bounded sequences of IVFNs, respectively, that is

$$c_0(E^2) = \{u = ([u_k^-, u_k^+]) \in w(E^2) : \lim_k \max\left\{\overline{d}(u_k^-, \overline{0}), \overline{d}(u_k^+, \overline{0})\right\} = \theta\},$$

$$c(E^2) = \{u = ([u_k^-, u_k^+]) \in w(E^2) : \lim_k \max\left\{\overline{d}(u_k^-, u_0^-), \overline{d}(u_k^+, u_0^+)\right\} = \theta\},$$

$$cs(E^2) = \{u = (u_k) \in w(E^2) : \lim_n \max\{\overline{d}(\sum_{k=0}^{n} u_k^-, u^-), \overline{d}(\sum_{k=0}^{n} u^+, u^+)\} = \theta\},$$

$$bs(E^2) = \{u = (u_k) \in w(E^1) : \sup_n \max\{\overline{d}(\sum_{k=0}^{n} u_k^-, \overline{0}), \overline{d}(\sum_{k=0}^{n} u^+, \overline{0})\} < \infty\},$$

$$\ell_p(E^2) = \{u = (u_k) \in w(E^2) : \left(\sum_k (\max\{\overline{d}(u_k^-, \overline{0}), \overline{d}(u_k^+, \overline{0})\})^p\right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty\},$$

$$\ell_\infty(E^2) = \{u = ([u_k^-, u_k^+]) \in w(E^2) : \sup_k \max\left\{\overline{d}(u_k^-, \overline{0}), \overline{d}(u_k^+, \overline{0})\right\} < \infty\}.$$

We may begin with the following results which are essential in the text.

Theorem 3.1. *The spaces $c_0(E^2)$, $c(E^2)$ and $\ell_\infty(E^2)$ are complete module sequence space of the* IVFN*s with the module defined by*

$$||u||_{\ell_\infty(E^2)} = \sup_k \max\left\{\overline{d}(u_k^-, \overline{0}), \overline{d}(u_k^+, \overline{0})\right\} \tag{3.1}$$

Proof. We shall only consider $c(E^2)$. It is very easy to see that, on $c(E^2)$, $||.||$ is a module defined by (3.1). To show that $c(E^2)$ is complete in this module, suppose that $(u_k^n) = (u_0^n, u_1^n, u_2^n, ...)$ be a fundamental sequence in $c(E^2)$ for each n. Then, we have

$$||u_k^n - u_k^m||_{\ell_\infty(E^2)} = \sup_k \max\left\{\overline{d}(u_k^{n-}, u_k^{m-}), \overline{d}(u_k^{n+}, u_k^{m+})\right\} < \epsilon.$$

Hence we obtain $\overline{d}(u_k^{n-}, u_k^{m-}) < \epsilon$ and $\overline{d}(u_k^{n+}, u_k^{m+}) < \epsilon$. This shows that (u_k^{n-}) and (u_k^{n+}) are fundamental sequence of fuzzy numbers in E^1. However, E^1 is complete and so (u_k^{n-}) and (u_k^{n+}) are convergent in E^1 for all $n \in \mathbb{N}$. Let us suppose that

$\lim_n x_k^{-n} = u_k^-$ and $\lim_n x_k^{+n} = u_k^+$ for each $k \in \mathbb{N}$. Since $D(u_k^n, u_k^m) < \epsilon$ for all $n, m \geq k$,

$$\lim_{m\to\infty} \overline{d}(u_k^{n-}, u_k^{m-}) = \overline{d}(u_k^{n-}, \lim_m u_k^{m-}) = \overline{d}(u_k^{n-}, u_k^-) < \epsilon \text{ and}$$
$$\lim_{m\to\infty} \overline{d}(u_k^{n+}, u_k^{m+}) = \overline{d}(u_k^{n+}, \lim_m u_k^{m+}) = \overline{d}(u_k^{n+}, u_k^+) < \epsilon.$$

This means that $u_k^{n-} \to u_k^-$ and $u_k^{n+} \to u_k^+$ as $(n \to \infty)$ that is $(u_k^n) \to u_k$, $(n \to \infty)$ in $\ell_\infty(E^2)$. On the other hand, since

$$\begin{aligned} ||u_k - (u_k^n - u_k^n)||_{\ell_\infty(E^2)} &= \sup_k \max\left\{\overline{d}(u_k^-, u_k^{n-} - u_k^{n-}), \overline{d}(u_k^+, u_k^{n+} - u_k^{n+})\right\} \\ &\leq \sup_k \max\left\{\overline{d}(u_k^-, u_k^{n-}), \overline{d}(u_k^+, u_k^{n+})\right\} \\ &+ \sup_k \max\left\{\overline{d}(\overline{0}, u_k^{n-}), \overline{d}(\overline{0}, u_k^{n+})\right\} \\ &\leq ||u - u_k||_{\ell_\infty(E^2)} + ||u_k||_{\ell_\infty(E^2)} < \infty \end{aligned}$$

this shows that $(u_k) \in \ell_\infty(E^2)$. □

Proposition 3.2. *The space $\ell_p(E^2)$ is complete module sequence spaces of the* `IVFN`*s with respect to module*

$$||u||_{\ell_p(E^2)} = \left(\sum_k (\max\{\overline{d}(u_k^-, \overline{0}), \overline{d}(u_k^+, \overline{0})\})^p\right)^{\frac{1}{p}} < \infty, \quad \textit{where} \quad p \geq 1.$$

Proof. Clearly the space $\ell_p(E^2)$ is a module sequence space of the `IVFN`s with the function $||u||_{\ell_p(E^2)}$. Let us suppose that $u = (u_i)$ be a fundamental sequence in $\ell_p(E^2)$, where $(u_k) = (u_0^{(i)}, u_1^{(i)}, u_2^{(i)}, ...)$. Then, for any $\epsilon > 0$ there exists an integer n_0 such that

$$(3.2) \qquad ||u_k^i - u_k^j||_{\ell_p(E^2)} = \left(\sum_k (\max\{\overline{d}(u_k^{i-}, u_k^{j-}), \overline{d}(u_k^{i+}, u_k^{j+})\})^p\right)^{\frac{1}{p}} \leq \epsilon$$

for $i, j \geq n_0$. It follows that for every $k = 0, 1, 2, ...$ have $\overline{d}(u_k^{i-}, u_k^{j-}) < \epsilon$ and $\overline{d}(u_k^{i+}, u_k^{j+}) < \epsilon$ This shows that the sequences $(u_k^{i-}) = (u_k^{(0-)}, u_k^{(1-)}, u_k^{(2-)}, ...)$, $(u_k^{i+}) = (u_k^{(0+)}, u_k^{(1+)}, u_k^{(2+)}, ...)$ are fundamental sequence in $\ell_p(E^1)$. It converges since the space $\ell_p(E^1)$ is complete [14]. Let us suppose that $u_k^{(i-)} \to u_k^-$ and $u_k^{(i+)} \to u_k^+$ for $i \to \infty$. Now using these limit points, define the sequences $z^- = (u_0^-, u_1^-, u_2^-, ...)$, $z^+ = (u_0^+, u_1^+, u_2^+, ...)$ and we show that $z^-, z^+ \in \ell_p(E^1)$ and $z_k^- \to z^-$, $z_k^+ \to z^+$. If we consider (3.2) then we have $\left(\sum_{k=0}^n (\max\{\overline{d}(u_k^{i-}, u_k^{j-}), \overline{d}(u_k^{i+}, u_k^{j+})\})^p\right)^{\frac{1}{p}} \leq \epsilon$. From here, for $i \to \infty$ and $i \geq n_0$, we have $\left(\sum_{k=0}^n (\max\{\overline{d}(u_k^-, u_k^{j-}), \overline{d}(u_k^+, u_k^{j+})\})^p\right)^{\frac{1}{p}} \leq \epsilon$. If $n \to \infty$ then we obtain

$$\left(\sum_k(\max\{\overline{d}(u_k^-,u_k^{j-},),\overline{d}(u_k^+,u_k^{j+})\})^p\right)^{\frac{1}{p}}\le\epsilon.$$ This step shows that $(u_i^- - u^-)\in\ell_p(E^1)$ and $(u_i^+ - u^+)\in\ell_p(E^1)$. Furthermore, for $(u_k^{(i)})\in\ell_p(E^2)$,

$$\begin{aligned}||u-(u_k^{(i)}-u_k^{(i)})||_{\ell_p(E^2)} &= \left(\sum_k(\max\{\overline{d}(u_k^-,u_k^{j-}-u_k^{j-}),\overline{d}(u_k^+,u_k^{j+}-u_k^{j+})\})^p\right)^{\frac{1}{p}}\\ &\le\left(\sum_k(\max\{\overline{d}(u_k^-,u_k^{j-}),\overline{d}(u_k^+,u_k^{j+})\})^p\right)^{\frac{1}{p}}\\ &+\left(\sum_k(\max\{\overline{d}(\theta_k^-,u_k^{j-}),\overline{d}(\theta_k^+,u_k^{j+})\})^p\right)^{\frac{1}{p}}\\ &\le||u-u_k^{(i)}||_{\ell_p(E^2)}+||u_k^{(i)}||_{\ell_p(E^2)}<\infty.\end{aligned}$$

This step completes the proof. □

Theorem 3.3. *The inclusions* $c_0(E^2)\subset c(E^2)\subset\ell_\infty(E^2)$ *strictly hold.*

Proof. The inclusion $c_0(E^2)\subset c(E^2)$ is clear. To show the validity of the inclusion relation $c(E^2)\subset\ell_\infty(E^2)$, let $u\in c(E^2)$ then;

$$\begin{aligned}\lim_k D([u_k^-,u_k^+],[u^-,u^+])=\theta &\Rightarrow\lim_k\left(\max\{\overline{d}(u_k^-,u^-),\overline{d}(u_k^+,u^+)\}\right)=\theta\\ &\Rightarrow\overline{d}(u_k^-,u^-)<\epsilon,\overline{d}(u_k^+,u^+)<\epsilon\\ &\Rightarrow(u_k^-)\in c(E^1),\quad(u_k^+)\in c(E^1).\end{aligned}$$

Since $c(E^1)\subset\ell_\infty(E^1)$, we see that $(u_k^-)\in\ell_\infty(E^1)$ and $(u_k^+)\in\ell_\infty(E^1)$, that is $u=([u_k^-,u_k^+])\in\ell_\infty(E^2)$. Hence inclusion $c(E^2)\subset\ell_\infty(E^2)$ holds.

Consider the sequence (u_k) of IVNFs defined by

$$u_k=\left[\begin{cases}x+3, & x\in[-3,-2]\\ -x-1, & x\in[-2,-1]\\ 0, & otherwise\end{cases},\begin{cases}x+2, & x\in[-2,-1]\\ -x, & x\in[-1,0]\\ 0, & otherwise\end{cases}\right],\quad if\quad k=2n,\quad n\in\mathbb{N},$$

and

$$u_k=\left[\begin{cases}x, & x\in[-3,-2]\\ -x+1, & x\in[-2,-1]\\ 0, & otherwise\end{cases},\begin{cases}x-1, & x\in[-2,-1]\\ -x+2, & x\in[-1,0]\\ 0, & otherwise\end{cases}\right],\quad if\quad k=2n+1,\quad n\in\mathbb{N}.$$

Since

$$\lim_k u_k=\begin{cases}\left[\begin{cases}x+3, & x\in[-3,-2]\\ -x-1, & x\in[-2,-1]\\ 0, & otherwise\end{cases},\begin{cases}x+2, & x\in[-2,-1]\\ -x, & x\in[-1,0]\\ 0, & otherwise\end{cases}\right], & if\quad k=2n,\quad n\in\mathbb{N},\\ \left[\begin{cases}x, & x\in[-3,-2]\\ -x+1, & x\in[-2,-1]\\ 0, & otherwise\end{cases},\begin{cases}x-1, & x\in[-2,-1]\\ -x+2, & x\in[-1,0]\\ 0, & otherwise\end{cases}\right], & if\quad k=2n+1,\quad n\in\mathbb{N}\end{cases}$$

the sequence (u_k) is not convergent sequence of IVFNs, i.e., $(u_k)\notin c(E^2)$ but $(u_k)\in\ell_\infty(E^2)$. That is to say that the inclusion $c(E^2)\subset\ell_\infty(E^2)$ strictly holds. □

Theorem 3.4. *The spaces $c_0(E^1)$, $c(E^1)$ and $\ell_\infty(E^1)$ consisting of the null, convergent and bounded sequences of fuzzy numbers are subsets of the spaces $c_0(E^2)$, $c(E^2)$ and $\ell_\infty(E^2)$, respectively.*

Proof. Since every element of $c_0(E^1)$(*or* $c(E^1)$, $\ell_\infty(E^1)$) is a degenerate sequences `IVFNs` by Definition 2.2 and Proposition 2.3, the proof is clear. □

Theorem 3.5. *For any $(u_k),(v_k) \in w(E^2)$, if $(u_k) \to u_0$ and $(v_k) \to u_0$ then we have*

(1) $u_k + v_k \to u_0 + v_0$, *as* $k \to \infty$.
(2) $u_k - v_k \to u_0 - v_0$, *as* $k \to \infty$.
(3) $u_k v_k \to u_0 v_0$, *as* $k \to \infty$.

Proof. Since the proof can also be obtained in a similar way for (2) and (3) we will only deal with (1). Let us suppose that $\lim_k u_k = u$ and $\lim_k v_k = v$. From definition (2.8) we have $D(u_k,u) = \sup_k \max\left\{\overline{d}(u_k^-,u^-),\overline{d}(u_k^+,u^+)\right\} < \frac{\varepsilon}{2}$ and $D(v_k,v) = \sup_k \max\left\{\overline{d}(v_k^-,v^-),\overline{d}(v_k^+,v^+)\right\} < \frac{\varepsilon}{2}$ for all $k \geq m$.

$$\begin{aligned} D(u_k+v_k,u+v) &= \sup_k \max\left\{\overline{d}(u_k^- + v_k^-, u^- + v^-), \overline{d}(u_k^+ + v_k^+, u^+ + v^+)\right\} \\ &\leq \sup_k \max\left\{\overline{d}(u_k^-,u^-) + \overline{d}(v_k^-,v^-), \overline{d}(u_k^+,u^+) + \overline{d}(v_k^+,v^+)\right\} \\ &\leq \sup_k \max\left\{\overline{d}(u_k^-,u^-), \overline{d}(u_k^+,u^+)\right\} + \sup_k \max\left\{\overline{d}(v_k^-,v^-), \overline{d}(v_k^+,v^+)\right\} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This step completes proof. □

Definition 3.6. A sequence space $\lambda(E^2)$ is said to be symmetric if, when u in $\lambda(E^2)$, then v is in $\lambda(E^2)$ when the coordinates of v are those of u, but in a different order.

Theorem 3.7. *The spaces $c_0(E^2)$ and $c(E^2)$ are symmetric spaces.*

Proof. Since the proof can also be obtained in a similar way for $c_0(E^2)$ we consider only $c(E^2)$. Let us consider the sequence (u_k) in $c(E^2)$ defined by

$$(u_k) = \left(\left[\left\{\begin{array}{cc} \frac{k}{2k-1}x, & x \in [0, \frac{2k-1}{k}] \\ 1, & x \in [\frac{2k-1}{k}, \frac{2k+1}{k}] \\ -\frac{k}{2k+1}(x-4), & x \in [\frac{2k+1}{k}, 4] \\ 0, & \textit{otherwise} \end{array}\right., \left\{\begin{array}{cc} \frac{k}{2k-1}x - 1, & x \in [1, \frac{3k-1}{k}] \\ 1, & x \in [\frac{3k-1}{k}, \frac{3k+1}{k}] \\ -\frac{k}{2k+1}(x-5), & x \in [\frac{3k+1}{k}, 5] \\ 0, & \textit{otherwise} \end{array}\right.\right]\right),$$

Clearly we see that

$$\begin{aligned} \lim_k u_k &= \lim_k \left[\left\{\begin{array}{cc} \frac{k}{2k-1}x, & x \in [0, \frac{2k-1}{k}] \\ 1, & x \in [\frac{2k-1}{k}, \frac{2k+1}{k}] \\ -\frac{k}{2k+1}(x-4), & x \in [\frac{2k+1}{k}, 4] \\ 0, & \textit{otherwise} \end{array}\right., \left\{\begin{array}{cc} \frac{k}{2k-1}x - 1, & x \in [1, \frac{3k-1}{k}] \\ 1, & x \in [\frac{3k-1}{k}, \frac{3k+1}{k}] \\ -\frac{k}{2k+1}(x-5), & x \in [\frac{3k+1}{k}, 5] \\ 0, & \textit{otherwise} \end{array}\right.\right] \\ &= \left[\left\{\begin{array}{cc} \frac{1}{2}x, & x \in [0,2] \\ -\frac{1}{2}(x-4), & x \in [2,4] \\ 0, & \textit{otherwise} \end{array}\right., \left\{\begin{array}{cc} \frac{1}{2}x - 1, & x \in [1,3] \\ -\frac{1}{2}(x-5), & x \in [3,5] \\ 0, & \textit{otherwise} \end{array}\right.\right]. \end{aligned}$$

Let (v_k) be a rearrangement of (u_k) which is defined by $(v_k) = (u_1, u_3, u_2, u_4, u_5, u_7, u_6, u_8, ...)$. Then

$$\lim_k v_k = \left(\left[\left\{\begin{array}{cc} \frac{1}{2}x, & x \in [0,2] \\ -\frac{1}{2}(x-4), & x \in [2,4] \\ 0, & otherwise \end{array}\right. , \left\{\begin{array}{cc} \frac{1}{2}x-1, & x \in [1,3] \\ -\frac{1}{2}(x-5), & x \in [3,5] \\ 0, & otherwise \end{array}\right.\right]\right),$$

i.e the sequences (u_k) and (v_k) have the same limit points. Therefore, from definition 3.6, we see that $c(E^2)$ is symmetric space. □

Let us give a theorem about the metric defined by (3.1).

Theorem 3.8. *If $u, v, w, r \in c(E^2)$, (or $c_0(E^2), \ell_\infty(E^2)$) and $\rho \in \mathbb{R}$ then the following cases hold:*

(1) $D(u+w, v+w) \leq D(u,w) + D(w,v)$,
(2) $D(\rho u, \rho v) = |\rho| D$,
(3) $D(u+v, w+z) \leq D(u,w) + D(v,z)$ *and*
(4) $D(uv, \theta) = D(u,\theta)D(v,\theta)$.

Proof. We will consider only (3) since the proof of others are similar to this one.

$$\begin{aligned} D(u+v, w+z) &= \sup_k \max\left\{\overline{d}(u_k^- + v_k^-, w_k^- + z_k^-), \overline{d}(u_k^+ + v_k^+, w_k^+ + z_k^+)\right\} \\ &\leq \sup_k \max\left\{(\overline{d}(u_k^-, w_k^-), \overline{d}(u_k^+, w_k^+)) + (\overline{d}(v_k^-, z_k^-), \overline{d}(v_k^+, z_k^+))\right\} \\ &\leq \sup_k \max\left\{(\overline{d}(u_k^-, w_k^-), \overline{d}(u_k^+, w_k^+))\right\} + \sup_k \max\left\{(\overline{d}(v_k^-, z_k^-), \overline{d}(v_k^+, z_k^+))\right\} \\ &= D(u,w) + D(v,z). \end{aligned}$$

□

4. The duals of the sequence spaces of the IVFNs

In this section, by using techniques in [15], we have stated and proved the theorems determining the $\alpha-$, $\beta-$ and $\gamma-$duals of the spaces $\lambda(E^2)$.

For the sequence spaces $\lambda(E^2)$ and $\mu(E^2)$, define the set $S(\lambda(E^2), \mu(E^2))$ by

(4.1)
$$S(\lambda(E^2), \mu(E^2)) = \left\{z = (z_k) \in w(E^2) : (x_k z_k) \in \mu(E^2) \text{ for all } x = (x_k) \in \lambda(E^2)\right\}.$$

With the notation of (4.1), the $\alpha-$, $\beta-$ and $\gamma-$duals of a sequence space $\lambda(E^2)$, which are respectively denoted by $\lambda^\alpha(E^2)$, $\lambda^\beta(E^2)$ and $\lambda^\gamma(E^2)$ are defined by $\lambda^\alpha(E^2) = S(\lambda(E^2)\ , \ell_1(E^2))$, $\lambda^\beta(E^2) = S(\lambda(E^2), cs(E^2))$ and $\lambda^\gamma(E^2) = S(\lambda(E^2), bs(E^2))$.

Definition 4.1. Let us suppose that $\lambda(E^2)$, $\mu(E^2)$ are sets of the sequences of IVFNs and $\lambda(E^2) \subset \mu(E^2)$. Then $\lambda(E^2)$ is called cofinal in $\mu(E^2)$ if for $(u_k) \in \lambda(E^2)$ there is $(v_k) \in \mu(E^2)$ such that $D(u_k, \theta) \leq D(v_k, \theta)$ for all $k \in \mathbb{N}$.

If $\lambda(E^2)$ is cofinal in $\mu(E^2)$ then $\lambda^\alpha(E^2) = \mu^\alpha(E^2)$; the converse of this assertion is not true.

Now, we may give results concerning the α-, β- and γ-duals of the sets $c_0(E^2)$, $c(E^2)$ and $\ell_\infty(E^2)$.

Theorem 4.2. *The α-dual of the set $\ell_\infty(E^2)$ of sequence spaces* IVFNs *is the set $\ell_1(E^2)$.*

Proof. Let $(u_k) \in \ell_\infty^\alpha(E^2)$. If we consider $(v_k) = ([\overline{1}, \overline{1}]) \in \ell_\infty(E^2)$ then the series
$$\sum_k \max\{\overline{d}(u_k^- v_k^-, \overline{0}), \overline{d}(u_k^+ v_k^+, \overline{0})\} = \sum_k \max\{\overline{d}(u_k^-, \overline{0}), \overline{d}(u_k^+, \overline{0})\}$$ converges, that is to say that $(u_k) \in \ell_1(E^2)$. Therefore we have

$$\ell_\infty^\alpha(E^2) \subseteq \ell_1(E^2). \tag{4.2}$$

Conversely, let us suppose that $(u_k) \in \ell_\infty(E^2)$ and $(v_k) \in \ell_1(E^2)$. Then there exists a $K > \theta$ such that $K = \sup_k \max\left\{\overline{d}(u_k^-, \overline{0}), \overline{d}(u_k^+, \overline{0})\right\} < \infty$. From here we have

$$\begin{aligned}\sum_k \max\{\overline{d}(u_k^- v_k^-, \overline{0}), \overline{d}(u_k^+ v_k^+, \overline{0})\} &= \sum_k \max\{\overline{d}(u_k^-, \overline{0}), \overline{d}(u_k^+, \overline{0})\} \sum_k \max\{\overline{d}(v_k^-, \overline{0}), \overline{d}(v_k^+, \overline{0})\} \\ &\leq K \sum_k \max\{\overline{d}(v_k^-, \overline{0}), \overline{d}(v_k^+, \overline{0})\} = K||v||_{\ell_\infty(E^2)} < \infty\end{aligned}$$

which gives that

$$\ell_1(E^2) \subseteq \ell_\infty^\alpha(E^2). \tag{4.3}$$

From (4.2) and (4.3) we see that $\ell_1(E^2) = \ell_\infty^\alpha(E^2)$.

□

Theorem 4.3. *The interval valued fuzzy sequence spaces $c_0(E^2)$, $c(E^2)$ are cofinal in $\ell_\infty(E^2)$.*

Proof. Denote any of the spaces $c_0(E^2)$ and $c(E^2)$ by $\lambda(E^2)$, and suppose that $D(u_k, \theta) \leq D(v_k, \theta)$ holds for some $(v_k) \in \mu(E^2)$. Then we can easily see that $\sup_k D(u_k, \theta) \leq \sup_k D(v_k, \theta)$, $\lim_k D(u_k, \theta) \leq \sup_k D(v_k, \theta)$ which lead us to the desired results. □

Theorem 4.4. *The α-dual of the sets $c_0(E^2)$ and $c(E^2)$ of sequence spaces* IVFNs *are the set $\ell_1(E^2)$.*

Proof. Since the sets $c_0(E^2)$ and $c(E^2)$ are cofinal in $\ell_\infty(E^2)$ (see, Theorem 4.3), the proof is clear. □

Theorem 4.5. *The β-dual of the sets $c(E^2)$ and $\ell_\infty(E^2)$ of sequence spaces* IVFNs *are the set $\ell_1(E^2)$.*

Proof. We give the proof only for the set $\ell_\infty(E^2)$. Let $(u_k) \in \ell_1(E^2)$ and $(v_k) \in \ell_\infty(E^2)$. Then there exists a $M > \theta$ such that $M = \sup_k \max\left\{\overline{d}(v_k^-, \overline{0}), \overline{d}(v_k^+, \overline{0})\right\} < \infty$. Since $(v_k) \in \ell_\infty(E^2)$, we have following equality:

$$\begin{aligned}\sum_k \max\{\overline{d}(u_k^- v_k^-, \overline{0}), \overline{d}(u_k^+ v_k^+, \overline{0})\} &= \sum_k \max\{\overline{d}(u_k^-, \overline{0})\overline{d}(v_k^-, \overline{0}), \overline{d}(u_k^+, \overline{0})\overline{d}(v_k^+, \overline{0})\} \\ &\leq \sum_k \max\{M_1\overline{d}(u_k^-, \overline{0}), M_2\overline{d}(u_k^+, \overline{0})\} \\ &\leq M \sum_k \max\{\overline{d}(u_k^-, \overline{0}), \overline{d}(u_k^+, \overline{0})\} = ||u||_{\in \ell_1(E^2)},\end{aligned}$$

where $M = \max\{M_1, M_2\}$.

From here, we have $\ell_1(E^2) \subseteq \ell_\infty^\beta(E^2)$.

Finally, we will show that the inclusion $\ell_\infty^\beta(E^2) \subseteq \ell_1(E^2)$ holds. Let us suppose that $(v_k) \in \ell_\infty^\beta(E^2)$. Then we have $\sum_k \max\{\overline{d}(u_k^- v_k^-, \overline{0}), \overline{d}(u_k^+ v_k^+, \overline{0})\} < \infty$ for all $(u_k) \in \ell_\infty(E^2)$. This holds for the sequence $(u_k) = ([\overline{1}, \overline{1}]) \in \ell_\infty(E^2)$. Then, since $u_{\ell k}^- = u_{\ell k}^+ = -1$ and $u_{rk}^- = u_{rk}^+ = 1$, we can write

$$\begin{aligned}
&\sum_k \max\{\overline{d}(u_k^- v_k^-, \overline{0}), \overline{d}(u_k^+ v_k^+, \overline{0})\} \\
&= \sum_k \max\{\sup_{\alpha\in[0,1]} d(u_k^-(\alpha) v_k^-(\alpha), \overline{0}(\alpha)), \sup_{\alpha\in[0,1]} d(u_k^+(\alpha) v_k^+(\alpha), \overline{0}(\alpha))\} \\
&\leq \sum_k \max\{\sup_{\alpha\in[0,1]} \{\max\{|v_{\ell k}^-(\alpha)|, |v_{rk}^-(\alpha)|\}\}, \sup_{\alpha\in[0,1]} \{\max\{|v_{\ell k}^+(\alpha)|, |v_{rk}^+(\alpha)|\}\}\} \\
&= ||v||_{\ell_1(E^2)}.
\end{aligned}$$

This shows that $v \in \ell_1(E^2)$. □

Corollary 4.6. *The γ-dual of the set $\ell_\infty(E^2)$ of sequence spaces* IVFNs *is the set $\ell_1(E^2)$.*

5. Matrix transformations on sequence spaces of IVFNs

Let $\lambda(E^2)$ and $\mu(E^2)$ be two sequence spaces of IVFNs and $A = (a_{nk})$ be an infinite matrix of IVFNs a_{nk} and $u = (u_k) \in \lambda(E^2)$, where $n, k \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Then, we can say that A defines a matrix mapping from $\lambda(E^2)$ to $\mu(E^2)$, and we denote it by writing $A : \lambda(E^2) \to \mu(E^2)$, if for every sequence $u = (u_k) \in \lambda(E^2)$ the sequence $Au = \{(Au)_n\}$, the A-transform of u, is in $\mu(E^2)$, where

$$A_n(u) = \sum_k a_{nk} u_k \quad = \sum_k [a_{nk}^-, a_{nk}^+][u_k^-, u_k^+] = \sum_k [\min\{R\}, \max\{R\}], \tag{5.1}$$

$R = \{a_{nk}^- u_k^-, a_{nk}^- u_k^+, a_{nk}^+ u_k^-, a_{nk}^+ u_k^+\}$ and $a_{nk}^-, u_k^-, u_k^+, a_{nk}^+ \in E^1$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda(E^2) : \mu(E^2))$, we denote the class of matrices A such that $A : \lambda(E^2) \to \mu(E^2)$. Thus, $A \in (\lambda(E^2) : \mu(E^2))$ if and only if the series on the right side of (5.1) converges for each $n \in \mathbb{N}$ and every $u \in \lambda(E^2)$, we have $Au = \{(Au)_n\}_{n\in\mathbb{N}} \in \mu(E^2)$ for all $u \in \lambda(E^2)$.

When does $A \in (\ell_\infty(E^2) : \ell_\infty(E^2))$? In the following theorem the necessary and sufficient condition i s obtained For this question, the necessary and sufficient condition is given by following theorem:

Theorem 5.1. $A = ([a_{nk}^-, a_{nk}^+]) \in (\ell_\infty(E^2) : \ell_\infty(E^2))$ *if and only if*

$$||A|| = \sup_n \sum_k \max\{\overline{d}(a_{nk}^-, \overline{0}), \overline{d}(a_{nk}^+, \overline{0})\} < \infty. \tag{5.2}$$

Proof. Let us suppose that (5.2) holds and $u \in \ell_\infty(E^2)$. Then,

$$\begin{aligned}||Au||_{\ell_\infty(E^2)} &= \sup_n \max\{\overline{d}(\sum_k a^-_{nk}u^-_k, \overline{0}), \overline{d}(\sum_k a^+_{nk}u^+_k, \overline{0})\}\\ &\leq \sup_n \sum_k \max\{\overline{d}(a^-_{nk}, \overline{0}), \overline{d}(a^+_{nk}, \overline{0}) \max\{\overline{d}(u^-_k, \overline{0}), \overline{d}(u^+_k, \overline{0})\}\\ &\leq M||u||_{\ell_\infty(E^2)} < \infty,\end{aligned}$$

that is $Au \in \ell_\infty(E^2)$.

Conversely, let us suppose that $A = ([a^-_{nk}, a^+_{nk}]) \in (\ell_\infty(E^2) : \ell_\infty(E^2))$ and $u \in \ell_\infty(E^2)$. Then, since $Au \in \ell_\infty(E^2)$ exists, the series $\sum_k [a^-_{nk}, a^+_{nk}][u^-_k, u^+_k]$ converges for each fixed $n \in \mathbb{N}$. and hence $A \in \ell^\beta_\infty(E^2)$. This holds for the sequence $(u_k) = ([-\overline{1}, \overline{1}]) \in \ell_\infty(E^2)$. Then, since $u^-_{\ell k}(\alpha) = u^+_{\ell k}(\alpha) = -1$ and $u^-_{rk}(\alpha) = u^+_{rk}(\alpha) = 1$ we can write

$$\begin{aligned}||Au||_{\ell_\infty(E^2)} &= \sup_n \max\{\overline{d}(\sum_k a^-_{nk}u^-_k, \overline{0}), \overline{d}(\sum_k a^+_{nk}u^+_k, \overline{0})\}\\ &\leq \sup_n \sum_k \max\{\overline{d}(a^-_{nk}, \overline{0}), \overline{d}(a^+_{nk}, \overline{0}) \max\{\overline{d}(u^-_k, \overline{0}), \overline{d}(u^+_k, \overline{0})\} < \infty\end{aligned}$$

which means that (5.2) holds. □

Example 5.2. Now let us show that there exists a matrix $A = ([a^-_{nk}, a^+_{nk}])$ which satisfies condition of the Theorem 5.1. Define the matrix $A = ([a^-_{nk}, a^+_{nk}])$ by

$$a^-_{nk} = \begin{cases} u^-_k, & n = k \\ \overline{0}, & otherwise \end{cases} \text{ with } u^-_k(x) = \begin{cases} 1 + (k+1)x, & -\frac{1}{k+1} \leq x \leq 0 \\ 1 - (k+1)x, & 0 \leq x \leq \frac{1}{k+1} \\ 0, & otherwise \end{cases} \quad (\text{ see, [15]})$$

and

$$a^+_{nk} = \begin{cases} u^+_k, & n = k \\ \overline{0}, & otherwise \end{cases} \text{ with } u^+_k(x) = \begin{cases} (k+1)x - k, & 1 - \frac{1}{k+1} \leq x \leq 1 \\ -kx + k + 1, & 1 \leq x \leq \frac{k+2}{k+1} \\ 0, & otherwise \end{cases}$$

for all $k, n \in \mathbb{N}$. Since

$$\overline{d}(a^-_{nk}, \overline{0}) = \begin{cases} \frac{1}{n+1}, & k = n \\ 0, & otherwise \end{cases} \text{ and } \overline{d}(a^+_{nk}, \overline{0}) = \begin{cases} 1 - \frac{1}{n+1} + \frac{2}{n+1}, & k = n \\ 0, & otherwise \end{cases},$$

we have $\sup_n \sum_k \max\{\overline{d}(a^-_{nk}, \overline{0}), \overline{d}(a^+_{nk}, \overline{0})\} < \infty$.

Theorem 5.3. *Let* $\lim_n \max\{\overline{d}(a^-_{nk}, \overline{0}), \overline{d}(a^+_{nk}, \overline{0})\} = 0$, *(k fixed) and suppose (5.2) holds. Then* $A = (a_{nk})$ *defines a bounded operator on* $c_0(E^2)$ *into itself, where* $a_{nk} \in E^2$ *for all* $n, k \in \mathbb{N}$.

Proof. Let $u = (u_k) \in c_0(E^2)$. If $u = (u_k) = ([\overline{0}^-, \overline{0}^+]) = \theta$ then, $A_n(u) = \sum_k a_{nk}u_k = \sum_k [a^-_{nk}, a^+_{nk}][u^-, u^+] = \sum_k [a^-_{nk}, a^+_{nk}][\overline{0}^-, \overline{0}^+] = \theta$, for all $n \in \mathbb{N}$. Hence

$A(u) \in c_0(E^2)$. Now we suppose that $u \neq \theta$. Then,

$$\begin{aligned}
||A_n(u)||_{c_0(E^2)} &= ||\sum_k a_{nk}u_k||_{c_0(E^2)} = ||\sum_{k=1}^{N} a_{nk}u_k + \sum_{k\geq N+1} a_{nk}u_k||_{c_0(E^2)} \\
&\leq \sum_{k=1}^{N} ||a_{nk}u_k||_{c_0(E^2)} + \sum_{k\geq N+1} ||a_{nk}u_k||_{c_0(E^2)} \\
&= \sum_{k=1}^{N} \max\left\{\overline{d}(a_{nk}^{-}u_k^{-}, \overline{0}), \overline{d}(a_{nk}^{+}u_k^{+}, \overline{0})\right\} \\
&+ \sum_{k\geq N+1} \max\left\{\overline{d}(a_{nk}^{-}u_k^{-}, \overline{0}), \overline{d}(a_{nk}^{+}u_k^{+}, \overline{0})\right\} \\
&= \sum_{k=1}^{N} \max\left\{\overline{d}(a_{nk}^{-}, \overline{0})\overline{d}(u_k^{-}, \overline{0}), \overline{d}(a_{nk}^{+}, \overline{0})\overline{d}(u_k^{+}, \overline{0})\right\} \\
&+ \sum_{k\geq N+1} \max\left\{\overline{d}(a_{nk}^{-}, \overline{0})\overline{d}(u_k^{-}, \overline{0}), \overline{d}(a_{nk}^{+}, \overline{0})\overline{d}(u_k^{+}, \overline{0})\right\} \\
&\leq \sum_{k=1}^{N} \max\left\{\overline{d}(a_{nk}^{-}, \overline{0}), \overline{d}(a_{nk}^{+}, \overline{0})\right\} \max\left\{\overline{d}(u_k^{-}, \overline{0}), \overline{d}(u_k^{+}, \overline{0})\right\} \\
&+ \sum_{k\geq N+1} \max\left\{\overline{d}(a_{nk}^{-}, \overline{0}), \overline{d}(a_{nk}^{+}, \overline{0})\right\} \max\left\{\overline{d}(u_k^{-}, \overline{0}), \overline{d}(u_k^{+}, \overline{0})\right\}.
\end{aligned}$$

Since $u \in c_0(E^2)$, we take $k > N$ so large that $\max\left\{\overline{d}(u_k^{-}, \overline{0}), \overline{d}(u_k^{+}, \overline{0})\right\} < \frac{\epsilon}{2M}$ and from

$\lim_n \max\{\overline{d}(a_{nk}^{-}, \overline{0}), \overline{d}(a_{nk}^{+}, \overline{0})\} = 0$ (k fixed) we take n so large that

$$\sum_{k=1}^{N} \max\{\overline{d}(a_{nk}^{-}, \overline{0}), \overline{d}(a_{nk}^{+}, \overline{0})\} \leq \frac{\epsilon}{2\max\left\{\overline{d}(u_k^{-}, \overline{0}), \overline{d}(u_k^{+}, \overline{0})\right\}}.$$

Hence, we have shown that $Au \in c_0(E^2)$. Finally, we will show that A is bounded:

$$\begin{aligned}
||Au||_{c_0(E^2)} &= \sup_n \max\{\overline{d}(\sum_k a_{nk}^{-}u_k^{-}, \overline{0}), \overline{d}(\sum_k a_{nk}^{+}u_k^{+}, \overline{0})\} \\
&\leq \sup_n \sum_k \max\{\overline{d}(a_{nk}^{-}, \overline{0}), \overline{d}(a_{nk}^{+}, \overline{0})\}\{\overline{d}(u_k^{-}, \overline{0}), \overline{d}(u_k^{+}, \overline{0})\} \\
&\leq \sup_n \sum_k \max\{\overline{d}(a_{nk}^{-}, \overline{0}), \overline{d}(a_{nk}^{+}, \overline{0})\} \max\{\overline{d}(u_k^{-}, \overline{0}), \overline{d}(u_k^{+}, \overline{0})\} \\
&= M||u||_{c_0(E^2)}.
\end{aligned}$$

□

The above-mentioned theorem shows that a certain type of matrix of `IVFN`s defines a linear operator on $c_0(E^2)$ into itself.

References

[1] Y. Altın, M. Mursaleen and H. Altınok, Statistical summability (C, 1) for sequences of fuzzy real numbers and a Tauberian theorem, J. of Intelligent -Fuzzy Systems 21 (6) (2010) 379–384.

[2] H. Altınok, R. Çolak and Y. Altın, On the class of λ-statistically convergent difference sequences of fuzzy numbers, Soft Computing 16 (6) (2012) 1029–1034

[3] H. Altınok, R. Çolak and M. Et, $\lambda-$difference sequence spaces of fuzzy numbers, Fuzzy sets and Systems 160 (2009) 3128–3139.

[4] S. J. Chen and S. M. Chen, Fuzzy risk analysis based on measures of similarity between interval-valued fuzzy numbers, Computers and Mathematics with Applications 55 (2008) 1670–1685.

[5] S. J. Chen and S. M. Chen, Handling information filtering problems based on interval-valued fuzzy numbers, Journal of the Chinese Institute of Engineers 9 (1) (2006) 83–96

[6] K. P. Chiao, Fundamental properties of interval vector max-norm, Tamsui Oxford Journal of Mathematical Sciences 18 (2) (2002) 219–233.

[7] B. Gorzalczany, Approximate inference with interval-valued fuzzy sets, an outline, in: Proc. Polish Symp. on Interval and Fuzzy Mathematics, Poznan (1983) 89–95.

[8] D. H. Hong and S. Lee, Some algebraic properties and a distance measure for interval-valued fuzzy numbers, Inform. Sci. 148 (2002) 1-10.

[9] C. Li, Distance between interval-valued fuzzy sets, The 28^{th} North American Fuzzy Information Processing Society Annual Conference, Cincinnati, Ohio, USA 2009.

[10] S. Markov, Quasilinear spaces and their relation to vector spaces, Eletronic Journal on Mathematic of Computation 2 (1) (2005) 1–21.

[11] M. Matloka, Sequence of fuzzy numbers, BUSEFAL 28 (1986) 28–37.

[12] A. R. Meenakshi and M. Kaliraja, Regular interval valued fuzzy matrices, Advanced in Fuzzy Mathematics 5 (2010) 7–15.

[13] R. E. Moore, Automatic error analysis in digital computation, LSMD- 48421, Lockheed Missiles and Space Company 1959.

[14] S. Nanda, On sequence spaces of fuzzy numbers, Fuzzy Sets and Systems 33 (1989) 123–126.

[15] Ö. Talo and F. Başar, Determination of the duals of classical sets of sequences of fuzzy numbers and related matrix transformations, Computers and Mathematics with Applications 58 (2009) 717–733.

[16] B. Turksen, Interval valud fuzzy sets based on normal forms, Fuzzy Sets and Systems 20 (1986) 191–210.

[17] G. Wang and X. Li, The application to interval-valued fuzzy numbers and interval distribution numbers, Fuzzy Sets And Systems 98 (1998) 331–335.

[18] J. S. Yao and F. T. Lin, Fuzzy shortest-path network problems, JISE J. Inf. Sci. Eng. 19 (2003) 329–351.

[19] L. A. Zadeh, Fuzzy Sets, Information and Control 8 (1965) 338–353.

Mehmet Şengönül (msengonul@nevsehir.edu.tr)
Department of Mathematics, Nevşehir Hacı Bektaş Veli University, 50300 Nevşehir, Turkey