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Some characterizations of intra-regular ordered left almost semigroups by their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals

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Abstract. In this paper, we have given the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals in ordered $\mathcal{LA}$-semigroups and characterized an intra-regular ordered $\mathcal{LA}$-semigroups by using the properties of $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy subsets and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals

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1. Introduction

The theory of fuzzy sets which was introduced by Zadeh in 1965 [19], has an important role for solving problems in real life involving uncertainties. Recently many theories like fuzzy set theory, theory of vague sets, theory of soft ideals, theory of intuitionistic fuzzy sets, and theory of rough set have been developed for handling such uncertainties (see [3, 5, 6, 17]). These theories can be used for latest development in almost every branch of science. In 1971, Rosenfeld introduced the concept of fuzzy set theory in groups [16]. Mordeson et. al. [10] have discussed the applications of fuzzy set theory in fuzzy coding, fuzzy automata and finite state machines. The theory of soft sets (see [4, 5]) has many applications in many fields such as the smoothness of function, game theory, operation research, Riemann integration etc.

Fuzzy set theory on semigroups has already been developed [19, 4]. In [11] Murali defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalance on a fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set was defined in [13]. Bhakat and Das (see [1, 2]) gave the concept of (α, β)-fuzzy subgroups by using the "belongs to" relation $\in$ and "quasi-coincident with" relation q between a fuzzy point and fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)-$fuzzy subgroup, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. Regular semigroups are characterized by the properties of their $(\in, \in \vee q)$-fuzzy ideals in [15]. In [14] semi groups were characterized by the properties of $(\in, \in \vee q)$- fuzzy ideals.

(i) A groupoid S is called an $\mathcal{LA}$-semigroup (left almost semigroup) if it satisfies the left invertive law $(ab)c = (cb)a$, for all a, b and $c \in S$.

(ii) In an $\mathcal{LA}$-semigroup medial law, $(ab)(cd) = (ac)(bd)$, holds for all a, b, c, $d \in S$.

(iii) An $\mathcal{LA}$-semigroup S with left identity satisfies the paramedial law $(ab)(cd) = (db)(ca)$, for all a, b, c, $d \in S$.

(iv) In the case of an $\mathcal{LA}$-semigroup which contains left identity, the following law holds,

$$a(bc) = b(ac), \text{for all } a, b, c \in S. \tag{1}$$

In this paper, we have introduced $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals in ordered $\mathcal{LA}$-semigroup and observed a variety of some new results. We have characterized an intra-regular ordered $\mathcal{LA}$-semigroup by the properties of its $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals.

2. Preliminaries

Definition 2.1 ([9]). An ordered $\mathcal{LA}$-semigroup (po-$\mathcal{LA}$-semigroup) is a structure $(S, \cdot, \leq)$ in which the following conditions hold :

(i) $(S, \cdot)$ is an $\mathcal{LA}$-semigroup.

(ii) $(S, \leq)$ is a poset.

(iii) $\forall\ a, b, x \in S,\ a \leq b \Longrightarrow ax \leq bx$ and $xa \leq xb$.

Example 2.2. *Let* $S = \{a, b, c\}$ *be an* $\mathcal{LA}$-semigroup *with the following multiplication table.*

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	a	a

Then S is an ordered $\mathcal{LA}$-semigroup with the following two orders:

$$\leq := \{(a, a), (b, b), (c, c), (c, a), (c, b)\}, \tag{1}$$

$$\leq := \{(a, a), (b, b), (c, c), (a, c), (a, b)\}. \tag{2}$$

Let S be an ordered $\mathcal{LA}$-semigroup. A fuzzy subset f of a non-empty set S is described as an arbitrary mapping $f : S \to [0, 1]$, where $[0, 1]$ is a usual closed interval of real numbers. A fuzzy subset f is a class of objects with a grades of membership having the form $f = \{(s,\ f(s))\ /s \in S\}$. Let $x \in A$, then

$$A_x = \{(y, z) \in S \times S \setminus x \leq yz\}.$$

The product of any two fuzzy subsets f and g of S is defined as:

$$(f\circ g)(x)=\begin{cases} \bigvee\limits_{(y,z)\in A_X}\{f(y)\wedge g(z)\}\ if\ x\leq yz & \text{if } A_x\neq\varnothing. \\ 0 & \text{if } A_x=\varnothing. \end{cases}$$

The order relation $\subseteq$ between any two fuzzy subsets f and g of S is defined as:

$$f\subseteq g \text{ if and only if } f(a)\leq g(a)\ \forall\ a\in S.$$

The symbols $f\wedge g$ and $f\vee g$ means the following fuzzy subsets of an ordered $\mathcal{LA}$-semigroup S.

$$\begin{aligned}(f\wedge g)(a) &= \min\{f(a),g(a)\},\ \text{for all } a\in S.\\ (f\vee g)(a) &= \max\{f(a),g(a)\},\ \text{for all } a\in S.\end{aligned}$$

Let A be a non-empty subset of an ordered $\mathcal{LA}$-semigroup S. Then

$$(A]=\{t\in S\mid t\leq a,\ for\ some\ a\in A\}.$$

For $A=\{a\}$, we usually written as $(a]$. Let A be a non-empty subset of an ordered $\mathcal{LA}$-semigroup S. Then,

- A is called an $\mathcal{LA}$-subsemigroup of S if $(A^2]\subseteq A$.
- A is called left ideal if $(SA]\subseteq A$.
- A is called right ideal if $(AS]\subseteq A$.
- A is is called ideal of S if it is both left ideal as well as right ideal.
- A is called a bi-ideal of S if $((AS)A]\subseteq A$ and $(A^2]\subseteq A$.
- A is called a generalized bi-ideal of S if $((AS)A]\subseteq A$.
- A is called quasi-ideal of S if $(QS]\cap(SQ]\subseteq Q$.

Obviously, every one sided ideal of an ordered $\mathcal{LA}$-semigroup S is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal.
Let f be a fuzzy subset of an ordered $\mathcal{LA}$-semigroup S. Then,

- f is called a fuzzy $\mathcal{LA}$-subsemigroup of S if

(i) $a\leq b\Rightarrow f(a)\geq f(b)$,for all a, $b\in S$,
(ii) $f(ab)\geq\min\{f(a),f(b)\}$, for all a, b and $c\in S$.

- f is called a fuzzy left (right) ideal of S if

(i) $a\leq b\Rightarrow f(a)\geq f(b)$,for all a, $b\in S$,
(ii) $f(ab)\geq f(b)$ $(f(ab)\geq f(a))$ for all a, $b\in S$

- f is called a fuzzy ideal of S if it is both fuzzy left and fuzzy right ideal of S.
- f is called a fuzzy generalized bi-ideal of S if

(i) $a\leq b\Rightarrow f(a)\geq f(b)$,for all a, $b\in S$,
(ii) $f((ax)b)\geq\min\{f(a),f(b)\}$, for all a, x and $b\in S$.

- A fuzzy ordered $\mathcal{LA}$-subsemigroup f is called a fuzzy bi-ideal of S if

(i) $x\leq y\Rightarrow f(x)\geq f(y)$,for all x, $y\in S$,
(ii) $f((xa)y)\geq\min\{f(x),f(y)\}$, for all x,a and $y\in S$.

- f is called a fuzzy quasi-ideal of S if $(f\circ\varsigma)\wedge(\varsigma\circ f)\leq f$, where ς is the fuzzy subset of S mapping every element of S on 1.

A fuzzy subset f of X of the form

$$f(y)=\begin{cases} t\in(0,1] & \text{if } y\leq x,\\ 0 & \text{otherwise,}\end{cases}$$

is said to be fuzzy point with support x and value t and is denoted by x_t.

3. $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Ideals of ordered $\mathcal{LA}$-semigroups

Let $\gamma,\delta\in[0,1]$ be such that $\gamma<\delta$. Let A and B be any sub sets of an ordered $\mathcal{LA}$-semigroup S such that $B\subseteq A$. Then $X^\delta_{\gamma B}$ the fuzzy subset of X is defined as: $X^\delta_{\gamma B}(x)\geq\delta$ for all $x\in B$ and $X^\delta_{\gamma B}(x)\leq\gamma$ if $x\notin B$. Clearly, we see that $X^\delta_{\gamma B}$ is the characteristic function of B if $\gamma=0$ and $\delta=1$. Let f be a fuzzy subset of X.Then for a fuzzy point x_t, we say that

- $x_t\in_\gamma f$ if $f(x)\geq t>\gamma$.
- $x_t q_\delta f$ if $f(x)+t>2\delta$.
- $x_t\in_\gamma\vee q_\delta f$ if $x_t\in_\gamma f$ or $x_t q_\delta f$.

Now we introduce a new relation on $\mathcal{F}(X)$, denoted by "$\subseteq\vee q_{(\gamma,\delta)}$". Let f and g be any fuzzy subsets of $\mathcal{F}(X)$. Then by $f\subseteq\vee q_{(\gamma,\delta)}g$ we mean that $x_t\in_\gamma f$ implies $x_t\in_\gamma\vee q_\delta g$ for all $x\in X$ and $t\in(\gamma,1]$. Moreover f and g are said to be (γ,δ)-equal, denoted by $f=_{(\gamma,\delta)}g$, if $f\subseteq\vee q_{(\gamma,\delta)}g$ and $g\subseteq\vee q_{(\gamma,\delta)}f$. The following definitions can be found in [18]

Lemma 3.1 ([8])**.** *In an odered $\mathcal{LA}$-semigroup S, the following are true.*
(i) $A\subseteq(A]$, *for all* $A\subseteq S$.
(ii) $A\subseteq B\subseteq G\Rightarrow(A]\subseteq(B]$, *for all* $A,B\subseteq S$.
(iii) $(A](B]\subseteq(AB]$, *for all* $A,B\subseteq S$.
(iv) $(A]=((A]]$, *for all* $A\subseteq S$.
(v) $((A](B]]=(AB]$, *for all* $A,B\subseteq S$.

Lemma 3.2 ([18])**.** *Let f and g be the fuzzy subsets of $\mathcal{F}(X)$. Then $f\subseteq\vee q_{(\gamma,\delta)}g$ if and only if*

$$\max\{g(x),\gamma\}\geq\min\{f(x),\delta\}\textit{ for all } x\in X.$$

Lemma 3.3. *Let f, g and h be the fuzzy subsets of $\mathcal{F}(X)$. If $f\subseteq\vee q_{(\gamma,\delta)}g$ and $g\subseteq\vee q_{(\gamma,\delta)}h$, then $f\subseteq\vee q_{(\gamma,\delta)}h$*

Proof. It is straightforward by Lemma 3.2. □

It is shown in [18] that "$=_{(\gamma,\delta)}$" is equivalence relation on $\mathcal{F}(X)$. It is also notified that $f=_{(\gamma,\delta)}g$ if and only if

$$\max\{\min\{f(x),\delta\},\gamma\}=\max\{\min\{g(x),\delta\},\gamma\} forall x\in X.$$

The proof of the following Lemma is same as in [18], therefore we omit its proof.

Lemma 3.4. *Let S be an ordered $\mathcal{LA}$-semigroup with left identity and A, B be any non-empty subsets of S. Then, we have*
(i) $A\subseteq B$ *if and only if* $X_A\subseteq\vee q_{(\gamma,\delta)}X^\delta_{\gamma B}$, *where* $r\in(\gamma,1]$ *and* $\gamma,\delta\in[0,1]$.
(ii) $X^\delta_{\gamma A}\cap X^\delta_{\gamma B}=_{(\gamma,\delta)}X^\delta_{\gamma(A\cap B)}$.
(iii) $X_A\circ X^\delta_{\gamma B}=_{(\gamma,\delta)}X_{(AB]}$.

Lemma 3.5. *In an ordered $\mathcal{LA}$-semigroup S with left identity* $(ab)^2 = a^2b^2 = b^2a^2$.

Proof. It can be followed by using medial law and paramedial law. □

Definition 3.6. Let S an ordered $\mathcal{LA}$-semigroup and f be a fuzzy subset of S.Then f is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy $\mathcal{LA}$-subsemigroup of S if for all $x, y \in S$ and $t, s \in (\gamma, 1]$, it satisfies

$$x_t \in_\gamma f,\ y_s \in_\gamma f \text{ implies that } (xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f.$$

Theorem 3.7. *Let f be a fuzzy subset of an ordered $\mathcal{LA}$-semigroup S. Then, f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy $\mathcal{LA}$-subsemigroup of S if and only if*

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}, where \gamma, \delta \in [0, 1]$$

Proof. Let S be an ordered $\mathcal{LA}$-semigroup and f be a fuzzy subset of S such that f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy $\mathcal{LA}$-subsemigroup of S. Assume that $x, y \in S$ and $t \in (\gamma, 1]$, such that

$$max\{f(xy), \gamma\} < t \leq \min\{f(x), f(y), \delta\}.$$

This implies that $f(xy) < t$. This further shows that $(xy)_{\min t}\overline{\in_\gamma \vee q_\delta} f$ and $\min\{f(x), f(y), \delta\} \geq t \Rightarrow f(x) \geq t > \gamma$, $f(y) \geq t > \gamma$ So $x_t \in_\gamma f$ and $y_s \in_\gamma f$. But $(xy)_{\min\{t,s\}}\overline{\in_\gamma \vee q_\delta} f$, which contradiction to the definition. Hence

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\} \text{ for all } x, y \in S.$$

Conversely, let there exist $x, y \in S$ and $t, s \in (\gamma, 1]$ such that $x_t \in_\gamma f$ and $y_s \in_\gamma f$ but $(xy)_{\min\{t,s\}}\overline{\in_\gamma \vee q_\delta} f$. Then $f(x) \geq t > \gamma$, $f(y) \geq s > \gamma$, $f(xy) < \min\{f(x), f(y), \delta\}$ and $f(xy) + \min\{t, s\} \leq 2\delta$. It follows that $f(xy) < \delta$ and so $max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$, which is contradiction. Hence

$$x_t \in_\gamma f \text{ and } y_s \in_\gamma f \text{ for all } x, y \in S \text{ and } t, s \in (0, 1].$$

This shows that $(xy)_{\min\{t,s\}} \in_\gamma \vee_{q\delta} f$ for all $x, y \in S$. □

Definition 3.8. A fuzzy subset of an ordered $\mathcal{LA}$-semigroup with left identity is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp.right) ideal of S if for all $x, y \in S$, we have

$$y_t \in_\gamma f \text{ implies that } (xy)_t \in_\gamma \vee_{q\delta} f \ (resp\ x_t \in_\gamma f \text{ implies that } (xy)_t \in_\gamma \vee_{q\delta} f).$$

Theorem 3.9. *Let S be an ordered $\mathcal{LA}$-semigroup with left identity and f be a fuzzy subset of S. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S if and only if*

$$\max\{f(xy), \gamma\} \geq \min\{f(x), \delta\} \textit{ for all } x, y \in S.$$

Proof. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S. Assume that there exist $a, b \in S$ and $t \in (\gamma, 1]$, such that

$$\max\{f(xy), \gamma\} < t \leq \min\{f(x), \delta\}.$$

Then, $\max\{f(xy), \gamma\} < t \leq \min\{f(x), \delta\}$ implies that $(xy)_t\overline{\in_\gamma} f$ which further shows that $(xy)_t\overline{\in_\gamma \vee_{q\delta}} f$. From $\min\{f(x), f(y), \delta\} \geq t > \gamma$ it follows that $f(x) \geq t > \gamma$, which implies that $x_t \in_\gamma f$. But $(xy)_t\overline{\in_\gamma \vee_{q\delta}}f$ a contradiction to the definition. Thus

$$\max\{f(xy), \gamma\} \geq \min\{f(x), \delta\}.$$

Conversely, suppose that there exists $x, y \in S$ and $t, s \in (\gamma, 1]$ such that $x_s \in_\gamma f$.but $(xy)_t\overline{\in_\gamma \vee_{q\delta}}f$. Then $f(x) \geq t > \gamma$, $f(xy) < \min\{f(x), \delta\}$ and $f(xy) + t \leq 2\delta$. It folows that $f(xy) < \delta$ and so $\max\{f(xy), \gamma\} < \min\{f(x), \delta\}$, which is contradiction. Hence $x_t \in_\gamma f$ this implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee_{q\delta}f$ (respectively $x_t \in_\gamma f$.implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee_{q\delta}f$ for all $x, y \in S$. □

Example 3.10. Let $S = \{1, 2, 3\}$ be an ordered $\mathcal{LA}$-semigroup with the following multiplication table and the order given below:

$\cdot$	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

$$\leqslant := \{(1,1), (2,2), (3,3), (3,1), (3,2)\}.$$

Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.4 & \text{if } x = 1 \\ 0.41 & \text{if } x = 2 \\ 0.38 & \text{if } x = 3. \end{cases}$$

Then, we have

- f is an $(\in_{0.2}, \in_{0.2} \vee q_{0.22})$-fuzzy right ideal,
- f is not an $(\in, \in \vee q_{0.22})$-fuzzy right ideal, because $f(2.3) < \min\{f(2), \frac{1-0.22}{2} = .39\}$.
- f is not a fuzzy right ideal because $f(2.3) < f(2)$.

Lemma 3.11. *R is a right ideal of an ordered $\mathcal{LA}$-semigroup S if and only if $X^\delta_{\gamma R}$ is an $(\in_\gamma, \in_\gamma \vee_{q\delta})$-fuzzy right ideal of S.*

Proof. (i) Let $x, y \in R$ it means that $xy \in R$. Then $X^\delta_{\gamma R}(xy) \geq \delta$, $X^\delta_{\gamma R}(x) \geq \delta$ and $X^\delta_{\gamma R}(y) \geq \delta$ but $\delta > \gamma$.
Thus

$$\max\{X^\delta_{\gamma R}(xy), \gamma\} = X^\delta_{\gamma R}(xy) \text{ and } \min\{X^\delta_{\gamma R}(x), \delta\} = \delta.$$

This shows that

$$\max\{X^\delta_{\gamma R}(xy), \gamma\} \geq \min\{X^\delta_{\gamma R}(x), \delta\}.$$

(ii) Ler $x \notin R$, $y \in R$

$Case(a)$: If $xy \notin R$. Then $X^\delta_{\gamma R}(x) \leq \gamma$, $X^\delta_{\gamma R}(y) \geq \delta$ and $X^\delta_{\gamma R}(xy) \leq \gamma$.
Therefore

$$\max\{X^\delta_{\gamma R}(xy), \gamma\} = \gamma \text{ and } \min\{X^\delta_{\gamma R}(x), \delta\} = X^\delta_{\gamma R}(x).$$

This implies that
$$\max\{X^{\delta}_{\gamma R}(xy), \gamma\} \geq \min\{X^{\delta}_{\gamma R}(x), \delta\}.$$
$Case(b)$: If $xy \in R$. Then $X^{\delta}_{\gamma R}(xy) \geq \delta$, $X^{\delta}_{\gamma R}(x) \leq \gamma$ and $X^{\delta}_{\gamma R}(y) \geq \delta$.
Thus

$$\max\{X^{\delta}_{\gamma R}(xy), \gamma\} = X^{\delta}_{\gamma R}(xy) \text{ and } \min\{X^{\delta}_{\gamma R}(x), \delta\} = X^{\delta}_{\gamma R}(x).$$

So
$$\max\{X^{\delta}_{\gamma R}(xy), \gamma\} > \min\{X^{\delta}_{\gamma R}(x), \delta\}$$
(iii) Let $x \in R$, $y \notin R$. Then $xy \in R$. Thus $X^{\delta}_{\gamma R}(xy) \geq \delta$, $X^{\delta}_{\gamma R}(y) \leq \gamma$ and $X^{\delta}_{\gamma R}(x) \geq \delta$.
Thus

$$\max\{X^{\delta}_{\gamma R}(xy), \gamma\} = X^{\delta}_{\gamma R}(xy) \text{ and } \min\{X^{\delta}_{\gamma R}(x), \delta\} = \delta.$$

Hence
$$\max\{X^{\delta}_{\gamma R}(xy), \gamma\} \geq \min\{X^{\delta}_{\gamma R}(x), \delta\}.$$
(iv) Let $x, y \notin R$, then $Case(a)$: Suppose that $xy \notin R$. Then by definition we have $X^{\delta}_{\gamma R}(xy) \leq \gamma$, $X^{\delta}_{\gamma R}(y) \leq \gamma$ and $X^{\delta}_{\gamma R}(x) \leq \gamma$.
Thus

$$\max\{X^{\delta}_{\gamma R}(xy), \gamma\} = \gamma \text{ and } \min\{X^{\delta}_{\gamma R}(x), \delta\} = X^{\delta}_{\gamma R}(x).$$

This shows that
$$\max\{X^{\delta}_{\gamma R}(xy), \gamma\} \geq \min\{X^{\delta}_{\gamma R}(x), \delta\}.$$
$Case(b)$: Suppose that $xy \in R$. Then by definition we have $X^{\delta}_{\gamma R}(xy) \geq \gamma$, $X^{\delta}_{\gamma R}(y) \leq \gamma$ and $X^{\delta}_{\gamma R}(x) \leq \gamma$.
Thus

$$\max\{X^{\delta}_{\gamma R}(xy), \gamma\} = X^{\delta}_{\gamma R}(xy) \text{ and } \min\{X^{\delta}_{\gamma R}(x), \delta\} = X^{\delta}_{\gamma R}(x).$$

Therefore
$$\max\{X^{\delta}_{\gamma R}(xy), \gamma\} > \min\{X^{\delta}_{\gamma R}(x), \delta\}.$$
Coversely, assume that $rs \in RS$, where $r \in R$ and $s \in S$. By assumption,
$$\max\{X^{\delta}_{\gamma R}(rs), \gamma\} \geq \min\{X^{\delta}_{\gamma R}(r), \delta\}.$$
Since $r \in R$, $X^{\delta}_{\gamma R}(r) \geq \delta$. This implies that $\min\{X^{\delta}_{\gamma R}(r), \delta\} = \delta$.
Thus
$$\max\{X^{\delta}_{\gamma R}(rs), \gamma\} \geq \delta.$$
This shows that $X^{\delta}_{\gamma R}(rs) \geq \delta$ this implies that $rs \in R$. Hence R is a right ideal of S. □

We now introduces the notion of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprine ideal in an ordered $\mathcal{LA}$-semigroups.

Definition 3.12. Let S be an ordered $\mathcal{LA}$-semigroup and f be a fuzzy subset of S.Then f is called $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprine if $x_t^2 \in_\gamma f$ implies that $x_t \in_\gamma \vee q_\delta f$ for all $x \in S$ and $t \in (\gamma, 1]$.

Example 3.13. Consider an ordered $\mathcal{LA}$-semigroup $S = \{1, 2, 3, 4, 5\}$ withe the following multiplication table and ordered given below:

$\bullet$	1	2	3	4	5
1	4	5	1	2	3
2	3	4	5	1	2
3	2	3	4	5	1
4	1	2	3	4	5
5	5	1	2	3	4

$$\leqslant := \{(1,1), (2,2), (2,1), (3,3), (4,4), (5,5)\}.$$

It is easy to see that S intra-regular. Define a fuzzy subset f on S as below:

$$f(x) = \begin{cases} 0.6 & \text{if } x = 1 \\ 0.7 & \text{if } x = 2 \\ 0.68 & \text{if } x = 3 \\ 0.63 & \text{if } x = 4 \\ 0.53 & \text{if } x = 5 \end{cases}$$

Then f an $(\in_{0.4}, \in_{0.4} \vee_{q0.5})$ fuzzy semiprime.

Theorem 3.14. *A fuzzy subset f of an ordered $\mathcal{LA}$-semigroup S is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime if and only if*

$$\max\{f(x), \gamma\} \geq \min\{f(x^2), \delta\}.$$

Proof. Let f be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime. Assume that there exists $x \in S$ and $t \in (\gamma, 1]$, such that

$$\max\{f(x), \gamma\} < t \leqslant \min\{f(x^2), \delta\}.$$

Then $\max\{f(x), \gamma\}< t$. This implies that $f(x) < t > \gamma$. Now since $\delta \geq \gamma$, so $f(x) + t < 2\delta$. Thus $x_t \overline{\in_\gamma \vee q_\delta} f$. Also since $\min\{f(x^2), \delta\} \geq t$, so $f(x^2) \geq t > \gamma$. This implies that $x_t^2 \in_\gamma f$. Thus by definition of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime $x_t \in_\gamma \vee q_\delta f$ which is contradiction that $x_t \overline{\in_\gamma \vee q_\delta} f$. Hence

$$\max\{f(x), \gamma\} \geq \min\{f(x^2), \delta\}, \text{ for all } x \in S.$$

Conversely, suppose that there exist $x \in S$ and $t \in (\gamma, 1]$ such that $x_t^2 \in_\gamma f$.
Then

$$f(x^2) \geq t > \gamma.$$

Thus

$$\max\{f(x),\gamma\} \geq \min\{f(x^2),\delta\} \geq \min\{t,\delta\}.$$

We consider to discuss two cases here, $Case(a)$: If $t \leqslant \delta$, then $f(x) \geq t > \gamma$, this shows that $x_t \in_\gamma f$.

$Case(b)$: if $t > \delta$, then $f(x) + t > 2\delta$. Thus $x_t \vee q_\delta f$. Hence from (a) and (b) we get $x_t \in_\gamma \vee q_\delta f$, this implies that f is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime. □

Theorem 3.15. *Let S be an ordered $\mathcal{LA}$-semigroup with a left identity and R be a right ideal of S. Then, the following conditions are equivalent:* (i) *R is semiprime.*

(ii) $X^{\delta}_{\gamma R}$ *is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.*

Proof. $(i) \Rightarrow (ii)$ Let R be a semiprime ideal of an ordered $\mathcal{LA}$-semigroup S. Let a be an orbitrary element of S such that $a \in R$. Then $a^2 \in R$. Hence $X^{\delta}_{\gamma R}(a) \geq \delta$ and $X^{\delta}_{\gamma R}(a^2) \geq \delta$ this implies that $\max\{X^{\delta}_{\gamma R}(a),\gamma\} \geq \min\{X^{\delta}_{\gamma R}(a^2),\delta\}$. Now let $a \notin R$. Since R is semiprime, we have $a^2 \notin R$. This implies that $X^{\delta}_{\gamma R}(a) \leqslant \gamma$ and $X^{\delta}_{\gamma R}(a^2) \leqslant \gamma$. Therefore

$$\max\{X^{\delta}_{\gamma R}(a),\gamma\} \geq \{\min X^{\delta}_{\gamma R}(a^2),\delta\}.$$

Hence

$$\max\{X^{\delta}_{\gamma R}(a),\gamma\} \geq \{\min X^{\delta}_{\gamma R}(a^2),\delta\} \text{ for all } a \in S.$$

$(ii) \Rightarrow (ii)$ Let $X^{\delta}_{\gamma R}$ be a fuzzy semiprime. If $a^2 \in R$, for some a in S, then $X^{\delta}_{\gamma R}(a^2) \geq \delta$. Since $X^{\delta}_{\gamma R}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime, therefore $\max\{X^{\delta}_{\gamma R}(a),\gamma\} \geq \{\min X^{\delta}_{\gamma R}(a^2),\delta\}$. This implies that $\max\{X^{\delta}_{\gamma R}(a),\gamma\} \geq \delta$. But $\delta > \gamma$, so $X^{\delta}_{\gamma R}(a) \geq \delta$. Thus $a \in R$. Hence R is semiprime. □

4. Characterizations of Intra-regular ordered $\mathcal{LA}$-semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals

An element a of an ordered $\mathcal{LA}$-semigroup S is called intra-regular if there exist $x, y \in S$ such that $a \leq (xa^2)y$ and S is called intra-regular, if every element of S is intra-regular. In this section, we discuss the characterizations of intra-regular ordered AG-groupoids.

Theorem 4.1. *For an ordered $\mathcal{LA}$-semigroup S with left identity the following conditions are equivalent :* (i) *S is intra-regular.*

(ii) *For any right ideal R of an ordered $\mathcal{LA}$-semigroup S, $R \subseteq (R^2]$ and R is a semiprime.*

(iii) *For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f of S, we have $f \subseteq \vee q_{(\gamma,\delta)} f \circ f$ and f is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.*

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of an intra-regular ordered $\mathcal{LA}$-semigroup S with left identity. Since S is intra-regular therefore for any $a \in S$ there exist $x, y \in S$ such that $a \leq (xa^2)y$. Now, by using (1), para medial law, medial law and left invertive law we obtain

$$\begin{aligned} a &\leq (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a. \\ y(xa) &\leq y(x((xa^2)y)) = [y((xa^2)(xy))] = [(xa^2)(xy^2)] = [(y^2x)(a^2x)] \\ &= [a^2((y^2x)x)] = [\{a(y^2x)\}(ax)]. \end{aligned}$$

If for any $a \in S$ there exist $p, q \in S$ such that $a \leq pq$, then

$$\begin{aligned}\max\{(f \circ f)(a), \gamma\} &= \max\left\{\bigvee_{a \leq pq} \{f(p) \wedge f(q)\}, \gamma\right\} \\ &\geq \max\{\min\{f(a(y^2x)), f(ax)\}, \gamma\} \\ &\geq \max\{\min\{f(a(y^2x)), f(ax)\}, \gamma\} \\ &= \min\{\max\{f(a(y^2x)), \gamma\}, \max\{f(ax), \gamma\}\} \\ &\geq \min\{\min\{f(a), \delta\}, \min\{f(a), \delta\}\} \\ &= \min\{f(a), \delta\}.\end{aligned}$$

Thus, by Lemma 3.2, $f \subseteq \vee q_{(\gamma,\delta)} f \circ f$.Now we show that f is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime. Since $S = S^2$, for each $y \in S$ there exist $y_1, y_2 \in S$ such that $y \leq y_1 y_2$. Thus using (1), para medial law and medial law we obtain

$$\begin{aligned} a &\leq (xa^2)y = (xa^2)(y_1y_2) \\ &= (y_2y_1)(a^2x) = a^2[(y_2y_1)x] \\ &= a^2t, \text{ where } [(y_2y_1)x] = t.\end{aligned}$$

So

$$\begin{aligned}\max\{f(a), \gamma\} &\leq \max\{f(a^2t), \gamma\} \\ &\geq \min\{f(a^2), \delta\}.\end{aligned}$$

$(iii) \Rightarrow (ii)$ Suppose that R is any right ideal of S. Then, by (iii) $X^\delta_{\gamma R}$ is semiprime and by Theorem 3.15,R is semiprime.Now using (iii) and Lemma 3.4 we get

$$X^\delta_{\gamma R} = X^\delta_{\gamma R \cap R} =_{(\gamma,\delta)} X^\delta_{\gamma R} \cap X^\delta_{\gamma R} \subseteq \vee q_{(\gamma,\delta)} X^\delta_{\gamma R} \circ X^\delta_{\gamma R} =_{(\gamma,\delta)} X^\delta_{\gamma (R^2]}.$$

Hence, by Lemma 3.4, we have $R \subseteq (R^2]$.

$(ii) \Rightarrow (i)$ Since Sa^2 is a right ideal of S containing a so using (ii) it is semiprime. Now by using Lemma 3.1 we get

$$a \in (Sa^2]^2 = (Sa^2](Sa^2] \subseteq ((Sa^2)(Sa^2)] \subseteq ((Sa^2)S].$$

Hence, S is intra-regular. □

Theorem 4.2. *Let S be an ordered $\mathcal{LA}$-semigroup with left identity. Then, the following conditions are equivalent : (i) S is intra-regular.*

(ii) For every right ideal R and any subset A of S, $R \cap A \subseteq (RA]$.

(iii) For every $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy right ideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset g, we have $f \cap g \subseteq \vee q_{(\gamma,\delta)}(f \circ g)$ and f $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and g be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset of an intra-regular ordered $\mathcal{LA}$-semigroup S. Since S is intra-regular, therefore for any $a \in S$ there exist $x, y \in S$ such that $a \leq (xa^2)y$. Now, by using medial, para medial laws and (1) we obtain

$$a \leq (xa^2)y = [(x(aa))y] = [(a(xa))y] = [(y(xa))]a.$$

$$\begin{aligned} y(xa) &\leq [y\{x((xa^2)y\}] = [y\{(xa^2)(xy)\}] = [(xa^2)(xy^2)] \\ &= [(y^2x)(a^2x)] = [a^2(y^2x^2)]. \end{aligned}$$

Thus

$$a \leq (a^2t)a \text{ where, } (y^2x^2) = t.$$

If for any $a \in S$ there exist $b, c \in S$ such that $a \leq bc$, then

$$\begin{aligned} \max\{(f \circ g)(a), \gamma\} &= \max\left\{ \bigvee_{a \leq bc} \{f(b) \wedge g(c)\}, \gamma \right\} \\ &\geq \max\{\min\{f(a^2t), g(a)\}, \gamma\} \\ &= \min[\max\{f(a^2t), \gamma\}, \max\{g(a), \gamma\}] \\ &\geq \min\{\min\{f(a), \delta\}, \min\{g(a), \delta\}\} \\ &= \min\{(f \cap g)(a), \delta\}. \end{aligned}$$

So, by Lemma 3.4, we have $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$.To prove that f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime see Theorem 4.1.

$(iii) \Rightarrow (ii)$ Let R be a right ideal and A be any subset of S . Then, by Lemma 3.4 and (iii), we get

$$X^\delta_{\gamma(R \cap A)} =_{(\gamma,\delta)} X^\delta_{\gamma R} \cap X^\delta_{\gamma A} \subseteq \vee q_{(\gamma,\delta)} X^\delta_{\gamma R} \circ X^\delta_{\gamma A} =_{(\gamma,\delta)} X^\delta_{\gamma(RA]}.$$

Hence, by Lemma 3.4, we get $R \cap A \subseteq (RA]$. See Theorem 4.1 for the proof of R is semiprime.

$(ii) \Rightarrow (i)$ Since $(Sa^2]$ is a right ideal containing a^2 . By (ii), it is semiprime. Now by using Lemma 3.1, we get

$$a \in (Sa^2] \cap (Sa] = ((Sa^2](Sa]] = (Sa^2](Sa] \subseteq ((Sa^2)(Sa)] \subseteq ((Sa^2)S].$$

Hence, S is an intra-regular. □

Theorem 4.3. *Let S be an ordered $\mathcal{LA}$-semigroup with left identity. Then the following conditions are equivalent : (i) S is intra-regular.*

(ii) For every right ideal R and any subset A of S, $R \cap A \subseteq (AR]$ and R is semiprime.

(iii) For every $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy right ideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset g, we have $f \cap g \subseteq \vee q_{(\gamma,\delta)}(g \circ f)$ and f is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and g be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset of an intra-regular ordered $\mathcal{LA}$-semigroup S. Since S is intra-regular, then for any $a \in S$ there exist $x, y \in S$ such that $a \leq (xa^2)y$. Now, by using medial, para medial laws, (1) we obtain

$$\begin{aligned} a &\leq (xa^2)y = (xa^2)(y_1y_2) = (y_2y_1)(a^2x) = a^2[(y_2y_1)x] \\ &= [x((y_2y_1)]a^2 = a[\{x(y_2y_1)\}a] \leq a[x\{(y_2y_1)\}\{(xa^2)y\}] \\ &= a[(xa^2)[\{x(y_2y_1)\}y]] = a[[y\{x(y_2y_1)\}](a^2x)] \\ &= a[a^2([y\{x(y_2y_1)\}]x] = a[(x[y\{x(y_2y_1)\}]a^2] \\ &= a[(x[y\{x(y_2y_1)\}])(aa)] \\ &= a[a((x[y\{x(y_2y_1)\}])a)] = a(au), \text{where } u = (x[y\{x(y_2y_1)\}])a. \end{aligned}$$

If for any $a \in S$ there exist $b, c \in S$ such that $a \leq bc$, then

$$\begin{aligned} \max\{(f \circ g)(a), \gamma\} &= \max\left\{\bigvee_{a \leq bc} \{g(b) \wedge f(c)\}, \gamma\right\} \\ &\geq \max\{\min\{g(a), f(au)\}, \gamma\} \\ &= \min[\max\{g(a), \gamma\}, \max\{f(au), \gamma\}] \\ &\geq \min\{\min\{g(a), \delta\}, \min\{f(a), \delta\}\} \\ &= \min[g(a), f(a), \delta] \\ &= \min\{(f \cap g)(a), \delta\}. \end{aligned}$$

Hence, by Lemma 3.2, we have $f \cap g \subseteq \vee q_{(\gamma,\delta)} g \circ f$. The rest of the proof is the same as in Theorem 4.1.

$(iii) \Rightarrow (ii)$ Let R be an right ideal and A any subset of S. Then, by Lemma 3.4 and (iii), we get

$$X^{\delta}_{\gamma(R \cap A)} = X^{\delta}_{\gamma(A \cap R)} =_{(\gamma,\delta)} X^{\delta}_{\gamma A} \cap X^{\delta}_{\gamma R} \subseteq \vee q_{(\gamma,\delta)} X^{\delta}_{\gamma A} \circ X^{\delta}_{\gamma R} =_{(\gamma,\delta)} X^{\delta}_{\gamma(AR]}.$$

So, by Lemma 3.4, we get $R \cap A \subseteq (RA]$. The rest of the proof is the same as in Theorem 4.1.

$(ii) \Rightarrow (i)$ Since $a^2 \in (Sa^2]$, where $(Sa^2]$ is a right ideal of S. So, by (ii),it is semiprime.Now by using Lemma 3.1, we get

$$a \in (Sa^2] \cap (Sa] = ((Sa^2](Sa]] = (Sa^2](Sa] \subseteq ((Sa^2)(Sa)] \subseteq ((Sa^2)S].$$

This shows that S is an intra-regular. □

Theorem 4.4. *Let S be an ordered $\mathcal{LA}$-semigroup with left identity. Then, the following conditions are equivalent : (i) S is intra-regular.*

(ii) For every right ideal R and any subset A of S, $A \cap R \subseteq (AR]$ and R is a semiprime.

(iii) For every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal g and for every $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy subset f of S, we have $f \cap g \subseteq \vee q_{(\gamma,\delta)} (f \circ g)$ and g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime

Proof. $(i) \Rightarrow (iii)$ Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset of an intra-regular ordered $\mathcal{LA}$-semigroup S. Since S is intra-regular, therefore for any $a \in S$ there exist $x, y \in S$ such that $a \leq (xa^2)y$. Now, by using medial, para medial laws and (1) we obtain

$$\begin{aligned} a &\leq (xa^2)y = [(xa^2)(y_1y_2)] = [(y_2y_1)(a^2x)] \\ &= [a^2((y_2y_1)x)] = [(x(y_2y_1))(aa)] \\ &= [a\{(x(y_2y_1))a\}] = a(ta), \\ \text{where } x(y_2y_1) &= t, \text{ and } ta = t[(xa^2)y] = (xa^2)(ty) = (yt)(a^2x) = a^2[(yt)x]. \end{aligned}$$

Thus

$$a \leq a(a^2v), \text{ where } (yt)x = v \text{ and } x(y_2y_1) = t.$$

If for any $a \in S$ there exist $b, c \in S$ such that $a \leq bc$, then

$$\begin{aligned} \max\{(f \circ g)(a), \gamma\} &= \max\left\{\bigvee_{a\leq bc}\{f(b) \wedge g(c)\}, \gamma\right\} \\ &\geq \max\{\min\{f(a), g(a^2v)\}, \gamma\} \\ &= \min\{\max\{f(a), \gamma\}, \max\{f(a^2v), \gamma\}\} \\ &\geq \min\{\min\{f(a), \delta\}, \min\{g(a), \delta\}\} \\ &= \min\{(f \cap g)(a), \delta\}. \end{aligned}$$

So, by Lemma 3.2, we have $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$.To prove that g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime seeTheorem 4.1.

$(iii) \Rightarrow (ii)$ Let R be an right ideal and A any subset of S. Then, by Lemma 3.4 and (iii), we get

$$X^{\delta}_{\gamma(A\cap R)} =_{(\gamma,\delta)} X^{\delta}_{\gamma A} \cap X^{\delta}_{\gamma R} \subseteq \vee q_{(\gamma,\delta)} X^{\delta}_{\gamma A} \circ X^{\delta}_{\gamma R} =_{(\gamma,\delta)} X^{\delta}_{\gamma(AR]}.$$

Hence, by Lemma 3.4, we get $A \cap R \subseteq (AR]$. For the rest of the proof see Theorem 4.1.

$(ii) \Rightarrow (i)$ Since $(Sa^2]$ is a right ideal containing a^2. By (ii),it is semiprime.Now by using Lemma 3.1, we get

$$\begin{aligned} a &\in (Sa] \cap (Sa^2] = ((Sa](Sa^2]] = (Sa](Sa^2] \subseteq ((Sa)(Sa^2)] \\ &= ((a^2S)(aS)] \subseteq ((a^2S)S] = (a^2(SS)S] = (((SS)a^2)S] \subseteq ((Sa^2)S]. \end{aligned}$$

Hence, S is an intra-regular. □

Theorem 4.5. *Let S be an ordered $\mathcal{LA}$-semigroup with left identity. Then, the following conditions are equivalent : (i) S is intra-regular.*

(ii) For any subsets A, B and any right ideal R of S, we have $A \cap B \cap R \subseteq ((AB)R]$ and R is semiprime ideal.

(iii) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets f, g and for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal h, we have $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}((f \circ g) \circ h)$ and h is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideal of S.

Proof. $(i) \Rightarrow (iii)$ Let f, g be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets and h be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of intra-regular ordered $\mathcal{LA}$-semigroup S with left identity. Since S is intra-regular, then for any $a \in S$ there exist $x, y \in S$ such that $a \leq (xa^2)y$. Now, by using medial law, para medial law and (1) we get

$$\begin{aligned} a &\leq (xa^2)y = (y_2y_1)(a^2x) = a^2[(y_2y_1)x] = [x(y_2y_1)]a^2 \\ &= a[\{x(y_2y_1)\}a] = a(pa), \text{ where } x(y_2y_1) = p, \text{ and} \\ pa &\leq p[(xa^2)y] = (xa^2)(py) = [(yp)(a^2x)] \\ &= a^2[(yp)x] = [x(yp)](aa) = a[\{(yp)\}a] \\ &= [a(qa)], \text{ where } x(yp) = q, \text{ and} \\ qa &= q[(xa^2)y] = (xa^2)(qy) = (yq)(a^2x) \\ &= a^2[(yq)x]. \end{aligned}$$

Thus

$a \leq a[a(a^2c)] = a[a^2(ac)] = a^2[a(ac)]$, where $(yq)x = c$ and $x(yp) = q$ and $(y_2y_1) = p$.

For any $a \in S$ there exist $b, c \in S$ such that $a \leq bc$, then

$$\begin{aligned} \max\{((f \circ g) \circ h)(a), \gamma\} &= \max\left\{\bigvee_{a \leq bc} \{(f \circ g)(b) \wedge h(c)\}, \gamma\right\} \\ &\geq \max\{\min\{(f \circ g)(aa), h(a(ac))\}, \gamma\} \\ &= \max\left\{\bigvee_{aa \leq pq} \{\{(f(p) \wedge g(q)\}, h(a(ac))\}, \gamma\right\} \\ &\geq \max[\min\{f(a), g(a)\}, h(a(ac))\}, \gamma] \\ &= \min[\max\{f(a), \gamma\}, \max\{g(a), \gamma\}, \max\{h(a(ac)), \gamma\}] \\ &\geq \min[\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}] \\ &= \min[\min\{f(a), g(a), h(a)\}, \delta] \\ &= \min\{(f \cap g \cap h)(a), \delta\}. \end{aligned}$$

Thus, by Lemma 3.2, we have $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$. The rmaining part of the proof is the the same as in Theorem 4.1.

$(iii) \Rightarrow (ii)$ Let R be a right ideal of S and A, B be any subsets of S. Then, by Lemma 3.4 and (iii), we get

$X^{\delta}_{\gamma(A\cap B)\cap R} =_{(\gamma,\delta)} X^{\delta}_{\gamma A} \cap X^{\delta}_{\gamma B} \cap X^{\delta}_{\gamma R} \subseteq \vee q_{(\gamma,\delta)}(X^{\delta}_{\gamma A} \circ X^{\delta}_{\gamma B}) \circ X^{\delta}_{\gamma R} =_{(\gamma,\delta)} X^{\delta}_{\gamma((AB)R]}$.

Hence, by Lemma 3.4, we get $(A \cap B) \cap R \subseteq ((AB)R]$. For the remaining part of the proof see Theorem 4.1.

$(ii) \Rightarrow (i)$ Since we know that Sa^2 is a right ideal of an ordered AG-groupoid S containing a^2. Therefore, by (ii) it is semiprime. Now by using Lemma 3.1, we get

$$\begin{aligned} a &\in (Sa] \cap (Sa] \cap (Sa^2] = ((Sa](Sa](Sa^2]] = (Sa](Sa](Sa^2] \\ &\subseteq ((Sa)(Sa)(Sa^2)] = (\{(SS)(aa)\}(Sa^2)] \subseteq ((Sa^2)S]. \end{aligned}$$

Hence, S is an intra-regular. □

Theorem 4.6. *Let S be an ordered $\mathcal{LA}$-semigroup with left identity. Then, the following conditions are equivalent : (i) S is intra-regular.*

(ii) For any right ideal R and for any subsets A, B of S, we have $A \cap B \cap R \subseteq ((AR)B]$ and R is semiprime ideal.

(iii) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets f, h and for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal g, we have $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}((f \circ g) \circ h)$ and g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime of S.

Proof. $(i) \Rightarrow (iii)$ Let f, h be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of intra-regular ordered $\mathcal{LA}$-semigroup S with left identity. Since S is intra-regular, then for any $a \in S$ there exist $x, y \in S$ such that $a \leq (xa^2)y$. Now by using medial law, para medial law and (1), we get

$$\begin{aligned}
a &\leq (xa^2)y = [(a(ax))y] = [y(xa))a], \\
y(xa) &\leq y[x((xa^2)y)] = y[(xa^2)(yx)] = [(xa^2)(xy^2)] \\
&= (y^2x)(a^2x) = a^2(y^2x^2) = (aa)(y^2x^2) = (x^2y^2)(aa) = a[(x^2y^2)a], \\
(x^2y^2)a &\leq (x^2y^2)[(xa^2)y] = (xa^2)[(x^2y^2)y] = [y(y^2x^2)](a^2x) = a^2[\{(y(y^2x^2)\}x]
\end{aligned}$$

Thus

$$a \leq [a(a^2v)]a, \text{where } \{y(y^2x^2)\}x = v.$$

For any $a \in S$ there exist $b, c \in S$ such that $a \leq bc$. Then

$$\begin{aligned}
\max\{((f \circ g) \circ h)(a), \gamma\} &= \max\left\{\bigvee_{a \leq bc} \{(f \circ g)(b) \wedge h(c)\}, \gamma\right\} \\
&\geq \max\{\min\{(f \circ g)(a(a^2v)), h(a)\}, \gamma\} \\
&= \max\left\{\bigvee_{a(a^2v) \leq cd} \{\{(f(c) \wedge g(d)\}, h(a)\}, \gamma\right\} \\
&\geq \max[\min\{f(a) \wedge g(a^2v)\}, h(a)\}, \gamma] \\
&= \min[\max\{f(a), \gamma\}, \max\{g(a^2v), \gamma\}, \max\{h(a), \gamma\}] \\
&\geq \min[\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}] \\
&= \min[\min\{f(a), g(a), h(a)\}, \delta] \\
&= \min\{(f \cap g \cap h)(a), \delta\}.
\end{aligned}$$

So, by Lemma 3.2, we have $(f \cap g) \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$. The rest of the proof is the the same as in Theorem 4.1.

$(iii) \Rightarrow (ii)$ Let R be a right ideal and A, B be any subsets of S. Then, by Lemma 3.4 and (iii), we get

$$X^\delta_{\gamma(A \cap R) \cap B} =_{(\gamma,\delta)} X^\delta_{\gamma A} \cap X^\delta_{\gamma R} \cap X^\delta_{\gamma B} \subseteq \vee q_{(\gamma,\delta)}(X^\delta_{\gamma A} \circ X^\delta_{\gamma R}) \circ X^\delta_{\gamma B} =_{(\gamma,\delta)} X^\delta_{\gamma((AR)B]}.$$

Hence, by Lemma 3.4, we get $(A \cap R) \cap B \subseteq ((AR)B]$. The rest of the proof is the the same as in Theorem 4.1.

$(ii) \Rightarrow (i)$ We know $(Sa^2]$ is a right ideal of an ordered $\mathcal{LA}$-semigroup S containing a^2. Therefore, by (ii) it is semiprime. Now by using Lemma 3.1,we get

$$
\begin{aligned}
a &\in (Sa] \cap (Sa^2] \cap (Sa] = ((Sa](Sa^2](Sa]] = (Sa](Sa^2](Sa] \subseteq ((Sa)(Sa^2)(Sa)] \\
&\subseteq ((S(Sa^2)S)] = ((S(Sa^2)(SS))] = ((SS)\{(Sa^2)S\}] = (S\{(Sa^2)S\}) \\
&= ((Sa^2)(SS)]
\end{aligned}
$$

Hence, S is an intra-regular. □

Theorem 4.7. *Let S be an ordered $\mathcal{LA}$-semigroup with left identity. Then, the following conditions are equivalent : (i) S is intra-regular.*

(ii) For any right ideal R and for any subsets A, B of S, we have $R \cap A \cap B \subseteq ((RA)B]$ and R is semiprime ideal.

(iii) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets g, h and of S, we have $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}((f \circ g) \circ h)$ and f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideal of S.

Proof. $(i) \Rightarrow (iii)$ Let g, h be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets and f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of intra-regular ordered $\mathcal{LA}$-semigroup S with left identity. Since S is intra-regular, then for any $a \in S$ there exist $x, y \in S$ such that $a \leq (xa^2)y$. Now, by using medial law, para medial law and (1), we get

$$
\begin{aligned}
a &\leq (xa^2)y = (x(aa))y = (a(ax))y = (y(xa))a, \\
y(xa) &\leq y[x\{(xa^2)y\}] = y[(xa^2)(yx)] = (xa^2)(xy^2) = (y^2x)(a^2x) = a^2[(y^2x)x] \\
&= [(x^2y^2)a]a and \\
(x^2y^2)a &\leq (x^2y^2)[(xa^2)y] = (xa^2)[(x^2y^2)y] = [y(y^2x^2)](a^2x) = a^2[\{y(y^2x)\}x] \\
&= a^2v.
\end{aligned}
$$

Thus

$$a \leq [(a^2v)a]a, \text{ where } [y(y^2x)]x = v.$$

For any $a \in S$ there exist $b, c \in S$ such that $a \leq bc$. Then

$$
\begin{aligned}
\max\{((f \circ g) \circ h)(a), \gamma\} &= \max\left\{\bigvee_{a \leq bc} \{(f \circ g)(b) \wedge h(c)\}, \gamma\right\} \\
&\geq \max\{\min\{(f \circ g)((a^2v)a), h(a)\}, \gamma\} \\
&= \max\left\{\bigvee_{(a^2v)a \leq pq} \{\{(f(p) \wedge g(q)\}, h(a)\}, \gamma\right\} \\
&\geq \max[\min\{f(a^2v), g(a)\}, h(a)\}, \gamma] \\
&= \min[\max\{f(a^2v), \gamma\}, \max\{g(a), \gamma\}, \max\{h(a), \gamma\}] \\
&\geq \min[\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a(ac)), \delta\}] \\
&= \min[\min\{f(a), g(a), h(a)\}, \delta] \\
&= \min\{(f \cap g \cap h)(a), \delta\}.
\end{aligned}
$$

So, by using Lemma 3.2, we have $(f \cap g) \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$. The remaining part of the proof is the the same as in Theorem 4.1.

$(iii) \Rightarrow (ii)$ Let R be a right ideal and A, B be any orbitrary subsets of S. Then, by Lemma 3.4 and (iii), we get

$X^{\delta}_{\gamma(R\cap A)\cap B} =_{(\gamma,\delta)} X^{\delta}_{\gamma R} \cap X^{\delta}_{\gamma A} \cap X^{\delta}_{\gamma B} \subseteq \vee q_{(\gamma,\delta)} (X^{\delta}_{\gamma R} \circ X^{\delta}_{\gamma A}) \circ X^{\delta}_{\gamma B} =_{(\gamma,\delta)} X^{\delta}_{\gamma((RA)B]}$.

Therefore, by using Lemma 3.4, we get $(R \cap A) \cap B \subseteq ((RA)B]$. The rest of the proof is the the same as in Theorem 4.1.

$(ii) \Rightarrow (i)$ We know $(Sa^2]$ is a right ideal of an ordered $\mathcal{LA}$-semigroup S containing a^2. Therefore, by (ii) it is semiprime. Now by using Lemma 3.1, we get

$$\begin{aligned} a &\in (Sa^2] \cap (Sa] \cap (Sa] = ((Sa^2](Sa](Sa]] = ((Sa^2](Sa](Sa] \subseteq ((Sa^2)(Sa)(Sa)] \\ &\subseteq (((Sa^2)S)S] = ((SS)(Sa^2)] = ((SS)\{(SS(aa)\}] = (SS\{(aa)SS\}] = ((SS)(a^2S)] \\ &= ((Sa^2)(SS)] \subseteq (Sa^2)S. \end{aligned}$$

Hence, S is an intra-regular. □

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