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Strongly almost summable sequence space of fuzzy numbers defined by a modulus function

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ABSTRACT. The object of the present article is to introduce and study a generalized class $[\hat{A}, f, p](F)$ of sequences of fuzzy numbers using the modulus function f, where $p = (p_k)$ is a bounded sequence of positive real numbers, $A = (a_{nk})_{n,k=1,2,3...}$ is an infinite matrix of non-negative real numbers. With the help of the paranorm, it is proved that $[\hat{A}, f, p](F)$ is a complete paranormed space. Besides studying the topological properties of this class, some inclusion relations and the properties like, solidity, convergence free *etc.* also studied.

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1. INTRODUCTION

Zadeh [33] has introduced the concept of fuzzy set theory. The theory of fuzzy sets has become an active area of research for the last forty years. It has wide range of applications in the field of science and engineering e.g. export system, fuzzy control, operations research *etc.* In recent years, the topological aspects of fuzzy sets have received serious consideration from the wider mathematical community.

Recently, Matloka [17], Nanda [19], Mursaleen and Basarır [18], Talo and Basar [32], Raj et al. [23], Raj and Sharma [24], Raj et al. [25] and others have studied various classes of sequences of fuzzy numbers in an analogous way as Simons [29], Maddox [13, 16], Connor [1] and so on studied these spaces for scalar valued field of real or complex numbers.

Nakano [21] has introduced the concept of modulus function. Later on, Ruckle [26], Maddox [16] and several authors have constructed various types of sequence spaces by using modulus function. Using the concept of modulus function Talo and Basar [32], Sarma [27], Esi [4], Esi and Açik$\breve{g}$oz [7, 8, 9] and several authors have generalized different types of sequence spaces of fuzzy numbers.

A sequence $x \in \ell_\infty$ is said to be almost convergent if all Banach limits of x coincide. Lorentz [12] has defined

$$\hat{c} = \left\{ x \in w : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_{k+m} \text{ exists, uniformly in } m \right\}.$$

Maddox [15] has defined x to be strongly almost convergent to a number L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_{k+m} - L| = 0 \text{ uniformly in } m.$$

$[\hat{c}]$ denotes the space of all strongly almost convergent sequences. Later on, Nanda [20], Khan [11] and many authors have constructed different types of sequence spaces by using the concept of strongly almost convergent sequences of real or complex numbers.

Using the concept of strongly almost convergent sequences Esi [5, 6], Gökhan et al. [10], Savaş [28], Subramanian and Esi [30] and many authors have constructed various sequence spaces of fuzzy numbers.

Motivated by the development of various classes of sequence spaces of fuzzy numbers by the earlier authors, the present work is aimed to introduce a general class $[\hat{A}, f, p](F)$ of sequences of fuzzy numbers using infinite matrix and modulus function. Various topological properties and algebraic properties such as solidity, convergence free *etc.* and some inclusion relations are also obtained for this class.

2. Preliminaries

Let D denote the set of all closed and bounded intervals $A = [a_1, a_2]$ on the real line $\mathbb{R}$. For $A, B \in D$, we define

$$d(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$$

, where $A = [a_1, a_2]$ and $B = [b_1, b_2]$. It is known that (D, d) is a complete metric space.

A fuzzy number X is a fuzzy set on $\mathbb{R}$, i.e. a mapping $X : \mathbb{R} \to [0, 1]$ associating each real number t with its grade of membership $X(t)$, which is normal and fuzzy convex.

A fuzzy number X is said to be upper-semicontinuous if, for each $\varepsilon > 0$, $X^{-1}([0, a+\varepsilon))$, for all $a \in [0, 1]$ is open in the usual topology of $\mathbb{R}$.

The set of all upper-semicontinuous, normal, convex fuzzy numbers with compact support is denoted by $L(\mathbb{R})$. Throughout this paper, by a fuzzy number we mean that the number belongs to $L(\mathbb{R})$.

The set $\mathbb{R}$ of all real numbers can be embedded in $L(\mathbb{R})$. For $r \in \mathbb{R}$, $\overline{r} \in L(\mathbb{R})$ is defined by $\overline{r}(t) = 1$ for $r = t$ and 0 for $r \neq t$.

The $\alpha-$level set X^α of $X \in L(\mathbb{R})$ is defined as

$$X^\alpha = \begin{cases} t : X(t) \geq \alpha & \text{if } \alpha \in (0, 1], \\ t : X(t) > 0 & \text{if } \alpha = 0. \end{cases}$$

The set $L(\mathbb{R})$ forms a linear space under addition $X+Y$ and scalar multiplication $\lambda X,\ \lambda \in \mathbb{R}$ in terms of $\alpha-$ level sets as defined below:

$$[X+Y]^\alpha = [X]^\alpha + [Y]^\alpha \text{ and } [\lambda X]^\alpha = \lambda [X]^\alpha \quad \text{for each } 0 \leq \alpha \leq 1.$$

For each $\alpha \in [0,1]$, the set X^α is a closed, bounded and nonempty interval of $\mathbb{R}$.

Let $\overline{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ be defined by

$$\overline{d}(X,Y) = \sup_{0\leq\alpha\leq 1} d(X^\alpha, Y^\alpha).$$

Then Diamond and Kloeden [3] proved that $\overline{d}$ defines a metric on $L(\mathbb{R})$ and $(L(\mathbb{R}, \overline{d})$ is a complete metric space.

Definition 2.1. A metric $\overline{d}$ on $L(\mathbb{R})$ is said to be translation invariant if $\overline{d}(X+Z, Y+Z) = \overline{d}(X,Y)$ for all $X, Y, Z \in L(\mathbb{R})$.

Lemma 2.2 ([14]). *Let a_k, b_k for all k be sequences of complex numbers and (p_k) be a bounded sequence of positive real numbers, then*

$$|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k})$$

and

$$|\lambda|^{p_k} \leq \max(1, |\lambda|^H)$$

, where $C = \max(1, 2^{H-1}), H = \sup p_k$ and λ is any complex number.

Lemma 2.3 ([14]). *Let $a_k \geq 0,\ b_k \geq 0$ for all k be sequences of complex numbers and $1 \leq p_k \leq \sup p_k < \infty$, then*

$$\Big(\sum_k |a_k + b_k|^{p_k}\Big)^{\frac{1}{M}} \leq \Big(\sum_k |a_k|^{p_k}\Big)^{\frac{1}{M}} + \Big(\sum_k |b_k|^{p_k}\Big)^{\frac{1}{M}}$$

, where $M = \max(1, H),\ H = \sup p_k$.

Lemma 2.4 ([31]). *(i) $\overline{d}(XY, \overline{0}) \leq \overline{d}(X, \overline{0})\overline{d}(Y, \overline{0})$ for all $X,\ Y \in L(\mathbb{R})$.*
(ii) If $X_k \to X$ as $k \to \infty$ then $\overline{d}(X_k, \overline{0}) \to \overline{d}(X, \overline{0})$ as $k \to \infty$.

Lemma 2.5 ([31]). *Let $X, Y, Z, V \in L(\mathbb{R})$ and $k \in \mathbb{R}$. Then,*

(i) $\overline{d}(kX, kY) = |k|\overline{d}(X,Y)$.
(ii) $\overline{d}(X+Z, Y+Z) = \overline{d}(X,Y)$.
(iii) $\overline{d}(X+Z, Y+V) \leq \overline{d}(X,Y) + \overline{d}(Z,V)$.
(iv) $|\overline{d}(X,\overline{0}) - \overline{d}(Y,\overline{0})| \leq \overline{d}(X,Y) \leq \overline{d}(X,\overline{0}) + \overline{d}(Y,\overline{0})$.

Lemma 2.6 ([22]). *Let f be a modulus and let $0 < \delta < 1$. Then for each $x \geq \delta$, we have $f(x) \leq 2f(1)\delta^{-1}x$.*

Lemma 2.7 ([16]). *Let f be any modulus with $\lim_{t\to\infty} \dfrac{f(t)}{t} = \gamma > 0$. Then there is a constant $\beta > 0$ such that $f(t) \geq \beta t$ for all $t \geq 0$.*

Definition 2.8 ([2]). A sequence space E is said to be normal (or solid) if, whenever $x = (x_k)$ is in E and $|y_k| \leq |x_k|$ for every k, then $y = (y_k)$ is in E.

Definition 2.9 ([2]). A sequence space E is said to be convergence free when, if $x = (x_k)$ is in E and if $y_k = 0$ whenever $x_k = 0$ then the sequence $y = (y_k)$ is in E.

3. Major Section

Consider a new class of fuzzy numbers as follows:

$$[\hat{A}, f, p](F) = \Big\{ X = (X_k) \in W(F) : \sum_{k=1}^{\infty} a_{nk}(f(\overline{d}(X_{k+m}, X_0)))^{p_k} \text{ converges for each } n \text{ and } \sum_{k=1}^{\infty} a_{nk}(f(\overline{d}(X_{k+m}, X_0)))^{p_k} \to 0 \text{ as } n \to \infty, \text{ uniformly in } m \Big\}$$

where f is a modulus function, $p = (p_k)$ is a bounded sequence of positive real numbers and $A = (a_{nk})_{n,k=1,2,3...}$ is an infinite matrix of non-negative real numbers such that $\sum_{k=1}^{\infty} a_{nk}$ converges for each n and $\sup_n \sum_{k=1}^{\infty} a_{nk} < \infty$ but $\sum_{k=1}^{\infty} a_{nk} \nrightarrow 0$ as $n \to \infty$.
It can be seen that for $f(x) = x$ and A as a regular matrix, this space $[\hat{A}, f, p](F)$ reduces to the space $[\hat{A}, p](F)$ defined by Savaş [28].

Theorem 3.1. *The class $[\hat{A}, f, p](F)$ of sequences of fuzzy numbers is closed under addition and scalar multiplication over the real field $\mathbb{R}$.*

Proof. Using Lemma 2.2, Lemma 2.5, subadditivity property of modulus function f and the result $f(\lambda x) \leq (1 + [|\lambda|])f(x)$, it is easy to prove that the class $[\hat{A}, f, p](F)$ of sequences of fuzzy numbers is closed under addition and scalar multiplication over the real field $\mathbb{R}$. □

Lemma 3.2. *For any modulus function f, the class $[\hat{A}, f, p](F)$ is a complete paranormed space with respect to the paranorm $g(X-Y) = \sup_{n,m} \left(\sum_{k=1}^{\infty} a_{nk} \Big(f(\overline{d}(X_{k+m}, Y_{k+m})) \Big)^{p_k} \right)^{\frac{1}{M}}$ where $(X_k) \in [\hat{A}, f, p](F)$ and (p_k) is a bounded sequence of positive real numbers such that* $\inf p_k > 0$.

Proof. Obviously $g(\theta) = 0, g(-X) = g(X)$ for all $X \in [\hat{A}, f, p](F)$ and subadditivity property of g follows from Lemma 2.3 and subadditivity property of f.

Next, we prove the continuity of scalar multiplication under g.
For $X \to \overline{0}$, $\lambda \to 0$ implies $\lambda X \to \overline{0}$ and also for $X \to \overline{0}$, λ fixed implies $\lambda X \to \overline{0}$. Now, let $\lambda \to 0$, X be fixed. For $|\lambda| < 1$, consider

$$\begin{aligned}
\sum_k a_{nk}\Big(f(\bar{d}(\lambda X_{k+m},\bar{0}))\Big)^{p_k} &\leq \sum_k a_{nk}\Big(f(\bar{d}(\lambda X_{k+m},\lambda X_0)+\bar{d}(\lambda X_0,\bar{0}))\Big)^{p_k}\Big) \\
&= \sum_k a_{nk}\Big(f(|\lambda|\bar{d}(X_{k+m},X_0)+|\lambda|\bar{d}(\lambda X_0,\bar{0}))\Big)^{p_k}\Big) \\
&\leq \sum_k a_{nk}\Big(f(|\lambda|\bar{d}(X_{k+m},X_0))+f(\bar{d}(\lambda X_0,\bar{0}))\Big)^{p_k}\Big) \\
&\leq C\sum_k a_{nk}\Big(f(|\lambda|\bar{d}(X_{k+m},X_0))\Big)^{p_k} \\
&\quad +C\sum_k a_{nk}\Big(f(|\lambda|\bar{d}(X_0,\bar{0}))\Big)^{p_k} \\
&\leq C(1+[|\lambda|])^H\sum_{k>N_0} a_{nk}\Big(f(\bar{d}(X_{k+m},X_0))\Big)^{p_k} \\
&\quad +C\sum_{k\leq N_0} a_{nk}\Big(f(|\lambda|\bar{d}(X_{k+m},X_0))\Big)^{p_k} \\
&\quad +C\sum_k a_{nk}\Big(f(|\lambda|\bar{d}(X_0,\bar{0}))\Big)^{p_k}. \qquad (3.1)
\end{aligned}$$

Since $X \in [\hat{A},f,p](F)$, so given $\varepsilon>0$, there exist a positive integer N_0 such that

$$\sum_{k>N_0} a_{nk}\Big(f(\bar{d}(X_{k+m},X_0))\Big)^{p_k} < \frac{\varepsilon}{2C(1+[|\lambda|])^H} \text{ for all } k>N_0 \text{ and for all } m. \qquad (3.2)$$

For each $k\leq N_0$, and for all m, by the continuity of f, as $\lambda\to 0$,

$$\sum_{k\leq N_0} a_{nk}\Big(f(|\lambda|\bar{d}(X_{k+m},X_0))\Big)^{p_k}+\sum_k a_{nk}\Big(f(|\lambda|\bar{d}(X_0,\bar{0}))\Big)^{p_k}\to 0.$$

Then choose $\delta<1$ such that $|\lambda|<\delta$ implies

$$C\sum_{k\leq N_0} a_{nk}\Big(f(|\lambda|\bar{d}(X_{k+m},X_0))\Big)^{p_k}+C\sum_k a_{nk}\Big(f(|\lambda|\bar{d}(X_0,\bar{0}))\Big)^{p_k} < \frac{\varepsilon}{2} \text{ for all } m. \qquad (3.3)$$

Using equation (3.2) and equation (3.3) in equation (3.1), we get

$$\sum_k a_{nk}\Big(f(\bar{d}(\lambda X_{k+m},\bar{0}))\Big)^{p_k} < \varepsilon \text{ for all } n \text{ and for all } m.$$

i.e.

$$\sup_{n,m}\left(\sum_k a_{nk}\Big(f(\bar{d}(\lambda X_{k+m},\bar{0}))\Big)^{p_k}\right)^{\frac{1}{M}} < \varepsilon$$

and hence $g(\lambda X)\to 0$ as $\lambda\to 0$ which implies that g is a paranorm.

To prove the completeness, let (X^u) be any Cauchy sequence in $[\hat{A},f,p](F)$ where $X^u=(X^u_k)\in[\hat{A},f,p](F)$. Then given $\varepsilon>0$, there exist $u_0\in\mathbb{N}$ such that

$$g(X^u-X^v)<\varepsilon \text{ for all } u,v\geq u_0.$$

i.e.

$$\sup_{n,m}\left(\sum_k a_{nk}\Big(f(\overline{d}(X^u_{k+m},X^v_{k+m}))\Big)^{p_k}\right)^{\frac{1}{M}} < \varepsilon \text{ for all } u,v\geq u_0. \tag{3.4}$$

i.e.

$$\sum_k a_{nk}\Big(f(\overline{d}(X^u_{k+m},X^v_{k+m}))\Big)^{p_k} < \varepsilon \text{ for all } u,v\geq u_0, \ \text{ for all } n \text{ and } m. \tag{3.5}$$

Proceeding successively, we get $g(X^u - X) < \varepsilon$ for all $u \geq u_0$.
Now to show $X = (X_k) \in [\hat{A}, f, p](F)$.
Since $X^u = (X^u_k) \in [\hat{A}, f, p](F)$, so for each u, there exist $X^u_0 \in L(\mathbb{R})$ and $n_u \in \mathbb{N}$ such that

$$\sum_k a_{nk}\Big(f(\overline{d}(X^u_{k+m},X^u_0))\Big)^{p_k} < \varepsilon \text{ for all } n\geq n_u \text{ and for all } m. \tag{3.6}$$

Similarly, $X^v = (X^v_k) \in [\hat{A}, f, p](F)$, so for each u, there exist $X^v_0 \in L(\mathbb{R})$ and $n_v \in \mathbb{N}$ such that

$$\sum_k a_{nk}\Big(f(\overline{d}(X^v_{k+m},X^v_0))\Big)^{p_k} < \varepsilon \text{ for all } n\geq n_v \text{ and for all } m. \tag{3.7}$$

Now let $u, v \geq u_0$ and $n_0 = \max(n_u, n_v)$. Then by using equation (3.4), equation (3.5), equation (3.7), subadditivity property of f and Lemma 2.2, we have

$$\begin{aligned}\sum_k a_{nk}\Big(f(\overline{d}(X^u_0,X^v_0))\Big)^{p_k} &\leq C\sum_k a_{nk}\Big(f(\overline{d}(X^u_0,X^u_{k+m}))\Big)^{p_k}\\ &\quad +C\sum_k a_{nk}\Big(f(\overline{d}(X^u_{k+m},X^v_{k+m}))\Big)^{p_k}\\ &\quad +C\sum_k a_{nk}\Big(f(\overline{d}(X^v_{k+m},X^v_0))\Big)^{p_k}\\ &< 3C\varepsilon \text{ for all } u,v\geq u_0, \ \text{for all } n\geq n_0 \text{ and for all } m.\end{aligned} \tag{3.8}$$

Now using the fact that modulus function is monotone and for suitable choice of $\varepsilon_3 > 0$, we have $\overline{d}(X^u_0, X^v_0) < \varepsilon_3$ for all $u, v \geq u_0$. $\Rightarrow (X^u_0)$ is a Cauchy sequence in $L(\mathbb{R})$. But $(L(\mathbb{R}), \overline{d})$ is a complete metric space. So let $X^u_0 \to X_0$ as $u \to \infty$. Substituting this value in equation (3.8), we get

$$\sum_k a_{nk}\Big(f(\overline{d}(X^u_0,X_0))\Big)^{p_k} < 3C\varepsilon \text{ for all } u,v\geq u_0 \text{ and for all } n\geq n_0. \tag{3.9}$$

Now consider

$$\begin{aligned}\sum_k a_{nk}\Big(f(\overline{d}(X_{k+m},X_0))\Big)^{p_k} &\leq C\sum_k a_{nk}\Big(f(\overline{d}(X_{k+m},X^{u_0}_{k+m}))\Big)^{p_k}\\ &\quad +C\sum_k a_{nk}\Big(f(\overline{d}(X^{u_0}_{k+m},X^{u_0}_0))\Big)^{p_k}\\ &\quad +C\sum_k a_{nk}\Big(f(\overline{d}(X^{u_0}_0,X_0))\Big)^{p_k}\\ &< 2C\varepsilon + 3C^2\varepsilon \text{ for all } n\geq n_0 \text{ and for all } m.\end{aligned}$$

Which implies $X = (X_k) \in [\hat{A}, f, p](F)$ and hence $[\hat{A}, f, p](F)$ is a complete paranormed space. □

Theorem 3.3. *If* $1 < h = \inf p_k \leq \sup p_k < \infty$*, then for any modulus function* f *and for any regular matrix A, we have* $[\hat{A}, p](F) \subseteq [\hat{A}, f, p](F)$ *where*

$$[\hat{A}, p](F) = \Big\{X = (X_k) : \sum_k a_{nk}[\overline{d}(X_k, \overline{0})]^{p_k} \to 0 \text{ as } n \to \infty, \text{ uniformly in } m\Big\}.$$

Proof. Since A is a regular matrix, by using Lemma 2.6, this theorem can be easily proved. □

Theorem 3.4. *If* $\gamma = \lim\limits_{t\to\infty} \frac{f(t)}{t} > 0$*, then for any non-negative infinite matrix* A *and any bounded sequence of positive real numbers* (p_k)*,* $[\hat{A}, f, p](F) = [\hat{A}, p](F)$ *where*

$$[\hat{A}, p](F) = \Big\{X = (X_k) : \sum_k a_{nk}[\overline{d}(X_k, \overline{0})]^{p_k} \to 0 \text{ as } n \to \infty, \text{ uniformly in } m\Big\}.$$

Proof. In Theorem 3.3, we have shown that $[\hat{A}, p](F) \subseteq [\hat{A}, f, p](F)$.

Next we prove that $[\hat{A}, f, p](F) \subseteq [\hat{A}, p](F)$. Let $X \in [\hat{A}, f, p](F)$ i.e. $\sum\limits_{k=1}^{\infty} a_{nk}\Big(f(\overline{d}(X_{k+m}, X_0))\Big)^{p_k} \to 0$ as $n \to \infty$, uniformly in m.

Since $\gamma = \lim\limits_{t\to\infty} \frac{f(t)}{t} > 0$, there is a constant $\beta > 0$ such that $f(t) > \beta t$ for every $t \in \mathbb{R}$. This gives

$$\beta\overline{d}(X_{k+m}, X_0) < f(\overline{d}(X_{k+m}, X_0)).$$

Now

$$\beta^{p_k} > \beta^H \text{ if } \beta < 1 \text{ and } \beta^{p_k} < \beta^H \text{ if } \beta > 1.$$

While

$$\beta^{p_k} > \beta^h \text{ if } \beta > 1 \text{ and } \beta^{p_k} < \beta^h \text{ if } \beta < 1.$$

So, consider $T = H$ if $\beta < 1$, otherwise $T = h$ where $h = \inf(p_k)$ and $H = \sup(p_k)$.

$$\Rightarrow \sum_{k=1}^{\infty} a_{nk}\beta^T\Big(\overline{d}(X_{k+m}, X_0)\Big)^{p_k} < \sum_{k=1}^{\infty} a_{nk}\Big(f(\overline{d}(X_{k+m}, X_0))\Big)^{p_k}$$
$$\Rightarrow \sum_{k=1}^{\infty} a_{nk}\Big(\overline{d}(X_{k+m}, X_0)\Big)^{p_k} < \frac{1}{\beta^T}\sum_{k=1}^{\infty} a_{nk}\Big(f(\overline{d}(X_{k+m}, X_0))\Big)^{p_k}$$

As $X \in [\hat{A}, f, p](F)$, which implies $\sum\limits_{k=1}^{\infty} a_{nk}\Big(\overline{d}(X_{k+m}, X_0)\Big)^{p_k} \to 0$ as $n \to \infty$.

That is $X \in [\hat{A}, p](F)$ and hence $[\hat{A}, f, p](F) \subseteq [\hat{A}, p](F)$. □

Theorem 3.5. *For any two modulus functions* f *and* g *we have*

(i) *If* $f(t) \leq g(t)$ *for all* $t \in [0, \infty)$*, then* $[\hat{A}, g, p](F) \subseteq [\hat{A}, f, p](F)$.

(ii) $[\hat{A}, f, p](F) \cap [\hat{A}, g, p](F) \subseteq [\hat{A}, f + g, p](F)$.

Proof. It can be established using standard technique. □

Theorem 3.6. *The class $[\hat{A}, f, p](F)$ is not solid, in general.*

Proof. The result follows from the following example. □

Example 3.7. Let A=I, the identity matrix, $f(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$. Consider the sequence $X = (X_k)$ as $X_k = \overline{2}$ for all $k \in \mathbb{N}$ and $X_0 = \overline{2}$. Then $X \in [\hat{A}, f, p](F)$.
Now consider the sequence $Y = (Y_k)$ as follows:

$$Y_k = \begin{cases} X_k & \text{if } k = 2^n \\ \overline{0} & \text{if } k \neq 2^n \end{cases}$$

Then $\overline{d}(Y_k, \overline{0}) \leq \overline{d}(X_k, \overline{0})$ for all $k \in \mathbb{N}$. But $Y \notin [\hat{A}, f, p](F)$.

Theorem 3.8. *The class $[\hat{A}, f, p](F)$ is not convergence free, in general.*

Proof. The result follows from the following example. □

Example 3.9. Let A=I, the identity matrix, $f(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$. Consider the sequence $X = (X_k)$ as follows:

$$X_k(t) = \begin{cases} kt - k + 1 & \text{if } 1 - \frac{1}{k} \leq t \leq 1 \\ k - kt + 1 & \text{if } 1 \leq t \leq 1 + \frac{1}{k} \\ 0 & \text{otherwise} \end{cases}$$

and the fuzzy number $X_0 = \overline{1}$. Then $\sum_{k=1}^{\infty} a_{nk} \Big(f(\overline{d}(X_{k+m}, X_0)) \Big)^{p_k} \to 0$ as $k \to \infty$. Which implies $X \in [\hat{A}, f, p](F)$. Now consider the sequence $Y = (Y_k)$ as follows:

$$Y_k(t) = \begin{cases} \frac{1}{k}(t - 1 + k) & \text{if } 1 - k \leq t \leq 1 \\ \frac{1}{k}(1 + k - t) & \text{if } 1 \leq t \leq 1 + k \\ 0 & \text{otherwise} \end{cases}$$

and the fuzzy number $Y_0 = \overline{1}$. Then $\sum_{k=1}^{\infty} a_{nk} \Big(f(\overline{d}(Y_{k+m}, Y_0)) \Big)^{p_k} \to 0$ as $k \to \infty$. Which implies $Y \notin [\hat{A}, f, p](F)$ and hence the sequence space $[\hat{A}, f, p](F)$ is not convergence free.

4. Conclusions

The sequence spaces of fuzzy numbers are introduced by Matloka [17]. In this paper, we have introduced a general class $[\hat{A}, f, p](F)$ of scalar valued sequence space of fuzzy numbers using infinite matrices and modulus function. It is shown that if (p_k) is a bounded sequence of positive real numbers such that $\inf p_k > 0$, then $[\hat{A}, f, p](F)$ is a complete paranormed space with respect to the paranorm
$g(X - Y) = \sup_{n,m} \Big(\sum_{k=1}^{\infty} a_{nk} \Big(f(\overline{d}(X_{k+m}, Y_{k+m})) \Big)^{p_k} \Big)^{\frac{1}{M}}$ where $(X_k) \in [\hat{A}, f, p](F)$.
Various topological properties and inclusion relations are obtained for this sequence space.

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