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Some results on fuzzy semi-inner product space

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Abstract. We have modified the concept of fuzzy real semi-inner product space to get more results of fuzzy semi-inner product space. Also we have illustrated the formation of real semi-inner product from fuzzy real semi-inner product and vice-versa. Finally, we have defined fuzzy semi-inner product on the cartesian product of two fuzzy semi-inner product spaces and also introduced fuzzy generalized semi-inner product space.

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1. Introduction and Preliminaries

The concept of fuzzy sets is first introduced by L. A. Zadeh [20]. Later on, Matloka[16], Nanda[17] discussed the sequences of fuzzy numbers and they have introduced the sequence spaces l_p^F, l_∞^F of fuzzy numbers. The concepts such as statistical convergence, summabilty etc. have also extended to fuzzy numbers by several mathematicians.

While studying fuzzy topological vector spaces, Katsaras [10] in 1984, first introduced the notion of fuzzy norm on a linear space. Later on, several mathematicians such as Felbin [8], Cheng and Mordeson[6], Bag and Samanta [1, 2, 3, 4] and others have defined fuzzy normed linear spaces using different approaches.

On the other hand, the study on fuzzy inner product spaces are relatively new and some work has been done in fuzzy inner product spaces. R. Biswas [5], A. M. El-Abyad and H. M. Hamouly [7] are among the first who gave a meaningful definition of fuzzy inner product space and the associated fuzzy norm functions. Recently, the fuzzy inner product is introduced by Kohli and Kumar [11], Majumder and Samanta [13], Goudarzi and Vaezpour [9] separately which is modified by Mukherjee and Bag [14, 15].

Lumer [12] has introduced the notion of semi-inner product space for real or complex numbers. Ramakrishnan [19] gave the idea of fuzzy points and fuzzy normed algebra

$C(I)$. He has defined fuzzy semi-inner product in terms of fuzzy points. We now give some definitions which are used in sequel.

Definition 1.1 ([1]). Let U be a linear space over a field F (real or complex). A fuzzy subset N of $U \times R$ is called a fuzzy norm on U if for all $x, u \in U$ and $c \in F$, the following conditions are satisfied:
(N1) For all $t \in R$, with $t \leq 0$, $N(x,t) = 0$.
(N2) $(\forall\, t \in R, t > 0, N(x,t) = 1)$ iff $x = \underline{0}$.
(N3) $\forall\, t \in R, t > 0, N(cx,t) = N(x, t/|c|)$ if $c \neq 0$.
(N4) $\forall\, s,t \in R,\ x,u \in U,\ \ N(x+u, s+t) \geq \min\{N(x,s), N(u,t)\}$.
(N5) $N(x,\cdot)$ is a non-decreasing function of R and $\lim\limits_{t\to+\infty} N(x,t) = 1$.

The pair (U,N) is referred to as a fuzzy normed linear space.

Definition 1.2 ([15]). Let X be a linear space over R. Then a fuzzy subset $F : X \times X \times R \to [0,1]$ is called fuzzy real inner product on X if for all $x,y,z \in X$ and $t \in R$, the following conditions hold.
(F1) $F(x,x,t) = 0\ \forall\, t < 0$.
(F2) $(F(x,x,t) = 1\ \forall\, t > 0)$ iff $x = \underline{0}$.
(F3) $F(x,y,t) = F(y,x,t)$.

(F4) $F(ax,y,t) = \begin{cases} F(x,y,\frac{t}{a}), & a > 0 \\ H(t), & a = 0 \\ 1 - F(x,y,\frac{t}{a}), & a < 0. \end{cases}$

Where, $H(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0. \end{cases}$

(F5) $F(x+y,z,t+s) \geq F(x,z,t) \wedge F(y,z,s)$.
(F6) $\lim\limits_{t\to+\infty} F(x,y,t) = 1$.

2. Fuzzy semi-inner product

Motivated by the work of Goudarzi and Vaezpour [9] and Mukherjee and Bag [15], we introduce the concept of fuzzy real semi-inner product on a linear space X as follows.

Definition 2.1. Let X be a linear space over R. Then a fuzzy subset $F : X \times X \times R \to [0,1]$ is called a fuzzy real semi-inner product on X if for all $x,y,z \in X$ and $t \in R$, the following conditions hold.

(1) (a) $F(x,x,t) = 0$ for all $t < 0$.
(b) $F(x,x,t) \neq H(t)$ for some t if $x \neq \underline{0}$, where, $H(t) = \begin{cases} 1 & t > 0 \\ 0, & t \leq 0. \end{cases}$

(2) (a) $F(ax,y,t) = \begin{cases} F(x,y,\frac{t}{a}), & a > 0 \\ H(t), & a = 0 \\ 1 - F(x,y,\frac{t}{a}), & a < 0. \end{cases}$

(b) $F(x+y,z,t+s) \geq F(x,z,t) \wedge F(y,z,s)$.

(3) $F(x, y, ts) \geq F(x, x, t^2) \wedge F(y, y, s^2)$ for $s, t > 0$.
(4) $\lim_{t \to +\infty} F(x, y, t) = 1$.

The pair (X, F) is said to be a fuzzy real semi-inner product space.

Lemma 2.1. *$F(x, y, \cdot)$ is a non-decreasing function in R.*

Proof. Let $t_1 > t_2$. Then $t_1 - t_2 > 0$. Thus

$$\begin{aligned} & F(0 + x, y, t_1 - t_2 + t_2) \geq F(0, y, t_1 - t_2) \wedge F(x, y, t_2) \\ \Rightarrow \quad & F(x, y, t_1) \geq 1 \wedge F(x, y, t_2) \\ \Rightarrow \quad & F(x, y, t_1) \geq F(x, y, t_2) \end{aligned}$$

$\Rightarrow F(x, y, \cdot)$ is a non-decreasing function in R. □

Example 2.1. Let $(X, < \cdot, \cdot >)$ be an real semi-inner product space. We define a mapping, $F(ax, y, t) = \begin{cases} \frac{t}{t + a|<x,y>|}, & a \geq 0, t > 0 \\ 1 - \frac{t}{t + a|<x,y>|}, & a < 0, t < 0 \\ 1, & a < 0, t > 0 \\ 0, & \text{otherwise.} \end{cases}$

Verification. We will verify conditions 1-4 as given in definition.
Putting $a = 1$, we get

$$F(x, y, t) = \begin{cases} \frac{t}{t + |<x,y>|}, & t > 0 \\ 0, & t \leq 0. \end{cases}$$

Now
(1a) The condition $F(x, x, t) = 0 \ \forall \ t < 0$ is obvious.
(1b) To show, $F(x, x, t) \neq H(t)$ for some t if $x \neq 0$ i.e. $F(x, x, t) \neq 1$ for some $t > 0$.
Assume that

$$\begin{aligned} & F(x, x, t) = 1 \ \forall \ t > 0 \\ \Rightarrow \ & \frac{t}{t + | < x, x > |} = 1 \ \forall \ t > 0 \\ \Rightarrow \ & | < x, x > | = 0 \\ \Rightarrow \ & x = \underline{0}. \end{aligned}$$

$\therefore$ If $x \neq \underline{0}$, then $F(x, x, t) \neq H(t)$ for some t.
(2a) Case I: $a > 0$.
Then for

$$\begin{aligned} &\text{For} \quad t > 0, \quad F(ax, y, t) = \frac{(t/a)}{(t/a) + | < x, y > |} = F(x, y, t/a). \\ &\text{For} \quad t < 0, \quad F(ax, y, t) = 0 = F(x, y, t/a) \text{ as } t/a < 0. \\ &\text{For} \quad t = 0, \quad F(ax, y, t) = 0 = F(x, y, t/a). \end{aligned}$$

Case II: $a = 0$.

$$F(ax, y, t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0, \end{cases}$$
$$\text{i.e. } F(ax, y, t) = H(t).$$

Case III: $a < 0$.

$$\text{For } \ t > 0, \quad F(ax, y, t) = 1 \text{ and } F(x, y, t/a) = 0 \text{ as } t/a < 0.$$

So, $F(ax, y, t) = 1 - F(x, y, t/a)$.

$$\begin{aligned} \text{For } \ t < 0, \ F(ax, y, t) &= 1 - \frac{t}{t + a|<x, y>|} \\ &= 1 - \frac{(t/a)}{(t/a) + |<x, y>|} \\ &= 1 - F(x, y, \frac{t}{a}) \ \ (\text{since } t/a > 0). \end{aligned}$$

Hence 2a holds.
(2b) Now we have to show that $F(x + y, z, t + s) \geq F(x, z, t) \wedge F(y, z, s)$.
The relation is obvious when (i) $s + t < 0$, (ii) $s + t > 0$; $s > 0, t < 0$; $s < 0, t > 0$.
(iii) Let $s, t > 0$, $s + t > 0$. Then

$$F(x + y, z, s + t) = \frac{s + t}{s + t + |<x + y, z>|} \geq \frac{s + t}{s + t + |<x, z>| + |<y, z>|}.$$

Assume that $F(x, z, t) \geq F(y, z, s)$. Then

$$\begin{aligned} \frac{t}{t + |<x, z>|} \geq \frac{s}{s + |<y, z>|} \quad &\Rightarrow \quad \frac{t}{t + |<x, z>|} - \frac{s}{s + |<y, z>|} \geq 0 \\ &\Rightarrow \quad t|<y, z>| - s|<x, z>| \geq 0 \end{aligned}$$

which implies

$$\frac{t + s}{t + s + |<x, z>| + |<y, z>|} - \frac{s}{s + |<y, z>|}$$

$$= \quad \frac{t|<y, z>| - s|<x, z>|}{(t + s + |<x, z>| + |<y, z>|)(s + |<y, z>|)}.$$

The right hand side is clearly greater than 0 i.e. $F(x + y, z, t + s) \geq F(y, z, s)$.
Similarly if we take, $F(y, z, s) \geq F(x, z, t)$, then $F(x + y, z, t + s) \geq F(x, z, t)$.
Combining, we get $F(x + y, z, t + s) \geq F(x, z, t) \wedge F(y, z, s)$.
(3) For $s, t > 0$, let us assume that,

$$\begin{aligned} & F(x, x, t^2) \geq F(y, y, s^2) \\ \Rightarrow \quad & \frac{t^2}{t^2 + |<x, x>|} \geq \frac{s^2}{s^2 + |<y, y>|} \\ \Rightarrow \quad & t^2 <y, y> - s^2 <x, x> \geq 0 \\ \Rightarrow \quad & t\sqrt{<y, y>} - s\sqrt{<x, x>} \geq 0. \end{aligned} \tag{2.1}$$

Now,

$$\begin{aligned} F(x,y,ts)-F(y,y,s^2) &= \frac{ts}{ts+|<x,y>|}-\frac{s^2}{s^2+|<y,y>|} \\ &= \frac{ts|<y,y>|-s^2|<x,y>|}{(ts+|<x,y>|)(s^2+|<y,y>|)} \\ &\geq \frac{s(t|<y,y>|-s\sqrt{|<x,x>|\cdot|<y,y>|})}{(ts+|<x,y>|)(s^2+|<y,y>|)} \end{aligned}$$

[using Schwartz inequality]

$$= \frac{s\sqrt{|<y,y>|}\cdot\left[t\sqrt{|<y,y>|}-s\sqrt{|<x,x>|}\right]}{(ts+|<x,y>|)(s^2+|<y,y>|)} \geq 0.$$

[by equation (2.1)]

$$F(x,y,ts) \geq F(y,y,s^2).$$

Also, by taking $F(y,y,s^2)\geq F(x,x,t^2)$, we can show, $F(x,y,ts)\geq F(x,x,t^2)$. Combining, we get, $F(x,y,ts)\geq F(x,x,t^2)\wedge F(y,y,s^2)$.
(4) Also, it is clear $\lim_{t\to+\infty} F(x,y,t)=1$.
Then F is a real fuzzy semi-inner product on X.

Theorem 2.1. *Let (X,F) be a real fuzzy semi-inner product space. Further assume that,*

(A) $\forall\ x,y\in X,\ t\in R,\ F(x+y,x+y,t^2)\geq F(x,x,t^2)$

Then X is a fuzzy normed space with the norm,

$$N(ax,t)=\begin{cases} F(a^2x,x,t^2), & a\neq 0, t>0\\ 0, & t\leq 0.\end{cases}$$

Proof. (1) $\forall\ t\leq 0,\ N(x,t)=0$.
(2)

$$\begin{aligned} & N(x,t)=1\ \forall\ t>0 \\ \Rightarrow\ & F(x,x,t^2)=1\ \forall\ t>0 \end{aligned}$$

If possible, let $x\neq \underline{0}$, then $F(x,x,t)\neq 1$ for some $t>0$ by definition of F which is a contradiction. Therefore, we have $x=\underline{0}$. Also, for $x=\underline{0}$, $N(x,t)=F(x,x,t)=H(t)$, i.e. $N(x,t)=1$ for all $t>0$.
(3) By the construction of N, it is easy to see that for all $a\neq 0$, $N(ax,t)=F(a^2x,x,t^2)=F(x,x,\frac{t^2}{a^2})=N(x,\frac{t}{|a|})$.
(4) Now for the triangle inequality, for any $s,t>0$,

$$\begin{aligned} N(x+y,t+s) &= F(x+y,x+y,(t+s)^2) \\ &\geq F(x+y,x+y,ts+st) \text{ (Since } F(x,y,\cdot) \text{ is non-decreasing by lemma 2.1)} \\ &\geq F(x,x+y,ts)\wedge F(y,x+y,st) \\ &\geq \{F(x,x,t^2)\wedge F(x+y,x+y,s^2)\}\wedge\{F(y,y,s^2)\wedge F(x+y,x+y,t^2)\} \\ &\quad \text{(By using condition (3) in definition 2.1)} \end{aligned}$$

By condition (A), $F(x+y,x+y,t^2)\geq F(x,x,t^2)$ and also $F(x+y,x+y,s^2)=F(y+x,y+x,s^2)\geq F(y,y,s^2)$.

Then we have, $N(x+y,t+s) \geq N(x,t) \wedge N(y,s)$. So, (N4) holds.
(5) Non-decreasing property of N follows from the property of $F(x,y,\cdot)$ and $\lim\limits_{t\to+\infty} N(x,t) = \lim\limits_{t\to+\infty} F(x,x,t^2) = 1$.
Thus $N(x,t)$ is a fuzzy norm on X. □

Example 2.2. Let $(X, <\cdot,\cdot>)$ be real semi-inner product space.

$$F(ax,y,t) = \begin{cases} \frac{\sqrt{t/a}}{\sqrt{t/a}+\sqrt{|<x,y>|}}, & a \geq 0, t > 0 \\ 1 - \frac{\sqrt{t/a}}{\sqrt{t/a}+\sqrt{|<x,y>|}}, & a < 0, t < 0 \\ 1, & a < 0, t > 0 \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that F is a real fuzzy semi-inner product on X from the previous example. Then by taking $a = 1$ and with the help of the previous theorem, we get,

$$N(x,t) = F(x,x,t^2) = \frac{t}{t+\sqrt{|<x,x>|}} = \frac{t}{t+\|x\|}.$$

which is similar to our known fuzzy norm function as defined by Bag and Samanta [1].

Theorem 2.2. *Let (X,F) be a fuzzy real semi-inner product space. Assume that,*

(B) For all $x,y \in X$, $\wedge\{t \in R : F(x,y,t) \geq \alpha\} < \infty \ \forall \ \alpha \in (0,1)$ and $F(x,x,t) > 0$ for all $t > 0 \Rightarrow x = \underline{0}$

Define for $x,y \in X$,

$$<x,y>_\alpha = \wedge\{t \in R : F(x,y,t) \geq \alpha\}, \alpha \in (0,1).$$

Then $\{<x,y>_\alpha : \alpha \in (0,1)\}$ is an ascending family of real semi-inner products on X.

Proof. (1) $<x,x>_\alpha = \wedge\{t \in R : F(x,x,t) \geq \alpha\}$. Then from 1a, we can conclude that $<x,x>_\alpha \geq 0 \ \forall \ \alpha \in (0,1)$.
Now, let for $\alpha \in (0,1), <x,x>_\alpha = 0$
$\Rightarrow \wedge\{t \in R : F(x,x,t) \geq \alpha\} = 0$
$\Rightarrow \ \forall \ t > 0, F(x,x,t) \geq \alpha > 0$.
By condition (B), we have $x = \underline{0}$.
i.e. for $x \neq \underline{0}$, $<x,x>_\alpha > 0$.
(2) To show $<x+y,z>_\alpha = <x,z>_\alpha + <y,z>_\alpha$.

$$\begin{aligned} <x,z>_\alpha + <y,z>_\alpha &= \wedge\{t : F(x,z,t) \geq \alpha\} + \wedge\{s : F(y,z,s) \geq \alpha\} \\ &= \wedge\{t+s : F(x,z,t) \geq \alpha, F(y,z,s) \geq \alpha\} \\ &= \wedge\{t+s : F(x,z,t) \wedge F(y,z,s) \geq \alpha\} \\ &\geq \wedge\{t+s : F(x+y,z,t+s) \geq \alpha\} \text{ (By 2b)} \\ &= \wedge\{r : F(x+y,z,r) \geq \alpha\},\ r = t+s \\ &= <x+y,z>_\alpha \end{aligned}$$

i.e. $<x,z>_\alpha + <y,z>_\alpha \geq <x+y,z>_\alpha$ .....(i)
Again, for any $\varepsilon > 0$, let

$$\begin{aligned} A &= 1 - \{1 - F(x,z,<x,z>_\alpha - \varepsilon/2)\} \wedge \{1 - F(y,z,<y,z>_\alpha - \varepsilon/2)\} \\ &= 1 - F(-x,z,-<x,z>_\alpha + \varepsilon/2) \wedge F(-y,z,-<y,z>_\alpha + \varepsilon/2) \\ &\geq 1 - F(-x-y,z,-<x,z>_\alpha - <y,z>_\alpha + \varepsilon) \text{ (By 2b)} \\ &= F(x+y,z,<x,z>_\alpha + <y,z>_\alpha - \varepsilon) \end{aligned}$$

Also, by the definition of infimum,

$$\begin{aligned} F(x,z,<x,z>_\alpha -\varepsilon/2) &< \alpha \\ \Rightarrow 1-F(x,z,<x,z>_\alpha -\varepsilon/2) &> 1-\alpha \end{aligned}$$

Similarly, $1-F(y,z,<y,z>_\alpha -\varepsilon/2) > 1-\alpha$.

$$\text{Then, } \{1-F(x,z,<x,z>_\alpha -\varepsilon/2)\} \wedge \{1-F(y,z,<y,z>_\alpha -\varepsilon/2)\} > 1-\alpha$$
$$\text{i.e. } 1-A > 1-\alpha$$
$$\text{i.e. } A < \alpha.$$

Which implies $F(x+y,z,<x,z>_\alpha + <y,z>_\alpha -\varepsilon) \le A < \alpha$
$\Rightarrow <x+y,z>_\alpha \ \ge\ <x,z>_\alpha + <y,z>_\alpha -\varepsilon$
Since $\varepsilon > 0$ is arbitrary, so $<x+y,z>_\alpha \ \ge\ <x,z>_\alpha + <y,z>_\alpha$ ..... (ii)
From (i) and (ii), we get $<x+y,z>_\alpha = <x,z>_\alpha + <y,z>_\alpha$.
(3) To prove, $<ax,y>_\alpha = a<x,y>_\alpha$.

$$\begin{aligned} \text{Let } a>0. \text{ Then } <ax,y>_\alpha &= \wedge\{t : F(ax,y,t) \ge \alpha\} \\ &= \wedge\{t : F(x,y,t/a) \ge \alpha\} \\ &= a(\wedge\{r : F(x,y,r) \ge \alpha\})\ ,\ r = t/a \\ &= a<x,y>_\alpha\ . \\ \text{Let } a=0,\ <ax,y>_\alpha &= \wedge\{t : F(0,y,t) \ge \alpha\} \\ &= \wedge\{t : H(t) \ge \alpha\} \\ &= \wedge\{t : t>0\} \\ &= 0 = a<x,y>_\alpha\ . \end{aligned}$$

For $a<0$, let $a=-b$, then,

$$\begin{aligned} <-bx+bx,y>_\alpha &= <-bx,y>_\alpha + <bx,y>_\alpha \\ \text{or, } <0,y>_\alpha = 0 &= <-bx,y>_\alpha + <bx,y>_\alpha \\ \text{or, } <-bx,y>_\alpha &= -<bx,y>_\alpha \end{aligned}$$

Therefore, $<ax,y>_\alpha = <-bx,y>_\alpha = -<bx,y>_\alpha = -b<x,y>_\alpha = a<x,y>_\alpha$.
(4) To prove, $<x,y>_\alpha^2 \le <x,x>_\alpha <y,y>_\alpha$.

$$\begin{aligned} <x,x>_\alpha<y,y>_\alpha &= \Big(\wedge\{t : F(x,x,t) \ge \alpha\}\Big)\Big(\wedge\{r : F(y,y,r) \ge \alpha\}\Big) \\ &= \wedge\{tr : F(x,x,t) \ge \alpha, F(y,y,r) \ge \alpha\} \\ &= \wedge\{tr : F(x,x,t) \wedge F(y,y,r) \ge \alpha\} \\ &\ge \{tr : F(x,y,\sqrt{tr}) \ge \alpha\} \text{ (By condition (3) from definition 2.1)} \\ &= \wedge\{p^2 : F(x,y,p) \ge \alpha\}\ ,\ p = \sqrt{tr} \\ &= \Big[\wedge\{p : F(x,y,p) \ge \alpha\}\Big]^2 \\ &= <x,y>_\alpha^2\ . \end{aligned}$$

$\therefore <\cdot,\cdot>_\alpha$ is a real semi-inner product for $\alpha \in (0,1)$. Now for $\alpha_1 > \alpha_2$,

$$\begin{aligned} \{t : F(x,y,t) \ge \alpha_1\} &\subseteq \{t : F(x,y,t) \ge \alpha_2\} \\ \Rightarrow \wedge\{t : F(x,y,t) \ge \alpha_1\} &\ge \wedge\{t : F(x,y,t) \ge \alpha_2\} \\ \Rightarrow <x,y>_{\alpha_1} &\ge <x,y>_{\alpha_2}\ . \end{aligned}$$

Therefore, $\{< \cdot,\cdot >_\alpha : \alpha \in (0,1)\}$ is an ascending family of real semi-inner products on X. □

Theorem 2.3. *Let $\{< \cdot,\cdot >_\alpha : \alpha \in (0,1)\}$ be an ascending family of real semi-inner products on X. Then $F : X \times X \times R \to [0,1]$ defined as,*

$$F(ax, y, t) = \begin{cases} \vee\{\alpha \in (0,1) :< ax, y >_\alpha \leq t\}, & a > 0 \\ H(t), & a = 0 \\ 1 - \vee\{\alpha \in (0,1) :< ax, y >_\alpha \geq t\}, & a < 0. \end{cases}$$

is a fuzzy real semi-inner product on X.

Proof. (1a) For all $t < 0$, $\{\alpha \in (0,1) :< x, x >_\alpha \leq t\} = \varnothing$. So, by construction, $F(x, x, t) = \vee\{\alpha \in (0,1) :< x, x >_\alpha \leq t\} = 0$.
(1b) Let $F(x, x, t) = H(t)$. Then $F(x, x, t) = 1$ for all $t > 0$. i.e. $\vee\{\alpha \in (0,1) :< x, x >_\alpha \leq t\} = 1$ for all $t > 0$. Let $0 < \varepsilon < 1$. Then by the definition of supremum $\exists\ \alpha_0$ lying between $\varepsilon < \alpha_0 \leq 1$ which gives $< x, x >_\varepsilon \ \leq \ < x, x >_{\alpha_0} \leq t$. Since ε, t are arbitrary, so $x = \underline{0}$. So, if $x \neq 0$, then $F(x, x, t) \neq H(t)$ for some $t \in R$.
(2a) Let, $a > 0$.

$$\begin{aligned} F(ax, y, t) &= \vee\{\alpha \in (0,1) :< ax, y >_\alpha \leq t\} \\ &= \vee\{\alpha \in (0,1) :< x, y >_\alpha \leq t/a\} \\ &= F(x, y, t/a) \end{aligned}$$

If $a = 0$, then $F(ax, y, t) = H(t)$ by construction.
Let $a < 0$,

$$\begin{aligned} F(ax, y, t) &= 1 - \vee\{\alpha \in (0,1) :< ax, y >_\alpha \geq t\} \\ &= 1 - \vee\{\alpha \in (0,1) : a < x, y >_\alpha \geq t\} \\ &= 1 - \vee\{\alpha \in (0,1) :< x, y >_\alpha \leq t/a\} \\ &= 1 - F(x, y, t/a) \end{aligned}$$

(2b) To prove $F(x + y, z, t + s) \geq F(x, z, t) \wedge F(y, z, s)$.
Let $F(x, z, t) = A$ and $F(y, z, s) = B$ and let $0 < r < A \leq B$. Then $\exists\ \alpha, \beta > r$ such that $< x, z >_\alpha \leq t$ and $< y, z >_\beta \leq s$. Let, $\gamma = \alpha \wedge \beta > r$. Then, $< x, z >_\gamma \leq < x, z >_\alpha \leq t$ and $< y, z >_\gamma \leq < y, z >_\beta \leq s$.
Which implies, $< x + y, z >_\gamma = < x, z >_\gamma + < y, z >_\gamma \leq t + s$.
i.e. $F(x + y, z, t + s) \geq \gamma = \alpha \wedge \beta > r$.
Since r is arbitrary, so $F(x + y, z, t + s) \geq A = F(x, z, t) \wedge F(y, z, s)$. Similarly if we take, $A \geq B$, then the relation also holds.
(3) To prove $F(x, y, ts) \geq F(x, x, t^2) \wedge F(y, y, s^2)$ for $t, s > 0$. Let $0 < r < F(x, x, t^2) \leq F(y, y, s^2)$. Then $\exists\ \alpha_0, \beta_0 < r$ such that,
$< x, x >_{\alpha_0} \leq t^2, < y, y >_{\beta_0} \leq s^2$.
Let $\gamma = \alpha_0 \wedge \beta_0 > r$. Then,

$$\begin{aligned} < x, y >_\gamma^2 &\leq < x, x >_\gamma < y, y >_\gamma \\ &\leq < x, x >_{\alpha_0} < y, y >_{\beta_0} \\ &\leq t^2 s^2 \\ \text{i.e. } < x, y >_\gamma &\leq ts \\ \text{i.e. } F(x, y, ts) \geq \gamma &= \alpha_0 \wedge \beta_0 > r \end{aligned}$$

Since r is arbitrary, $F(x,y,ts) \geq F(x,x,t^2) = F(x,x,t^2) \wedge F(y,y,s^2)$. Similarly by taking, $F(x,x,t^2) \geq F(y,y,s^2)$, the relation also follows.
(4) $\lim_{t\to+\infty} F(x,y,t) = 1$. $\therefore$ F is a real fuzzy semi-inner product on X. □

Theorem 2.4. *Let (X_1,F_1) and (X_2,F_2) be two semi-inner product spaces. Let*
$X = X_1 \times X_2 = \{(x_1,x_2) : x_i \in X_i\}$
under co-ordinate wise addition and multiplication over R. Define, for $x=(x_1,x_2), y=(y_1,y_2) \in X = X_1 \times X_2$

$$F(ax,y,t) = \begin{cases} F_1(ax_1,y_1,t) \wedge F_2(ax_2,y_2,t), & a \geq 0 \\ F_1(ax_1,y_1,t) \vee F_2(ax_2,y_2,t), & a < 0. \end{cases}$$

Then F is a fuzzy semi-inner product on X.

Proof. Proof is easy. So, we omit it. □

3. Generalized fuzzy semi-inner product

Also, motivated by the concept of generalized semi-inner product introduced by B. Nath [18] in 1971, we define a generalized fuzzy semi-inner product space which is defined as follows.

Definition 3.1. Let X be a linear space over R. Then a generalized fuzzy semi-inner product is a function $F : X \times X \times R \to [0,1]$ which satisfies the conditions (1), (2), (4) of Definition 2.1 and in place of (3), it satisfies the condition (modified),

$$(3^*)\ F(x,y,ts) \geq F(x,x,t^p) \wedge F(y,y,s^q) \text{ where } p>1 \text{ is an integer with } \frac{1}{p}+\frac{1}{q}=1.$$

It is clear that $p=2$ gives the space in definition 2.1. Now, depending on the above definition and we can generalize theorem 2.1 as :

Theorem 3.1. *Let (X,F) be a generalized fuzzy semi-inner product space and let p be an integer greater than 1 with $\frac{1}{p}+\frac{1}{q}=1$. In addition, assume,*

$$(C) \qquad \forall\ x,y \in X,\ t \in R,\ F(x+y,x+y,t^q) \geq F(x,x,t^p)$$

Then X is a fuzzy normed space with the norm,

$$N_p(ax,t) = \begin{cases} F(|a|^p x,x,t^p), & a \neq 0, t>0 \\ 0, & t \leq 0. \end{cases}$$

Proof. Proof follows the same lines. So, we omit it. □

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