Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 10, No. 4, (Octobrt 2015), pp. 657–680
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Approaches to interval-valued intuitionistic hesitant fuzzy soft sets based decision making

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Received 20 January 2015; Revised 2 April 2015; Accepted 29 April 2015

Abstract. The soft set theory is a newly mathematical tool to deal with uncertain problems. However, the classical soft sets are not appropriate to deal with imprecise and fuzzy parameters. In this paper, we introduce the average function and propose the interval-valued intuitionistic hesitant fuzzy soft sets (IVIHFSSs) which are a combination of the interval-valued intuitionistic hesitant fuzzy sets and soft sets. Then, the complement, AND, OR, union, intersection, restricted union, extended intersection, difference, necessity, possibility, power-λ, λ-multiply, average, and geometric operations are defined on the IVIHFSSs, and some basic properties are also discussed in detail. Finally, by means of level soft sets and aggregation operators, we presented two algorithms to IVIHFSSs based on decision making, and the practicality and effectiveness is proved by an illustrative example.

2010 AMS Classification: 03E72

Keywords: Interval-valued intuitionistic hesitant fuzzy soft sets, Level soft sets, Aggregation operators.

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1. Introduction

Many complicated problems in fields such as engineering, economics, medical science, and environmental science involving vagueness and fuzziness. While a wide variety of existing theories such as probability theory, fuzzy set theory, rough set theory [18], and interval mathematics [7] have been developed to model indeterminacy. However, each of these theories has its inherent difficulties as pointed out in [16]. The soft set theory, originally proposed by Molodtsov [16], is free from the inadequacy of the parameterization tools of those theories. It has been successfully applied in many different fields such as decision making, theory of measurement, and data analysis.

By combining soft sets with other mathematical models, many extended soft sets model have been developed recently. Maji et al.[13] firstly explored fuzzy soft sets, a more generalized notion combining fuzzy sets and soft sets. Çağman et al.[2] introduced fuzzy parameterized (FP) soft sets and discussed their related properties. Yang et al. [27] developed the concept of interval-valued fuzzy soft sets. Jiang et al. [9] defined interval-valued intuitionistic fuzzy soft sets which are an extension of interval-valued fuzzy soft sets or intuitionistic fuzzy soft sets [14]. Dinda et al.[4] proposed generalised intuitionistic fuzzy soft sets and gave an adjustable approach to decision making. Subsequently, Jiang [8] pointed out two uncorrected definitions in [9]. Feng et al.[6] established a colorful connection between rough sets and soft sets, and applied it to multicriteria group decision making [5]. Jun [11] studied the application of soft sets in BCK/BCI-algebras and initiated soft BCK/BCI-algebras. Combining soft set and cubic set [10], Muhiuddin and Al-roqi [17]developed cubic soft set and applied it to BCK/BCI-algebras. Maji et al. [15] explored neutrosophic soft set, and some definitions and operations have been introduced. Xiao et al. [24] investigated trapezoidal soft sets by combining trapezoidal fuzzy number and soft sets. Yang et al. [28] proposed the multi-fuzzy soft sets and applied it to decision making.

Hesitant fuzzy sets, proposed by Torra [20], permit the membership to have a series of possible values between 0 and 1 which can describe the human's hesitance more objectively and accurately. Xu et al. [26] developed a series of aggregation operators for hesitant fuzzy information with the aid of quasi-arithmetic means. Dual hesitant fuzzy sets were introduced by Zhu et al. [32], in which the membership degree and non-membership degree of an element to a given set are defined by two sets of several real values. It emphasizes the importance of the non-membership which can avoid information distortion and losing effectively in describing the vague decision making information. To accommodate more complex environment, Zhang [30] proposed the concept of interval-valued intuitionistic hesitant fuzzy sets (IVIHFSs) and applied them to group decision making. Recently, by combining soft sets with hesitant fuzzy sets, Das and Kar [3] and Wang et al. [21] proposed hesitant fuzzy soft sets (HFSSs). Zhang et al. [29] extended the HFSSs into interval-valued hesitant fuzzy soft sets (IVHFSSs). On the one hand, it is unreasonable to use IVHFSSs to handle some decision making problems because of insufficiency in describing the parameter of non-membership information. Instead, adopting several interval-valued intuitionistic numbers may overcome the difficulty. In that case, it is necessary to extend IVHFSSs into interval-valued intuitionistic hesitant fuzzy soft environment. On the other hand, by referring to a great deal of literature and expertise, we find that the discussions about fusions of IVIHFSs and soft sets do not also exist in the related literatures. Considering the above facts, it is necessary for us to investigate the combination of IVIHFSs and soft sets. The purpose of this paper is to initiate the concept of interval-valued intuitionistic hesitant fuzzy soft sets (IVIHFSSs). In order to illustrate the efficiency of the model, two effective algorithms are proposed.

The remainder of this paper is organized as follows: In Section 2, some basic definitions of soft sets, fuzzy soft sets, hesitant fuzzy sets, interval-valued intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy soft sets, interval-valued intuitionistic hesitant fuzzy sets are briefly reviewed and we propose average function. In Section

3, the concepts and operations of the IVIHFSSs are presented, especially some new operations on them are defined, meanwhile, their properties are discussed in detail. In Section 4, we apply the initiated interval-valued intuitionistic hesitant fuzzy soft sets to a decision making problems and give two effective algorithms. The paper is concluded in Section 5.

2. Preliminaries

2.1. Soft sets and fuzzy soft sets.

Definition 2.1 ([16]). A pair (F, A) is called a soft set over U, where F is a mapping given by $F: \ A \to P(U)$. In other words, the soft set is not a kind of set, but a parameterized family of subsets of the set U. For any parameter $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A).

Definition 2.2 ([13]). Let $\widetilde{P}(U)$ be the set of all fuzzy subsets of U. A pair (Υ, A) is called a fuzzy soft set over U, where Υ is a mapping given by $\Upsilon: \ A \to \widetilde{P}(U)$.

2.2. Hesitant fuzzy sets.

Definition 2.3 ([20]). A hesitant fuzzy set on U is defined in terms of a function that returns a subset of [0, 1] when it is applied to U, i.e.,

$$A = \{< x, h(x) >| \ x \in U\}$$

where $h(x)$ is a set of some different values in [0,1], representing the possible membership degrees of the element $x \in U$ to A. For convenience, Xia and Xu [23] called $h(x)$ a hesitant fuzzy element (HFE) and H the set of all hesitant fuzzy elements (HFEs).

Definition 2.4 ([23]). Let $h = \bigcup_{\gamma \in h}\{\gamma\}$ be a HFE, then the score function of h is defined as follows:

$$S(h) = \tfrac{1}{\#h} \sum_{\gamma \in h} \gamma$$

where $\#h$ is the number of the elements in h. For two HFEs h_1, h_2, if $S(h_1) > S(h_2)$, then $h_1 > h_2$; if $S(h_1) = S(h_2)$, then $h_1 = h_2$.

2.3. Hesitant fuzzy soft sets.

Definition 2.5 ([3]). Let $H(U)$ be the set of all hesitant fuzzy subsets of U. A pair (Γ, A) is called a hesitant fuzzy soft set over U, where Γ is a mapping given by $\Gamma: \ A \to H(U)$.

2.4. Interval-valued intuitionistic fuzzy sets.

Definition 2.6 ([1]). Let U be an ordinary nonempty set. An interval-valued intuitionistic fuzzy set $\widetilde{A}$ in U is an object that has the form:

$$\widetilde{A} = \{< x, \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x) >| \ x \in U\}$$

where $\mu_{\widetilde{A}}(x) = [\mu^-_{\widetilde{A}}(x), \mu^+_{\widetilde{A}}(x)]$ and $\nu_{\widetilde{A}}(x) = [\nu^-_{\widetilde{A}}(x), \nu^+_{\widetilde{A}}(x)]$ satisfy $0 \leq \mu^+_{\widetilde{A}}(x) + \nu^+_{\widetilde{A}}(x) \leq 1$ for all $x \in U$ and are, respectively, called the membership degree and the non-membership degree of the element $x \in U$ to $\widetilde{A}$.

Xu [25] called each pair $(\mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x))$ in $\widetilde{A}$ an interval-valued intuitionistic fuzzy number (IVIFN). For convenience, each IVIFN can be simply denoted by $\alpha = (\mu_\alpha, \nu_\alpha)$, where $\mu_\alpha = [\mu^-_\alpha, \mu^+_\alpha]$, $\nu_\alpha = [\nu^-_\alpha, \nu^+_\alpha]$, and $\mu^+_\alpha + \nu^+_\alpha \leq 1$.

Definition 2.7 ([25])**.** Let $\alpha = ([\mu^-_\alpha, \mu^+_\alpha], [\nu^-_\alpha, \nu^+_\alpha])$, $\alpha_1 = ([\mu^-_{\alpha_1}, \mu^+_{\alpha_1}], [\nu^-_{\alpha_1}, \nu^+_{\alpha_1}])$, and $\alpha_2 = ([\mu^-_{\alpha_2}, \mu^+_{\alpha_2}], [\nu^-_{\alpha_2}, \nu^+_{\alpha_2}])$ be any three IVIFNs, and $\lambda > 0$. Then,
(1) $\alpha^c = ([\nu^-_\alpha, \nu^+_\alpha], [\mu^-_\alpha, \mu^+_\alpha])$;
(2) $\alpha_1 \vee \alpha_2 = ([\mu^-_{\alpha_1} \vee \mu^-_{\alpha_2}, \mu^+_{\alpha_1} \vee \mu^+_{\alpha_2}], [\nu^-_{\alpha_1} \wedge \nu^-_{\alpha_2}, \nu^+_{\alpha_1} \wedge \nu^+_{\alpha_2}])$;
(3) $\alpha_1 \wedge \alpha_2 = ([\mu^-_{\alpha_1} \wedge \mu^-_{\alpha_2}, \mu^+_{\alpha_1} \wedge \mu^+_{\alpha_2}], [\nu^-_{\alpha_1} \vee \nu^-_{\alpha_2}, \nu^+_{\alpha_1} \vee \nu^+_{\alpha_2}])$;
(4) $\alpha_1 \oplus \alpha_2 = ([\mu^-_{\alpha_1} + \mu^-_{\alpha_2} - \mu^-_{\alpha_1}\mu^-_{\alpha_2}, \mu^+_{\alpha_1} + \mu^+_{\alpha_2} - \mu^+_{\alpha_1}\mu^+_{\alpha_2}], [\nu^-_{\alpha_1}\nu^-_{\alpha_2}, \nu^+_{\alpha_1}\nu^+_{\alpha_2}])$;
(5) $\alpha_1 \otimes \alpha_2 = ([\mu^-_{\alpha_1}\mu^-_{\alpha_2}, \mu^+_{\alpha_1}\mu^+_{\alpha_2}], [\nu^-_{\alpha_1} + \nu^-_{\alpha_2} - \nu^-_{\alpha_1}\nu^-_{\alpha_2}, \nu^+_{\alpha_1} + \nu^+_{\alpha_2} - \nu^+_{\alpha_1}\nu^+_{\alpha_2}])$;
(6) $\lambda\alpha = ([1 - (1 - \mu^-_\alpha)^\lambda, 1 - (1 - \mu^+_\alpha)^\lambda], [(\nu^-_\alpha)^\lambda, (\nu^+_\alpha)^\lambda])$;
(7) $\alpha^\lambda = ([(\mu^-_\alpha)^\lambda, (\mu^+_\alpha)^\lambda], [1 - (1 - \nu^-_\alpha)^\lambda, 1 - (1 - \nu^+_\alpha)^\lambda])$.

Definition 2.8 ([25])**.** Let $\alpha = ([\mu^-_\alpha, \mu^+_\alpha], [\nu^-_\alpha, \nu^+_\alpha])$, then the score function and accuracy function of α defined as follows:

$$S(\alpha) = (1/2)(\mu^-_\alpha - \nu^-_\alpha + \mu^+_\alpha - \nu^+_\alpha), \quad h(\alpha) = (1/2)(\mu^-_\alpha + \nu^-_\alpha + \mu^+_\alpha + \nu^+_\alpha).$$

For two IVIFNs α_1 and α_2,
(1) If $S(\alpha_1) > S(\alpha_2)$, then $\alpha_1 > \alpha_2$.
(2) If $S(\alpha_1) = S(\alpha_2)$, then the following hold,
  (a) If $h(\alpha_1) > h(\alpha_2)$, then $\alpha_1 > \alpha_2$.
  (b) If $h(\alpha_1) = h(\alpha_2)$, then $\alpha_1 = \alpha_2$.
  (c) If $h(\alpha_1) < h(\alpha_2)$, then $\alpha_1 < \alpha_2$.

2.5. **Interval-valued intuitionistic hesitant fuzzy sets.**

Definition 2.9 ([30])**.** Let U be a fixed set, then an interval-valued intuitionistic hesitant fuzzy set (IVIHFS) on U is given in terms of a function that when applied to returns a subset of Ω.

To be easily understood, we express the IVIHFS by a mathematical symbol as follows:

$$\widetilde{E} = \{< x, h_{\widetilde{E}}(x) >| \ x \in U\}$$

where $h_{\widetilde{E}}(x)$ is a set of some IVIFNs in Ω, denoting the possible membership degree intervals and nonmembership degree intervals of the element $x \in U$ to the set $\widetilde{E}$. For convenience, Zhang [30] call $\widetilde{h} = h_{\widetilde{E}(x)}$ an interval-valued intuitionistic hesitant fuzzy element (IVIHFE). The set of all interval-valued intuitionistic hesitant fuzzy sets on U is denoted by $\widetilde{H}(U)$. If $\alpha \in \widetilde{h}$, then α is an IVIFN, and it can be denoted by $\alpha = (\mu_\alpha, \nu_\alpha) = ([\mu^-_\alpha, \mu^+_\alpha], [\nu^-_\alpha, \nu^+_\alpha])$.

Definition 2.10 ([30])**.** Given three IVIHFEs represented by $\widetilde{h}$, $\widetilde{h}_1$, and $\widetilde{h}_2$, then the operational laws of IVIHFEs are defined as follows:
(1) $\widetilde{h}^c = \{([\nu^-_\alpha, \nu^+_\alpha], [\mu^-_\alpha, \mu^+_\alpha]) \mid \alpha \in \widetilde{h}\}$;

(2) $\widetilde{h}_1 \cup \widetilde{h}_2 = \{([\mu^-_{\alpha_1} \vee \mu^-_{\alpha_2}, \mu^+_{\alpha_1} \vee \mu^+_{\alpha_2}], [\nu^-_{\alpha_1} \wedge \nu^-_{\alpha_2}, \nu^+_{\alpha_1} \wedge \nu^+_{\alpha_2}]) \mid \alpha_1 \in \widetilde{h}_1, \alpha_2 \in \widetilde{h}_2\}$;
(3) $\widetilde{h}_1 \cap \widetilde{h}_2 = \{([\mu^-_{\alpha_1} \wedge \mu^-_{\alpha_2}, \mu^+_{\alpha_1} \wedge \mu^+_{\alpha_2}], [\nu^-_{\alpha_1} \vee \nu^-_{\alpha_2}, \nu^+_{\alpha_1} \vee \nu^+_{\alpha_2}]) \mid \alpha_1 \in \widetilde{h}_1, \alpha_2 \in \widetilde{h}_2\}$;
(4) $\widetilde{h}_1 \oplus \widetilde{h}_2 = \{([\mu^-_{\alpha_1} + \mu^-_{\alpha_2} - \mu^-_{\alpha_1}\mu^-_{\alpha_2}, \mu^+_{\alpha_1} + \mu^+_{\alpha_2} - \mu^+_{\alpha_1}\mu^+_{\alpha_2}], [\nu^-_{\alpha_1}\nu^-_{\alpha_2}, \nu^+_{\alpha_1}\nu^+_{\alpha_2}]) \mid \alpha_1 \in \widetilde{h}_1, \alpha_2 \in \widetilde{h}_2\}$;
(5) $\widetilde{h}_1 \otimes \widetilde{h}_2 = \{([\mu^-_{\alpha_1}\mu^-_{\alpha_2}, \mu^+_{\alpha_1}\mu^+_{\alpha_2}], [\nu^-_{\alpha_1} + \nu^-_{\alpha_2} - \nu^-_{\alpha_1}\nu^-_{\alpha_2}, \nu^+_{\alpha_1} + \nu^+_{\alpha_2} - \nu^+_{\alpha_1}\nu^+_{\alpha_2}]) \mid \alpha_1 \in \widetilde{h}_1, \alpha_2 \in \widetilde{h}_2\}$;
(6) $\lambda\widetilde{h} = \{([1-(1-\mu^-_\alpha)^\lambda, 1-(1-\mu^+_\alpha)^\lambda], [(\nu^-_\alpha)^\lambda, (\nu^+_\alpha)^\lambda]) \mid \alpha \in \widetilde{h}\}$;
(7) $\widetilde{h}^\lambda = \{([(\mu^-_\alpha)^\lambda, (\mu^+_\alpha)^\lambda], [1-(1-\nu^-_\alpha)^\lambda, 1-(1-\nu^+_\alpha)^\lambda]) \mid \alpha \in \widetilde{h}\}$.

Theorem 2.11 ([30])**.** *Let $\widetilde{h}$, $\widetilde{h}_1$, and $\widetilde{h}_2$ be three IVIHFEs, and λ, λ_1, and $\lambda_2 > 0$. Then,*
(1) $\widetilde{h}^c_1 \cup \widetilde{h}^c_2 = (\widetilde{h}_1 \cap \widetilde{h}_2)^c$. (2) $\widetilde{h}^c_1 \cap \widetilde{h}^c_2 = (\widetilde{h}_1 \cup \widetilde{h}_2)^c$.
(3) $\widetilde{h}^c_1 \oplus \widetilde{h}^c_2 = (\widetilde{h}_1 \otimes \widetilde{h}_2)^c$. (4) $\widetilde{h}^c_1 \otimes \widetilde{h}^c_2 = (\widetilde{h}_1 \oplus \widetilde{h}_2)^c$.
(5) $(\widetilde{h}^c)^\lambda = (\lambda\widetilde{h})^c$. (6) $\lambda(\widetilde{h}^c) = (\widetilde{h}^\lambda)^c$.
(7) $\widetilde{h}^{\lambda_1\lambda_2} = (\widetilde{h}^{\lambda_1})^{\lambda_2}$. (8) $\lambda_1\lambda_2(\widetilde{h}) = \lambda_1(\lambda_2\widetilde{h})$.
(9) $\lambda(\widetilde{h}_1 \oplus \widetilde{h}_2) = \lambda\widetilde{h}_1 \oplus \lambda\widetilde{h}_2$. (10) $\widetilde{h}^\lambda_1 \otimes \widetilde{h}^\lambda_2 = (\widetilde{h}_1 \otimes \widetilde{h}_2)^\lambda$.

Theorem 2.12. *Let $\widetilde{E} = \{< x, \widetilde{h}(x) >| x \in U\}, \widetilde{E}_1 = \{< x, \widetilde{h}_1(x) >| x \in U\}$, and $\widetilde{E}_2 = \{< x, \widetilde{h}_2(x) >| x \in U\}$ be three interval-valued intuitionistic hesitant fuzzy sets on U. Then,*
(1) $\widetilde{E}^c_1 \cup \widetilde{E}^c_2 = (\widetilde{E}_1 \cap \widetilde{E}_2)^c$. (2) $\widetilde{E}^c_1 \cap \widetilde{E}^c_2 = (\widetilde{E}_1 \cup \widetilde{E}_2)^c$.
(3) $\widetilde{E}^c_1 \oplus \widetilde{E}^c_2 = (\widetilde{E}_1 \otimes \widetilde{E}_2)^c$. (4) $\widetilde{E}^c_1 \otimes \widetilde{E}^c_2 = (\widetilde{E}_1 \oplus \widetilde{E}_2)^c$.
(5) $(\widetilde{E}^c)^\lambda = (\lambda\widetilde{E})^c$. (6) $\lambda(\widetilde{E}^c) = (\widetilde{E}^\lambda)^c$.
(7) $\widetilde{E}^{\lambda_1\lambda_2} = (\widetilde{E}^{\lambda_1})^{\lambda_2}$. (8) $\lambda_1\lambda_2(\widetilde{E}) = \lambda_1(\lambda_2\widetilde{E})$.
(9) $\lambda(\widetilde{E}_1 \oplus \widetilde{E}_2) = \lambda\widetilde{E}_1 \oplus \lambda\widetilde{E}_2$. (10) $\widetilde{E}^\lambda_1 \otimes \widetilde{E}^\lambda_2 = (\widetilde{E}_1 \otimes \widetilde{E}_2)^\lambda$.

Proof. In the following, we shall prove (1); (2)-(10) are proved analogously.
(1) For $\forall x \in U, (\widetilde{h}_1 \cap \widetilde{h}_2)(x) = \widetilde{h}_1(x) \cap \widetilde{h}_2(x)$, according to Theorem 2.11, we have $((\widetilde{h}_1 \cap \widetilde{h}_2)(x))^c = (\widetilde{h}_1(x) \cap \widetilde{h}_2(x))^c = \widetilde{h}_1(x)^c \cup \widetilde{h}_2(x)^c$. Hence, Proved.

□

Definition 2.13 ([30])**.** For an IVIHFE $\widetilde{h}, S(\widetilde{h}) = \sum_{\alpha\in\widetilde{h}} S(\alpha)/\#\widetilde{h}$ is called the score function of $\widetilde{h}$, where $\#\widetilde{h}$ is the number of the elements in $\widetilde{h}$. $h(\widetilde{h}) = \sum_{\alpha\in\widetilde{h}} h(\alpha)/\#\widetilde{h}$ is called the accuracy function of $\widetilde{h}$. For any two IVIHFEs $\widetilde{h}_1$ and $\widetilde{h}_2$,

(1) If $S(\widetilde{h}_1) > S(\widetilde{h}_2)$, then $\widetilde{h}_1 > \widetilde{h}_2$.
(2) If $S(\widetilde{h}_1) = S(\widetilde{h}_2)$, then the following hold,
  (*a*) If $h(\widetilde{h}_1) > h(\widetilde{h}_2)$, then $\widetilde{h}_1 > \widetilde{h}_2$.
  (*b*) If $h(\widetilde{h}_1) = h(\widetilde{h}_2)$, then $\widetilde{h}_1 = \widetilde{h}_2$.
  (*c*) If $h(\widetilde{h}_1) < h(\widetilde{h}_2)$, then $\widetilde{h}_1 < \widetilde{h}_2$.

Definition 2.14. For an IVIHFE $\widetilde{h}$,

$A(\widetilde{h}) = ([\frac{\sum_{\alpha\in\widetilde{h}}\mu_\alpha^-}{\#\widetilde{h}}, \frac{\sum_{\alpha\in\widetilde{h}}\mu_\alpha^+}{\#\widetilde{h}}], [\frac{\sum_{\alpha\in\widetilde{h}}\nu_\alpha^-}{\#\widetilde{h}}, \frac{\sum_{\alpha\in\widetilde{h}}\nu_\alpha^+}{\#\widetilde{h}}])$
is called the average function of $\widetilde{h}$, where $\#\widetilde{h}$ is the number of the elements in $\widetilde{h}$. It can be easily seen that $A(\widetilde{h})$ is an IVIFN.

Definition 2.15. Suppose that an IVIHFE $\widetilde{h}$, and stipulate that $\widetilde{h}^+$ and $\widetilde{h}^-$ are the maximum and the minimum interval-valued intuitionistic hesitant fuzzy values in the IVIHFE $\widetilde{h}$, then we call $\widetilde{h}^N = \eta\widetilde{h}^+ + (1-\eta)\widetilde{h}^-$ an extension interval value, where $\eta(0 \le \eta \le 1)$ is the parameter determined by the decision maker (DM) according to his/her principle preference.

Consequently, if $l_\alpha \neq l_\beta$, and α, β are corresponding IVIHFEs, then we need add different interval values to the IVIHFE which has the less elements using the parameter η via the DM's principle preference until both of them have the same length, i.e.

(1) when the DM's principle preference is optimistic, we can add the extension interval value $\widetilde{h}^N = \widetilde{h}^+$;

(2) when the DM's principle preference is neutral, we can add the extension interval value $\widetilde{h}^N = \frac{1}{2}(\widetilde{h}^+ + \widetilde{h}^-)$;

(3) when the DM's principle preference is pessimistic, we can add the extension interval value $\widetilde{h}^N = \widetilde{h}^-$.

Obviously, the parameter η provided by the DM reflects his/her principle preference which can affect the final results. In what follows, we take the optimistic principle.

Based on the well-known Hamming distance, as well as the above operational laws, analogous to the distance measure for HFEs in [19], we further propose a generalized distance measure between IVIHFEs:

$$d(\widetilde{h},\widetilde{g}) = \frac{1}{4l}\sum_{i=1}^{l}\Big(\mid \mu^-_{\widetilde{h}_{\sigma(i)}} - \mu^-_{\widetilde{g}_{\sigma(i)}} \mid + \mid \mu^+_{\widetilde{h}_{\sigma(i)}} - \mu^+_{\widetilde{g}_{\sigma(i)}} \mid + \mid \nu^-_{\widetilde{h}_{\sigma(i)}} - \nu^-_{\widetilde{g}_{\sigma(i)}} \mid + \mid \nu^+_{\widetilde{h}_{\sigma(i)}} - \nu^+_{\widetilde{g}_{\sigma(i)}} \mid \Big) \tag{2.1}$$

where l is the number of $\max\{\#\widetilde{h}, \#\widetilde{g}\}$.

2.6. **Interval-valued intuitionistic fuzzy soft sets.**

Definition 2.16 ([9])**.** Let $I(U)$ be the set of all interval-valued intuitionistic fuzzy subsets of U. A pair $(\widetilde{F}, A)$ is called an interval-valued intuitionistic fuzzy soft set over U, where $\widetilde{F}$ is a mapping given by $\widetilde{F} : A \to I(U)$.

Definition 2.17 ([9])**.** Let $L = \{(\alpha,\beta) \mid \alpha = [\alpha_1,\alpha_2] \in Int([0,1]), \beta = [\beta_1,\beta_2] \in Int([0,1]), \alpha_2+\beta_2 \le 1\}$, where $Int([0,1])$ denotes the set of all closed subintervals of [0,1]. Then a relation $\geqslant_L$ on L is defined as follows:
$\forall(\alpha,\beta),(\xi,\eta) \in L,\ \ (\alpha,\beta) \le_L (\xi,\eta) \Leftrightarrow \alpha \le \xi$ and $\beta \ge \eta \Leftrightarrow [\alpha_1,\alpha_2] \le [\xi_1,\xi_2]$ and $[\beta_1, \beta_2] \ge [\eta_1,\eta_2] \Leftrightarrow \alpha_1 \le \xi_1, \alpha_2 \le \xi_2, \beta_1 \ge \eta_1,$ and $\beta_2 \ge \eta_2$.

Definition 2.18 ([9])**.** Let $\mho = (\widetilde{F}, A)$ be an interval-valued intuitionistic fuzzy soft set over U, where $A \subseteq E$ and E is the parameter set. For $(\alpha,\beta) \in L$, the $(\alpha,\beta)-$level soft set of $\mho$ is a crisp soft set $L(\mho;\alpha,\beta)=(\widetilde{F}_{(\alpha,\beta)},A)$ defined by

$\widetilde{F}_{(\alpha,\beta)}(e)=L(\widetilde{F}(e);\alpha,\beta)=\{x\in U\mid \widetilde{F}(e)(x)\geqslant_L(\alpha,\beta)\}=\{x\in U\mid \mu_{\widetilde{F}(e)}(x)\geq \alpha,\nu_{\widetilde{F}(e)}(x)\leq\beta\}$ for all $e\in A$.

3. Interval-valued intuitionistic hesitant fuzzy soft sets

3.1. **The concept of interval-valued intuitionistic hesitant fuzzy soft sets.**

Definition 3.1. A pair $(\widetilde{F},A)$ is called an interval-valued intuitionistic hesitant fuzzy soft set over U, where $\widetilde{F}$ is a mapping given by $\widetilde{F}:\ A\to\widetilde{H}(U)$.

An interval-valued intuitionistic hesitant fuzzy soft set is a mapping from parameters to $\widetilde{H}(U)$, and it is not a set, but a parameterized family of interval-valued intuitionistic hesitant fuzzy subset of U. For $e\in A,\widetilde{F}(e)$ may be considered as the $e-$approximate elements of the interval-valued intuitionistic hesitant fuzzy soft set $(\widetilde{F},A)$.

Example 3.2. Suppose that $U=\{a_1,a_2,a_3\}$ is the set of software development projects under consideration, $A=\{e_1,e_2,e_3\}$ is the set of parameters, $A=\{e_1=\text{economic feasibility}, e_2=\text{technological feasibility}, e_3=\text{staff feasibility}\}$. We define an interval-valued intuitionistic hesitant fuzzy soft set as follows:

$\widetilde{F}(e_1)=\{<a_1,\{([0.5,0.6],[0.1,0.2]),([0.6,0.7],[0.2,0.3])\}>,<a_2,\{([0.4,0.7],[0.1,0.3])\}>,<a_3,\{([0.4,0.5],[0.1,0.2]),([0.5,0.6],[0.2,0.4])\}>\}$,
$\widetilde{F}(e_2)=\{<a_1,\{([0.5,0.6],[0.2,0.3]),([0.7,0.8],[0.1,0.2])\}>,<a_2,\{([0.5,0.7],[0.1,0.3])\}>,<a_3,\{([0.6,0.8],[0.1,0.2]),([0.8,0.9],[0,0.1])\}>\}$,
$\widetilde{F}(e_3)=\{<a_1,\{([0.5,0.7],[0.1,0.2]),([0.7,0.9],[0,0.1])\}>,<a_2,\{([0.65,0.7],[0.1,0.2]),([0.7,0.8],[0,0.1])\}>,<a_3,\{([0.5,0.6],[0.2,0.3])\}>\}$.

Definition 3.3. Let $A,B\subseteq E$. $(\widetilde{F},A)$ and $(\widetilde{G},B)$ be two interval-valued intuitionistic hesitant fuzzy soft sets over U. $(\widetilde{G},B)$ is called to be an interval-valued intuitionistic hesitant fuzzy soft subset of $(\widetilde{F},A)$ if

(1) $A\supseteq B$;
(2) $\forall e\in B,x\in U,\ S(\widetilde{h}_{\widetilde{F}(e)}(x))\geq S(\widetilde{h}_{\widetilde{G}(e)}(x))$.

In this case, we write $(\widetilde{F},A)\widetilde{\supseteq}(\widetilde{G},B)$.

Example 3.4. Let $U=\{x_1,x_2,x_3\},A=\{e_1,e_2\},B=\{e_1\}$. Suppose $(\widetilde{F},A)$ and $(\widetilde{G},B)$ be two interval-valued intuitionistic hesitant fuzzy soft sets over U defined as follows:

$\widetilde{F}(e_1)=\{<x_1,\{([0.6,0.7],[0,0.1]),([0.6,0.7],[0.2,0.3])\}>,<x_2,\{([0.6,0.7],[0.1,0.3])\}>,<x_3,\{([0.5,0.6],[0.1,0.2]),([0.5,0.6],[0.2,0.4])\}>\}$,
$\widetilde{F}(e_2)=\{<x_1,\{([0.7,0.8],[0.1,0.2]),([0.5,0.7],[0.2,0.3])\}>,<x_2,\{([0.5,0.7],[0.1,0.3])\}>,<x_3,\{([0.3,0.5],[0.1,0.2]),([0.4,0.6],[0.2,0.4])\}>\}$,
and,

$\widetilde{G}(e_1) = \{< x_1, \{([0.5, 0.6], [0.1, 0.2]), ([0.5, 0.6], [0.2, 0.3])\} >, < x_2, \{([0.4, 0.7], [0.2, 0.3])\} >, < x_3, \{([0.4, 0.5], [0.1, 0.2]), ([0.4, 0.5], [0.3, 0.4])\} >\}$.

For $\forall e \in A$ and $x \in U$, we have $S(\widetilde{h}_{\widetilde{F}(e)}(x)) \geq S(\widetilde{h}_{\widetilde{G}(e)}(x))$. Hence, $(\widetilde{F}, A)\widetilde{\supseteq}(\widetilde{G}, B)$.

Definition 3.5. Two interval-valued intuitionistic hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ over U are called to be interval-valued intuitionistic hesitant fuzzy soft equal, if $(\widetilde{F}, A)$ is an interval-valued intuitionistic hesitant fuzzy soft subset of $(\widetilde{G}, B)$, and $(\widetilde{G}, B)$ is an interval-valued intuitionistic hesitant fuzzy soft subset of $(\widetilde{F}, A)$.

In this case, we write $(\widetilde{F}, A)\widetilde{=}(\widetilde{G}, B)$.

Remark 3.6. Interval-valued intuitionistic hesitant fuzzy soft equal is different from equal of two interval-valued intuitionistic hesitant fuzzy soft sets. Obviously, equal is a special case of interval-valued intuitionistic hesitant fuzzy soft equal and interval-valued intuitionistic hesitant fuzzy soft equal is a generalization of equal. For example, let $U = \{x\},\ A = B = \{e\},\ \widetilde{F}(e) = \{< x, \{([0.5, 0.6], [0.2, 0.3]), ([0.5, 0.7], [0, 0.2])\} >\}, \widetilde{G}(e) = \{< x, \{([0.5, 0.6], [0.1, 0.2])\} >\}$. Obviously, $(\widetilde{F}, A)\widetilde{=}(\widetilde{G}, B)$, but $(\widetilde{F}, A) \neq (\widetilde{G}, B)$.

Definition 3.7. An interval-valued intuitionistic hesitant fuzzy soft set $(\widetilde{F}, A)$ over U is called the empty interval-valued intuitionistic hesitant fuzzy soft set if $\widetilde{F}(e) = \{\widetilde{[0, 0], [1, 1]}\}$ for all $e \in A$, denoted by $\widetilde{\Phi}_A$. An interval-valued intuitionistic hesitant fuzzy soft set on U is called the full interval-valued intuitionistic hesitant fuzzy soft set if $\widetilde{F}(e) = \{\widetilde{[1, 1], [0, 0]}\}$ for all $e \in A$, denoted by $\widetilde{U}_A$.

3.2. **Operations on interval-valued intuitionistic hesitant fuzzy soft sets.**

Definition 3.8. The complement of an interval-valued intuitionistic hesitant fuzzy soft set $(\widetilde{F}, A)$ over U is denoted by $(\widetilde{F}, A)^c$ and is defined by

$$(\widetilde{F}, A)^c = (\widetilde{F}^c, A)$$

where $\widetilde{F}^c :\ A \rightarrow \widetilde{H}(U)$ is a mapping given by $\widetilde{F}^c(e) = (\widetilde{F}(e))^c$ for $\forall e \in A$.

Obviously, $(\widetilde{F}^c)^c = \widetilde{F}$ and $((\widetilde{F}, A)^c)^c = (\widetilde{F}, A)$. It is worth noting that in the above definition, the parameter set of the complement $(\widetilde{F}, A)^c$ is still the original parameter set A, instead of $\neg A$.

Example 3.9. Reconsider Example 3.2, we have $(\widetilde{F}, A)^c$ as follows:
$\widetilde{F}^c(e_1) = \{< a_1, \{([0.1, 0.2], [0.5, 0.6]), ([0.2, 0.3], [0.6, 0.7])\} >, < a_2, \{([0.1, 0.3], [0.4, 0.7])\} >, < a_3, \{([0.1, 0.2], [0.4, 0.5]), ([0.2, 0.4], [0.5, 0.6])\} >\}$,
$\widetilde{F}^c(e_2) = \{< a_1, \{([0.2, 0.3], [0.5, 0.6]), ([0.1, 0.2], [0.7, 0.8])\} >, < a_2, \{([0.1, 0.3], [0.5, 0.7])\} >, < a_3, \{([0, 0.1], [0.8, 0.9]), ([0.1, 0.2], [0.6, 0.8])\} >\}$,
$\widetilde{F}^c(e_3) = \{< a_1, \{([0.1, 0.2], [0.5, 0.7]), ([0, 0.1], [0.7, 0.9])\} >, < a_2, \{([0.65, 0.7], [0.1, 0.2]), ([0, 0.1], [0.7, 0.8])\} >, < a_3, \{([0.2, 0.3], [0.5, 0.6])\} >\}$.

Definition 3.10. The AND of two interval-valued intuitionistic hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ over a common U which is denoted by $(\widetilde{F}, A) \wedge (\widetilde{G}, B)$ and is defined by

$$(\widetilde{F}, A) \wedge (\widetilde{G}, B) = (\widetilde{H}, A \times B)$$

where $\widetilde{H}(\alpha, \beta) = \widetilde{F}(\alpha) \cap \widetilde{G}(\beta)$, for $\forall(\alpha, \beta) \in A \times B, x \in U, \widetilde{H}(\alpha, \beta)(x) = \{([\inf(\underline{\mu}_{\widetilde{f}(\alpha)}(x), \underline{\mu}_{\widetilde{g}(\beta)}(x)), \inf(\overline{\mu}_{\widetilde{f}(\alpha)}(x), \overline{\mu}_{\widetilde{g}(\beta)}(x))], [\sup(\underline{\nu}_{\widetilde{f}(\alpha)}(x), \underline{\nu}_{\widetilde{g}(\beta)}(x)), \sup(\overline{\nu}_{\widetilde{f}(\alpha)}(x), \overline{\nu}_{\widetilde{g}(\beta)}(x))]) \mid \widetilde{f}(\alpha)(x) \in \widetilde{F}(\alpha)(x), \widetilde{g}(\beta)(x) \in \widetilde{G}(\beta)(x)\}$.

Definition 3.11. The OR of two interval-valued intuitionistic hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ over a common U which is denoted by $(\widetilde{F}, A) \vee (\widetilde{G}, B)$ and is defined by

$$(\widetilde{F}, A) \vee (\widetilde{G}, B) = (\widetilde{O}, A \times B)$$

where $\widetilde{O}(\alpha, \beta) = \widetilde{F}(\alpha) \cup \widetilde{G}(\beta)$, for $\forall(\alpha, \beta) \in A \times B, x \in U, \widetilde{O}(\alpha, \beta)(x) = \{([\sup(\underline{\mu}_{\widetilde{f}(\alpha)}(x), \underline{\mu}_{\widetilde{g}(\beta)}(x)), \sup(\overline{\mu}_{\widetilde{f}(\alpha)}(x), \overline{\mu}_{\widetilde{g}(\beta)}(x))], [\inf(\underline{\nu}_{\widetilde{f}(\alpha)}(x), \underline{\nu}_{\widetilde{g}(\beta)}(x)), \inf(\overline{\nu}_{\widetilde{f}(\alpha)}(x), \overline{\nu}_{\widetilde{g}(\beta)}(x))]) \mid \widetilde{f}(\alpha)(x) \in \widetilde{F}(\alpha)(x), \widetilde{g}(\beta)(x) \in \widetilde{G}(\beta)(x)\}$.

Example 3.12. Let $U = \{x_1, x_2, x_3\}, A = \{e_1, e_2\}, B = \{e_1, e_2, e_3\}$. Suppose $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ be two interval-valued intuitionistic hesitant fuzzy soft sets over U defined as follows:

$\widetilde{F}(e_1) = \{< x_1, \{([0.5, 0.6], [0.1, 0.2])\} >, < x_2, \{([0.4, 0.7], [0.1, 0.2]), ([0.6, 0.7], [0.1, 0.3])\} >, < x_3, \{([0.5, 0.7], [0.2, 0.3])\} >\}$,
$\widetilde{F}(e_2) = \{< x_1, \{([0.6, 0.7], [0.2, 0.3]), ([0.6, 0.8], [0, 0.1])\} >, < x_2, \{([0.4, 0.6], [0.2, 0.3])\} >, < x_3, \{([0.7, 0.9], [0, 0.1])\} >\}$,
$\widetilde{G}(e_1) = \{< x_1, \{([0.7, 0.8], [0.1, 0.2])\} >, < x_2, \{([0.6, 0.8], [0.1, 0.2])\} >, < x_3, \{([0.6, 0.7], [0.1, 0.3])\} >\}$,
$\widetilde{G}(e_2) = \{< x_1, \{([0.5, 0.7], [0.1, 0.2])\} >, < x_2, \{([0.6, 0.7], [0.1, 0.2])\} >, < x_3, \{([0.6, 0.8], [0.1, 0.2]), ([0.7, 0.8], [0, 0.1])\} >\}$,
$\widetilde{G}(e_3) = \{< x_1, \{([0.6, 0.8], [0, 0.1])\} >, < x_2, \{([0.6, 0.8], [0, 0.1])\} >, < x_3, \{([0.7, 0.9], [0, 0.1]), ([0.8, 0.9], [0, 0.1])\} >\}$.

Then, we have $(\widetilde{F}, A) \wedge (\widetilde{G}, B) = (\widetilde{H}, A \times B), (\widetilde{F}, A) \vee (\widetilde{G}, B) = (\widetilde{O}, A \times B)$, where the results are shown in TABLEs 1 and 2, respectively.

Theorem 3.13. *(De Morgan's Law) Let $(\widetilde{F}, A)$, $(\widetilde{G}, B)$ be two interval-valued intuitionistic hesitant fuzzy soft sets over U, then*

(1) $((\widetilde{F}, A) \wedge (\widetilde{G}, B))^c = (\widetilde{F}, A)^c \vee (\widetilde{G}, B)^c$.
(2) $((\widetilde{F}, A) \vee (\widetilde{G}, B))^c = (\widetilde{F}, A)^c \wedge (\widetilde{G}, B)^c$.

Proof. In the following, we shall prove (1); (2) is proved analogously.
(1) Suppose that $(\widetilde{F}, A) \wedge (\widetilde{G}, B) = (\widetilde{H}, A \times B)$. Therefore, $((\widetilde{F}, A) \wedge (\widetilde{G}, B))^c = (\widetilde{H}, A \times B)^c = (\widetilde{H}^c, A \times B)$. Similarly, $(\widetilde{F}, A)^c \vee (\widetilde{G}, B)^c = (\widetilde{F}^c, A) \vee (\widetilde{G}^c, B) = (\widetilde{O}, A \times$

Table 1. The results of "AND" operation on $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$

U	(e_1, e_1)	(e_1, e_2)
x_1	$\{([0.5, 0.6], [0.1, 0.2])\}$	$\{([0.5, 0.6], [0.1, 0.2])\}$
x_2	$\{([0.4, 0.7], [0.1, 0.2]), ([0.6, 0.7], [0.1, 0.3])\}$	$\{([0.4, 0.7], [0.1, 0.2]), ([0.6, 0.7], [0.1, 0.3])\}$
x_3	$\{([0.5, 0.7], [0.2, 0.3])\}$	$\{([0.5, 0.7], [0.2, 0.3])\}$
	(e_1, e_3)	(e_2, e_1)
x_1	$\{([0.5, 0.6], [0.1, 0.2])\}$	$\{([0.6, 0.7], [0.2, 0.3]), ([0.6, 0.8], [0.1, 0.2])\}$
x_2	$\{([0.4, 0.7], [0.1, 0.2]), ([0.6, 0.7], [0.1, 0.3])\}$	$\{([0.4, 0.6], [0.2, 0.3])\}$
x_3	$\{([0.5, 0.7], [0.2, 0.3])\}$	$\{([0.6, 0.7], [0.1, 0.3])\}$
	(e_2, e_2)	(e_2, e_3)
x_1	$\{([0.5, 0.7], [0.2, 0.3]), ([0.5, 0.7], [0.1, 0.2])\}$	$\{([0.6, 0.7], [0.2, 0.3]), ([0.6, 0.8], [0, 0.1])\}$
x_2	$\{([0.4, 0.6], [0.2, 0.3])\}$	$\{([0.4, 0.6], [0.2, 0.3])\}$
x_3	$\{([0.6, 0.8], [0.1, 0.2]), ([0.7, 0.8], [0, 0.1])\}$	$\{([0.7, 0.9], [0, 0.1])\}$

Table 2. The results of "OR" operation on $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$

U	(e_1, e_1)	(e_1, e_2)
x_1	$\{([0.7, 0.8], [0.1, 0.2])\}$	$\{([0.5, 0.7], [0.1, 0.2])\}$
x_2	$\{([0.6, 0.8], [0.1, 0.2])\}$	$\{([0.6, 0.7], [0.1, 0.2])\}$
x_3	$\{([0.6, 0.7], [0.1, 0.3])\}$	$\{([0.6, 0.8], [0.1, 0.2]), ([0.7, 0.8], [0, 0.1])\}$
	(e_1, e_3)	(e_2, e_1)
x_1	$\{([0.6, 0.8], [0, 0.1])\}$	$\{([0.7, 0.8], [0.1, 0.2]), ([0.7, 0.8], [0, 0.1])\}$
x_2	$\{([0.6, 0.8], [0, 0.1])\}$	$\{([0.4, 0.6], [0.2, 0.3])\}$
x_3	$\{([0.7, 0.9], [0, 0.1]), ([0.8, 0.9], [0, 0.1])\}$	$\{([0.6, 0.7], [0.1, 0.3])\}$
	(e_2, e_2)	(e_2, e_3)
x_1	$\{([0.6, 0.7], [0.1, 0.2]), ([0.6, 0.8], [0, 0.1])\}$	$\{([0.6, 0.8], [0, 0.1])\}$
x_2	$\{([0.4, 0.6], [0.2, 0.3])\}$	$\{([0.4, 0.6], [0.2, 0.3])\}$
x_3	$\{([0.6, 0.8], [0.1, 0.2]), ([0.7, 0.8], [0, 0.1])\}$	$\{([0.7, 0.9], [0, 0.1])\}$

$B)$. Now take $(\alpha, \beta) \in A \times B$, therefore, according to Theorem 2.12, $\widetilde{H}^c(\alpha, \beta) = (\widetilde{H}(\alpha, \beta))^c = (\widetilde{F}(\alpha) \cap \widetilde{G}(\beta))^c = \widetilde{F}^c(\alpha) \cup \widetilde{G}^c(\beta)$, Hence, $\widetilde{H}^c(\alpha, \beta) = \widetilde{O}(\alpha, \beta)$. Proved. □

Theorem 3.14. *(Associative Law) Let $(\widetilde{F}, A)$, $(\widetilde{G}, B)$, and $(\widetilde{H}, C)$ be three interval-valued intuitionistic hesitant fuzzy soft sets over U, then*
(1) $(\widetilde{F}, A) \wedge ((\widetilde{G}, B) \wedge (\widetilde{H}, C)) = ((\widetilde{F}, A) \wedge (\widetilde{G}, B)) \wedge (\widetilde{H}, C)$.
(2) $(\widetilde{F}, A) \vee ((\widetilde{G}, B) \vee (\widetilde{H}, C)) = ((\widetilde{F}, A) \vee (\widetilde{G}, B)) \vee (\widetilde{H}, C)$.

Proof. In the following, we shall prove (1); (2) is proved analogously.
(1) Suppose that $(\widetilde{G}, B) \wedge (\widetilde{H}, C) = (\widetilde{I}, B \times C)$, where $\widetilde{I}(\beta, \delta) = \widetilde{G}(\beta) \cap \widetilde{H}(\delta), \forall (\beta, \delta) \in B \times C$. Therefore, we have that $\widetilde{I}(\beta, \delta)(x) = \{([\inf(\underline{\mu}_{\widetilde{g}(\beta)}(x),\ \underline{\mu}_{\widetilde{h}(\delta)}(x)), \inf(\overline{\mu}_{\widetilde{g}(\beta)}(x), \overline{\mu}_{\widetilde{h}(\delta)}(x))], [\sup(\underline{\nu}_{\widetilde{g}(\beta)}(x), \underline{\nu}_{\widetilde{h}(\delta)}(x)), \sup(\overline{\nu}_{\widetilde{g}(\beta)}(x), \overline{\nu}_{\widetilde{h}(\delta)}(x))]) \mid \widetilde{g}(\beta)(x) \in \widetilde{G}(\beta)(x), \widetilde{h}(\delta)(x) \in \widetilde{H}(\delta)(x)\}$.

Since, $(\widetilde{F}, A) \wedge ((\widetilde{G}, B) \wedge (\widetilde{H}, C)) = (\widetilde{F}, A) \wedge (\widetilde{I}, B \times C)$, assume that $(\widetilde{F}, A) \wedge (\widetilde{I}, B \times C) = (\widetilde{J}, A \times (B \times C))$, where $\widetilde{J}(\alpha, \beta, \delta) = \widetilde{F}(\alpha) \cap \widetilde{I}(\beta, \delta), (\alpha, \beta, \delta) \in A \times (B \times C) = A \times B \times C$. Therefore,

$\widetilde{J}(\alpha,\beta,\delta)(x) = \{([\inf(\underline{\mu}_{\widetilde{f}(\alpha)}(x), \underline{\mu}_{\widetilde{i}(\beta,\delta)}(x)), \inf(\overline{\mu}_{\widetilde{f}(\alpha)}(x), \overline{\mu}_{\widetilde{i}(\beta,\delta)}(x))], [\sup(\underline{\nu}_{\widetilde{f}(\alpha)}(x), \underline{\nu}_{\widetilde{i}(\beta,\delta)}(x)), \sup(\overline{\nu}_{\widetilde{f}(\alpha)}(x), \overline{\nu}_{\widetilde{i}(\beta,\delta)}(x))]) \mid \widetilde{f}(\alpha)(x) \in \widetilde{F}(\alpha)(x), \widetilde{i}(\beta,\delta)(x) \in \widetilde{I}(\beta,\delta)(x)\}$
$= \{([\inf(\underline{\mu}_{\widetilde{f}(\alpha)}(x), \inf(\underline{\mu}_{\widetilde{g}(\beta)}(x), \underline{\mu}_{\widetilde{h}(\delta)}(x))), \inf(\overline{\mu}_{\widetilde{f}(\alpha)}(x), \inf(\overline{\mu}_{\widetilde{g}(\beta)}(x), \overline{\mu}_{\widetilde{h}(\delta)}(x)))], [\sup(\underline{\nu}_{\widetilde{f}(\alpha)}(x), \sup(\underline{\nu}_{\widetilde{g}(\beta)}(x), \underline{\nu}_{\widetilde{h}(\delta)}(x))), \sup(\overline{\nu}_{\widetilde{f}(\alpha)}(x), \sup(\overline{\nu}_{\widetilde{g}}(x), \overline{\nu}_{\widetilde{h}(\delta)}(x)))]) \mid \widetilde{f}(\alpha)(x) \in \widetilde{F}(\alpha)(x), \widetilde{g}(\beta)(x) \in \widetilde{G}(\beta)(x), \widetilde{h}(\delta)(x) \in \widetilde{H}(\delta)(x)\}$.

Assume that $(\widetilde{F}, A) \wedge (\widetilde{G}, B) = (\widetilde{K}, A \times B)$, where $\widetilde{K}(\alpha,\beta) = \widetilde{F}(\alpha) \cap \widetilde{G}(\beta), \forall(\alpha,\beta) \in A \times B$. Therefore, we have that $\widetilde{K}(\alpha,\beta)(x) = \{([\inf(\underline{\mu}_{\widetilde{f}(\alpha)}(x), \underline{\mu}_{\widetilde{g}(\beta)}(x)), \inf(\overline{\mu}_{\widetilde{f}(\alpha)}(x), \overline{\mu}_{\widetilde{g}(\beta)}(x))], [\sup(\underline{\nu}_{\widetilde{f}(\alpha)}(x), \underline{\nu}_{\widetilde{g}(\beta)}(x)), \sup(\overline{\nu}_{\widetilde{f}(\alpha)}(x), \overline{\nu}_{\widetilde{g}(\beta)}(x))]) \mid \widetilde{f}(\alpha)(x) \in \widetilde{F}(\alpha)(x), \widetilde{g}(\beta)(x) \in \widetilde{G}(\beta)(x)\}$.

Since, $((\widetilde{F}, A) \wedge (\widetilde{G}, B)) \wedge (\widetilde{H}, C) = (\widetilde{K}, A \times B) \wedge (\widetilde{H}, C)$, assume that $(\widetilde{K}, A \times B) \wedge (\widetilde{H}, C) = (\widetilde{L}, (A \times B) \times C)$, where $\widetilde{L}(\alpha,\beta,\delta) = \widetilde{K}(\alpha,\beta) \cap \widetilde{H}(\delta), (\alpha,\beta,\delta) \in (A \times B) \times C = A \times B \times C$. Therefore,

$\widetilde{L}(\alpha,\beta,\delta)(x) = \{([\inf(\underline{\mu}_{\widetilde{k}(\alpha,\beta)}(x), \underline{\mu}_{\widetilde{h}(\delta)}(x)), \inf(\overline{\mu}_{\widetilde{k}(\alpha,\beta)}(x), \overline{\mu}_{\widetilde{h}(\delta)}(x))], [\sup(\underline{\nu}_{\widetilde{k}(\alpha,\beta)}(x), \underline{\nu}_{\widetilde{h}(\delta)}(x)), \sup(\overline{\nu}_{\widetilde{k}(\alpha,\beta)}(x), \overline{\nu}_{\widetilde{h}(\delta)}(x))]) \mid \widetilde{k}(\alpha, \beta)(x) \in \widetilde{K}(\alpha, \beta)(x), \widetilde{h}(\delta)(x) \in \widetilde{H}(\delta)(x)\}$
$= \{([\inf(\inf(\underline{\mu}_{\widetilde{f}(\alpha)}(x), \underline{\mu}_{\widetilde{g}(\beta)}(x)), \underline{\mu}_{\widetilde{h}(\delta)}(x)), \inf(\inf(\overline{\mu}_{\widetilde{f}(\alpha)}(x), \overline{\mu}_{\widetilde{g}(\beta)}(x)), \overline{\mu}_{\widetilde{h}(\delta)}(x))], [\sup(\sup(\underline{\mu}_{\widetilde{f}(\alpha)}(x), \underline{\mu}_{\widetilde{g}(\beta)}(x)), \underline{\mu}_{\widetilde{h}(\delta)}(x)), \sup(\sup(\overline{\mu}_{\widetilde{f}(\alpha)}(x), \overline{\mu}_{\widetilde{g}(\beta)}(x)), \overline{\mu}_{\widetilde{h}(\delta)}(x))]) \mid \widetilde{f}(\alpha)(x) \in \widetilde{F}(\alpha)(x), \widetilde{g}(\beta)(x) \in \widetilde{G}(\beta)(x), \widetilde{h}(\delta)(x) \in \widetilde{H}(\delta)(x)\}$
$= \{([\inf(\underline{\mu}_{\widetilde{f}(\alpha)}(x), \inf(\underline{\mu}_{\widetilde{g}(\beta)}(x), \underline{\mu}_{\widetilde{h}(\delta)}(x))), \inf(\overline{\mu}_{\widetilde{f}(\alpha)}(x), \inf(\overline{\mu}_{\widetilde{g}(\beta)}(x), \overline{\mu}_{\widetilde{h}(\delta)}(x)))], [\sup(\underline{\nu}_{\widetilde{f}(\alpha)}(x), \sup(\underline{\nu}_{\widetilde{g}(\beta)}(x), \underline{\nu}_{\widetilde{h}(\delta)}(x))), \sup(\overline{\nu}_{\widetilde{f}(\alpha)}(x), \sup(\overline{\nu}_{\widetilde{g}}(x), \overline{\nu}_{\widetilde{h}(\delta)}(x)))]) \mid \widetilde{f}(\alpha)(x) \in \widetilde{F}(\alpha)(x), \widetilde{g}(\beta)(x) \in \widetilde{G}(\beta)(x), \widetilde{h}(\delta)(x) \in \widetilde{H}(\delta)(x)\}$.

Hence, $\widetilde{L}(\alpha,\beta,\delta)(x) = \widetilde{J}(\alpha,\beta,\delta)(x)$. Proved. □

Definition 3.15. Union of two interval-valued intuitionistic hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ over a common U is the interval-valued intuitionistic hesitant fuzzy soft set $(\widetilde{H}, C)$, where $C = A \cup B$, and $\forall e \in C$,

$$\widetilde{H}(e) = \begin{cases} \widetilde{F}(e), & \text{if } \ e \in A - B, \\ \widetilde{G}(e), & \text{if } \ e \in B - A, \\ \widetilde{F}(e) \cup \widetilde{G}(e), & \text{if } \ e \in A \cap B. \end{cases}$$

We write $(\widetilde{F}, A) \widetilde{\cup} (\widetilde{G}, B) = (\widetilde{H}, C)$.

Example 3.16. Reconsider Example 3.12, the results of union of interval-valued intuitionistic hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ are shown in TABLE 3.

Theorem 3.17. *Let $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ be two interval-valued intuitionistic hesitant fuzzy soft sets over U. Then,*

(1) $(\widetilde{F}, A) \,\widetilde{\cup}\, (\widetilde{F}, A) = (\widetilde{F}, A)$.

Table 3. The results of "union" operation on $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$

U	e_1	e_2
x_1	$\{([0.7, 0.8], [0.1, 0.2])\}$	$\{([0.6, 0.7], [0.1, 0.2]), ([0.6, 0.8], [0, 0.1])\}$
x_2	$\{([0.6, 0.8], [0.1, 0.2])\}$	$\{([0.4, 0.6], [0.2, 0.3])\}$
x_3	$\{([0.6, 0.7], [0.1, 0.3])\}$	$\{([0.6, 0.8], [0, 0.1])\}$
	e_3	
x_1	$\{([0.6, 0.8], [0, 0.1])\}$	
x_2	$\{([0.6, 0.8], [0, 0.1])\}$	
x_3	$\{([0.7, 0.9], [0, 0.1]), ([0.8, 0.9], [0, 0.1])\}$	

(2) $(\widetilde{F}, A) \,\widetilde{\cup}\, \widetilde{\Phi}_A = (\widetilde{F}, A)$.
(3) $(\widetilde{F}, A) \,\widetilde{\cup}\, \widetilde{U}_A = \widetilde{U}_A$.
(4) $(\widetilde{F}, A) \,\widetilde{\cup}\, \widetilde{U}_B = \widetilde{U}_B$ *iff* $A \subseteq B$.
(5) $(\widetilde{F}, A) \,\widetilde{\cup}\, \widetilde{\Phi}_B = (\widetilde{F}, A)$ *iff* $B \subseteq A$.
(6) $(\widetilde{F}, A) \,\widetilde{\cup}\, (\widetilde{G}, B) = (\widetilde{G}, B) \,\widetilde{\cup}\, (\widetilde{F}, A)$.

Proof. The proofs can be obtained from Definition 3.15. □

Definition 3.18. Intersection of two interval-valued intuitionistic hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ with $A \cap B \neq \phi$ over U is the interval-valued intuitionistic hesitant fuzzy soft set $(\widetilde{H}, C)$, where $C = A \cap B$, and $\forall e \in C$, $\widetilde{H}(e) = \widetilde{F}(e) \cap \widetilde{G}(e)$.

We write $(\widetilde{F}, A) \widetilde{\cap} (\widetilde{G}, B) = (\widetilde{H}, C)$.

Example 3.19. Reconsider Example 3.12, the results of intersection of interval-valued intuitionistic hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ are shown in TABLE 4.

Table 4. The results of "intersection" operation on $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$

U	e_1	e_2
x_1	$\{([0.5, 0.6], [0.1, 0.2])\}$	$\{([0.5, 0.7], [0.1, 0.2]), ([0.5, 0.7], [0.2, 0.3])\}$
x_2	$\{([0.4, 0.7], [0.1, 0.2]), ([0.6, 0.7], [0.1, 0.3])\}$	$\{([0.4, 0.6], [0.2, 0.3])\}$
x_3	$\{([0.5, 0.7], [0.2, 0.3])\}$	$\{([0.6, 0.8], [0.1, 0.2]), ([0.7, 0.8], [0, 0.1])\}$

Theorem 3.20. *Let $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ be two interval-valued intuitionistic hesitant fuzzy soft sets over U. Then,*
(1) $(\widetilde{F}, A) \,\widetilde{\cap}\, (\widetilde{F}, A) = (\widetilde{F}, A)$.
(2) $(\widetilde{F}, A) \,\widetilde{\cap}\, \widetilde{\Phi}_A = \widetilde{\Phi}_A$.
(3) $(\widetilde{F}, A) \,\widetilde{\cap}\, \widetilde{U}_A = (\widetilde{F}, A)$.
(4) $(\widetilde{F}, A) \,\widetilde{\cap}\, \widetilde{U}_B = (\widetilde{F}, A \cap B)$.
(5) $(\widetilde{F}, A) \,\widetilde{\cap}\, \widetilde{\Phi}_B = \widetilde{\Phi}_{A \cap B}$.
(6) $(\widetilde{F}, A) \,\widetilde{\cap}\, (\widetilde{G}, B) = (\widetilde{G}, B) \,\widetilde{\cap}\, (\widetilde{F}, A)$.

Proof. The proofs can be obtained from Definition 3.18. □

Definition 3.21. The restricted union of two interval-valued intuitionistic hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ with $A \cap B \neq \phi$ over U is the interval-valued intuitionistic hesitant fuzzy soft set $(\widetilde{H}, C)$, where $C = A \cap B$, and $\forall e \in C$, $\widetilde{H}(e) = \widetilde{F}(e) \cup \widetilde{G}(e)$.

We write $(\widetilde{F}, A)\widetilde{\cup}_R(\widetilde{G}, B) = (\widetilde{H}, C)$.

Example 3.22. Reconsider Example 3.12, the results of restricted union of interval-valued intuitionistic hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ are shown in TABLE 5.

TABLE 5. The results of "restricted union" operation on $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$

U	e_1	e_2
x_1	$\{([0.7, 0.8], [0.1, 0.2])\}$	$\{([0.6, 0.7], [0.1, 0.2]), ([0.6, 0.8], [0, 0.1])\}$
x_2	$\{([0.6, 0.8], [0.1, 0.2])\}$	$\{([0.4, 0.6], [0.2, 0.3])\}$
x_3	$\{([0.6, 0.7], [0.1, 0.3])\}$	$\{([0.6, 0.8], [0, 0.1])\}$

Theorem 3.23. *Let $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ be two interval-valued intuitionistic hesitant fuzzy soft sets over U. Then,*
(1) $(\widetilde{F}, A)\ \widetilde{\cup}_R\ (\widetilde{F}, A) = (\widetilde{F}, A)$.
(2) $(\widetilde{F}, A)\ \widetilde{\cup}_R\ \widetilde{\Phi}_A = (\widetilde{F}, A)$.
(3) $(\widetilde{F}, A)\ \widetilde{\cup}_R\ \widetilde{U}_A = \widetilde{U}_A$.
(4) $(\widetilde{F}, A)\ \widetilde{\cup}_R\ \widetilde{U}_B = \widetilde{U}_{A\cap B}$.
(5) $(\widetilde{F}, A)\ \widetilde{\cup}_R\ \widetilde{\Phi}_B = (\widetilde{F}, A \cap B)$.
(6) $(\widetilde{F}, A)\ \widetilde{\cup}_R\ (\widetilde{G}, B) = (\widetilde{G}, B)\ \widetilde{\cup}_R\ (\widetilde{F}, A)$.

Proof. The proofs can be obtained from Definition 3.21. □

Definition 3.24. The extended intersection of two interval-valued intuitionistic hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ over U is the interval-valued intuitionistic hesitant fuzzy soft set $(\widetilde{H}, C)$, where $C = A \cup B$, and $\forall e \in C$,

$$\widetilde{H}(e) = \begin{cases} \widetilde{F}(e), & \text{if} \quad e \in A - B, \\ \widetilde{G}(e), & \text{if} \quad e \in B - A, \\ \widetilde{F}(e) \cap \widetilde{G}(e), & \text{if} \quad e \in A \cap B. \end{cases}$$

We write $(\widetilde{F}, A)\widetilde{\cap}_E(\widetilde{G}, B) = (\widetilde{H}, C)$.

Example 3.25. Reconsider Example 3.12, the results of extended intersection of interval-valued intuitionistic hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ are shown in TABLE 6.

Theorem 3.26. *Let $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ be two interval-valued intuitionistic hesitant fuzzy soft sets over U. Then,*
(1) $(\widetilde{F}, A)\ \widetilde{\cap}_E\ (\widetilde{F}, A) = (\widetilde{F}, A)$.

TABLE 6. The results of "extended intersection" operation on $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$

U	e_1	e_2
x_1	$\{([0.5, 0.6], [0.1, 0.2])\}$	$\{([0.5, 0.7], [0.1, 0.2]), ([0.5, 0.7], [0.2, 0.3])\}$
x_2	$\{([0.4, 0.7], [0.1, 0.2]), ([0.6, 0.7], [0.1, 0.3])\}$	$\{([0.4, 0.6], [0.2, 0.3])\}$
x_3	$\{([0.5, 0.7], [0.2, 0.3])\}$	$\{([0.6, 0.8], [0.1, 0.2]), ([0.7, 0.8], [0, 0.1])\}$
	e_3	
x_1	$\{([0.6, 0.8], [0, 0.1])\}$	
x_2	$\{([0.6, 0.8], [0, 0.1])\}$	
x_3	$\{([0.7, 0.9], [0, 0.1]), ([0.8, 0.9], [0, 0.1])\}$	

(2) $(\widetilde{F}, A) \widetilde{\cap}_E \widetilde{\Phi}_A = \widetilde{\Phi}_A$.
(3) $(\widetilde{F}, A) \widetilde{\cap}_E \widetilde{U}_A = (\widetilde{F}, A)$.
(4) $(\widetilde{F}, A) \widetilde{\cap}_E \widetilde{U}_B = (\widetilde{F}, A)$ *iff* $B \subseteq A$.
(5) $(\widetilde{F}, A) \widetilde{\cap}_E \widetilde{\Phi}_B = \widetilde{\Phi}_B$ *iff* $A \subseteq B$.
(6) $(\widetilde{F}, A) \widetilde{\cap}_E (\widetilde{G}, B) = (\widetilde{G}, B) \widetilde{\cap}_E (\widetilde{F}, A)$.

Proof. The proofs can be obtained from Definition 3.24. □

Theorem 3.27. *(De Morgan's Law) Let $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ be two interval-valued intuitionistic hesitant fuzzy soft sets over U. Then,*
(1) $((\widetilde{F}, A) \widetilde{\cup} (\widetilde{G}, B))^c = (\widetilde{F}, A)^c \widetilde{\cap}_E (\widetilde{G}, B)^c$.
(2) $((\widetilde{F}, A) \widetilde{\cap}_E (\widetilde{G}, B))^c = (\widetilde{F}, A)^c \widetilde{\cup} (\widetilde{G}, B)^c$.
(3) $((\widetilde{F}, A) \widetilde{\cup}_R (\widetilde{G}, B))^c = (\widetilde{F}, A)^c \widetilde{\cap} (\widetilde{G}, B)^c$.
(4) $((\widetilde{F}, A) \widetilde{\cap} (\widetilde{G}, B))^c = (\widetilde{F}, A)^c \widetilde{\cup}_R (\widetilde{G}, B)^c$.

Proof. In the following, we shall prove (1); (2)-(4) are proved analogously.

Suppose that $(\widetilde{F}, A) \widetilde{\cup} (\widetilde{G}, B) = (\widetilde{H}, C)$, where $C = A \cup B$, and $\forall e \in C$,

$$\widetilde{H}(e) = \begin{cases} \widetilde{F}(e), & \text{if } \ e \in A - B, \\ \widetilde{G}(e), & \text{if } \ e \in B - A, \\ \widetilde{F}(e) \cup \widetilde{G}(e), & \text{if } \ e \in A \cap B. \end{cases}$$

Thus, $((\widetilde{F}, A) \widetilde{\cup} (\widetilde{G}, B))^c = (\widetilde{H}, C)^c = (\widetilde{H}^c, C)$, and $\forall e \in C$,

$$\widetilde{H}^c(e) = \begin{cases} \widetilde{F}^c(e), & \text{if } \ e \in A - B, \\ \widetilde{G}^c(e), & \text{if } \ e \in B - A, \\ \widetilde{F}^c(e) \cap \widetilde{G}^c(e), & \text{if } \ e \in A \cap B \text{ (Theorem 2.12)}. \end{cases}$$

Again suppose that $(\widetilde{F}, A)^c \widetilde{\cap}_E (\widetilde{G}, B)^c = (\widetilde{F}^c, A) \widetilde{\cap}_E (\widetilde{G}^c, B) = (\widetilde{I}, D)$, where $D = A \cup B$, and $\forall e \in D$,

$$\widetilde{I}(e) = \begin{cases} \widetilde{F}^c(e), & \text{if } \ e \in A - B, \\ \widetilde{G}^c(e), & \text{if } \ e \in B - A, \\ \widetilde{F}^c(e) \cap \widetilde{G}^c(e), & \text{if } \ e \in A \cap B. \end{cases}$$

We can conclude $\widetilde{H}^c(e) = \widetilde{I}(e)$. Proved.

□

Theorem 3.28. *Let* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$ *be two interval-valued intuitionistic hesitant fuzzy soft sets over* U*. Then,*
(1) $((\widetilde{F}, A) \,\widetilde{\cup}\, (\widetilde{G}, B))^c \widetilde{\subseteq} (\widetilde{F}, A)^c \,\widetilde{\cup}\, (\widetilde{G}, B)^c$.
(2) $(\widetilde{F}, A)^c \,\widetilde{\cap}\, (\widetilde{G}, B)^c \widetilde{\subseteq} ((\widetilde{F}, A) \,\widetilde{\cap}\, (\widetilde{G}, B))^c$.

Proof. In the following, we shall prove (1); (2) is proved analogously.

Suppose that $(\widetilde{F}, A) \,\widetilde{\cup}\, (\widetilde{G}, B) = (\widetilde{H}, C)$, where $C = A \cup B$, and $\forall e \in C$,

$$\widetilde{H}(e) = \begin{cases} \widetilde{F}(e), & \text{if} \quad e \in A - B, \\ \widetilde{G}(e), & \text{if} \quad e \in B - A, \\ \widetilde{F}(e) \cup \widetilde{G}(e), & \text{if} \quad e \in A \cap B. \end{cases}$$

Thus, $((\widetilde{F}, A) \,\widetilde{\cup}\, (\widetilde{G}, B))^c = (\widetilde{H}, C)^c = (\widetilde{H}^c, C)$, and $\forall e \in C$,

$$\widetilde{H}^c(e) = \begin{cases} \widetilde{F}^c(e), & \text{if} \quad e \in A - B, \\ \widetilde{G}^c(e), & \text{if} \quad e \in B - A, \\ \widetilde{F}^c(e) \cap \widetilde{G}^c(e), & \text{if} \quad e \in A \cap B \text{ (Theorem 2.12)}. \end{cases}$$

Again suppose that $(\widetilde{F}, A)^c \,\widetilde{\cup}\, (\widetilde{G}, B)^c = (\widetilde{F}^c, A) \,\widetilde{\cup}\, (\widetilde{G}^c, B) = (\widetilde{I}, C)$, where $C = A \cup B$, and $\forall e \in C$,

$$\widetilde{I}(e) = \begin{cases} \widetilde{F}^c(e), & \text{if} \quad e \in A - B, \\ \widetilde{G}^c(e), & \text{if} \quad e \in B - A, \\ \widetilde{F}^c(e) \cap \widetilde{G}^c(e), & \text{if} \quad e \in A \cap B \text{ (Theorem 2.12)}. \end{cases}$$

Obviously, $\forall e \in C, \widetilde{H}^c(e) \widetilde{\subseteq} \widetilde{I}(e)$. Proved. □

Theorem 3.29. *Let* $(\widetilde{F}, A)$*,* $(\widetilde{G}, B)$*, and* $(\widetilde{H}, C)$ *be three interval-valued intuitionistic hesitant fuzzy soft sets over* U*. Then,*
(1) $(\widetilde{F}, A) \widetilde{\cap}_E ((\widetilde{G}, B) \widetilde{\cap}_E (\widetilde{H}, C)) = ((\widetilde{F}, A) \widetilde{\cap}_E (\widetilde{G}, B)) \widetilde{\cap}_E (\widetilde{H}, C)$.
(2) $(\widetilde{F}, A) \widetilde{\cup} ((\widetilde{G}, B) \widetilde{\cup} (\widetilde{H}, C)) = ((\widetilde{F}, A) \widetilde{\cup} (\widetilde{G}, B)) \widetilde{\cup} (\widetilde{H}, C)$.

Proof. In the following, we shall prove (1), (2) is proved analogously.

Suppose that $(\widetilde{G}, B) \widetilde{\cap}_E (\widetilde{H}, C) = (\widetilde{I}, D)$, where $D = B \cup C$, and $\forall e \in D$,

$$\widetilde{I}(e) = \begin{cases} \widetilde{G}(e), & \text{if} \quad e \in B - C, \\ \widetilde{H}(e), & \text{if} \quad e \in C - B, \\ \widetilde{G}(e) \cap \widetilde{H}(e), & \text{if} \quad e \in B \cap C. \end{cases}$$

Since $(\widetilde{F}, A) \widetilde{\cap}_E ((\widetilde{G}, B) \widetilde{\cap}_E (\widetilde{H}, C)) = (\widetilde{F}, A) \widetilde{\cap}_E (\widetilde{I}, D)$, we assume $(\widetilde{F}, A) \widetilde{\cap}_E (\widetilde{I}, D) = (\widetilde{K}, E)$, where $E = A \cup D = A \cup B \cup C$, then we have the following,

$$\widetilde{K}(e)=\begin{cases}\widetilde{F}(e), & \text{if} \quad e\in A-B-C,\\ \widetilde{G}(e), & \text{if} \quad e\in B-A-C,\\ \widetilde{H}(e), & \text{if} \quad e\in C-A-B,\\ \widetilde{G}(e)\cap\widetilde{H}(e), & \text{if} \quad e\in B\cap C-A,\\ \widetilde{F}(e)\cap\widetilde{G}(e), & \text{if} \quad e\in A\cap B-C,\\ \widetilde{F}(e)\cap\widetilde{H}(e), & \text{if} \quad e\in A\cap C-B,\\ \widetilde{F}(e)\cap\widetilde{G}(e)\cap\widetilde{H}(e), & \text{if} \quad e\in A\cap B\cap C.\end{cases}$$

Again suppose that $(\widetilde{F},A)\widetilde{\cap}_E(\widetilde{G},B)=(\widetilde{S},X)$, where $X=A\cup B$, and $\forall e\in X$,

$$\widetilde{S}(e)=\begin{cases}\widetilde{F}(e), & \text{if} \quad e\in A-B,\\ \widetilde{G}(e), & \text{if} \quad e\in B-A,\\ \widetilde{F}(e)\cap\widetilde{G}(e), & \text{if} \quad e\in A\cap B.\end{cases}$$

Since $((\widetilde{F},A)\widetilde{\cap}_E\widetilde{G},B))\widetilde{\cap}_E(\widetilde{H},C)=(\widetilde{S},X)\widetilde{\cap}_E(\widetilde{H},C)$, we assume $(\widetilde{S},X)\widetilde{\cap}_E(\widetilde{H},C)=(\widetilde{L},Y)$, where $Y=X\cup C=A\cup B\cup C$, then we have the following,

$$\widetilde{L}(e)=\begin{cases}\widetilde{F}(e), & \text{if} \quad e\in A-B-C,\\ \widetilde{G}(e), & \text{if} \quad e\in B-A-C,\\ \widetilde{H}(e), & \text{if} \quad e\in C-A-B,\\ \widetilde{G}(e)\cap\widetilde{H}(e), & \text{if} \quad e\in B\cap C-A,\\ \widetilde{F}(e)\cap\widetilde{G}(e), & \text{if} \quad e\in A\cap B-C,\\ \widetilde{F}(e)\cap\widetilde{H}(e), & \text{if} \quad e\in A\cap C-B,\\ \widetilde{F}(e)\cap\widetilde{G}(e)\cap\widetilde{H}(e), & \text{if} \quad e\in A\cap B\cap C.\end{cases}$$

Therefore, $\widetilde{L}(e)=\widetilde{K}(e),\forall e\in A\cup B\cup C$. Proved.

□

Definition 3.30. The difference of interval-valued intuitionistic hesitant fuzzy soft sets $(\widetilde{F},A)$ and $(\widetilde{G},B)$ over a common U with $A\cap B\neq\phi$ which is denoted by $(\widetilde{F},A)-(\widetilde{G},B)$ and is defined by

$$(\widetilde{F},A)-(\widetilde{G},B)=(\widetilde{H},C)$$

where $C=A\cap B$ and for $\forall e\in C,\widetilde{H}(e)=\widetilde{F}(e)-\widetilde{G}(e)=\{d(\widetilde{h}_{ij},\widetilde{g}_{ij})\}$, $\widetilde{h}_{ij}$ denotes the IVIHFE of the i object under the j attribute in the interval-valued intuitionistic hesitant fuzzy soft set $(\widetilde{F},A)$, $\widetilde{g}_{ij}$ denotes the IVIHFE of the i object under the j attribute in the interval-valued intuitionistic hesitant fuzzy soft set $(\widetilde{G},B)$.

Example 3.31. Reconsider Example 3.12, the difference between $(\widetilde{F},A)$ and $(\widetilde{G},B)$ is:

$\widetilde{H}(e_1)=\{<x_1,0.1>,<x_2,0.0625>,<x_3,0.05>\}$,
$\widetilde{H}(e_2)=\{<x_1,0.0875>,<x_2,0.125>,<x_3,0.0625>\}$.

From Example 3.31, we can find that the difference between two interval-valued intuitionistic hesitant fuzzy soft sets is a fuzzy soft set.

According to this method, we can define that the two interval-valued intuitionistic hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ are equivalent if and only if $(\widetilde{H}, C) = \varnothing$.

Furthermore, suppose that $(\widetilde{F}, A), (\widetilde{G}, B)$, and $(\widetilde{H}, C)$ be three interval-valued intuitionistic hesitant fuzzy soft sets, the interval-valued intuitionistic hesitant fuzzy soft set $(\widetilde{G}, B)$ is closer to the interval-valued intuitionistic hesitant fuzzy soft set $(\widetilde{F}, A)$ than the interval-valued intuitionistic hesitant fuzzy soft set $(\widetilde{H}, C)$ with $(\widetilde{F}, A)$, if and only if $((\widetilde{G}, B) - (\widetilde{F}, A)) \widetilde{\subseteq} ((\widetilde{H}, C) - (\widetilde{F}, A))$.

Definition 3.32. If $(\widetilde{F}, A)$ is an interval-valued intuitionistic hesitant fuzzy soft set over U, the "power-λ" operation on interval-valued intuitionistic hesitant fuzzy soft set is defined as follows:

$$(\widetilde{F}, A)^{\lambda} = \{\widetilde{F}^{\lambda}(e) \mid e \in A\}, \text{ where } \widetilde{F}^{\lambda}(e) = \{\widetilde{h}^{\lambda}_{\widetilde{F}(e)}(x) \mid x \in U\}, \lambda > 0.$$

Definition 3.33. If $(\widetilde{F}, A)$ is an interval-valued intuitionistic hesitant fuzzy soft set over U, the "λ-multiply" operation on interval-valued intuitionistic hesitant fuzzy soft set is defined as follows:

$$\lambda(\widetilde{F}, A) = \{\lambda\widetilde{F}(e) \mid e \in A\}, \text{ where } \lambda\widetilde{F}(e) = \{\lambda\widetilde{h}_{\widetilde{F}(e)}(x) \mid x \in U\}.$$

Theorem 3.34. *Let $(\widetilde{F}, A)$ is an interval-valued intuitionistic hesitant fuzzy soft set over U, and $\lambda > 0, \lambda_1 > 0$. Then,*
(1) $((\widetilde{F}, A)^c)^{\lambda} = (\lambda(\widetilde{F}, A))^c$.
(2) $\lambda((\widetilde{F}, A)^c) = ((\widetilde{F}, A)^{\lambda})^c$.
(3) $(\widetilde{F}, A)^{\lambda\lambda_1} = ((\widetilde{F}, A)^{\lambda})^{\lambda_1}$.
(4) $(\lambda\lambda_1)(\widetilde{F}, A) = \lambda(\lambda_1(\widetilde{F}, A))$.

Proof. In the following, we shall prove (1); (2)-(4) are proved analogously.
(1) $\forall e \in A$, according to Theorem 2.12, we have $(\widetilde{F}^c(e))^{\lambda} = (\lambda\widetilde{F}(e))^c$. Hence, $((\widetilde{F}, A)^c)^{\lambda} = (\lambda(\widetilde{F}, A))^c$. Proved. □

Definition 3.35. The "average" operation on two interval-valued intuitionistic hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ over U which is denoted by $(\widetilde{F}, A) \oplus (\widetilde{G}, B)$ and is defined by

$$(\widetilde{F}, A) \oplus (\widetilde{G}, B) = (\widetilde{H}, A \times B)$$
where $\widetilde{H}(\alpha, \beta) = \widetilde{F}(\alpha) \oplus \widetilde{G}(\beta), \forall (\alpha, \beta) \in A \times B$.

Definition 3.36. The "geometric" operation on two interval-valued intuitionistic hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ over U which is denoted by $(\widetilde{F}, A) \otimes (\widetilde{G}, B)$ and is defined by

$$(\widetilde{F}, A) \otimes (\widetilde{G}, B) = (\widetilde{O}, A \times B)$$
where $\widetilde{O}(\alpha, \beta) = \widetilde{F}(\alpha) \otimes \widetilde{G}(\beta), \forall (\alpha, \beta) \in A \times B$.

Example 3.37. Reconsider Example 3.12, the results of "average" and "geometric" of interval-valued intuitionistic hesitant fuzzy soft sets $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ are shown in TABLEs 7 and 8, respectively.

TABLE 7. The results of "average" operation on $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$

U	(e_1, e_1)	(e_1, e_2)
x_1	$\{([0.85, 0.92], [0.01, 0.04])\}$	$\{([0.75, 0.88], [0.01, 0.04])\}$
x_2	$\{([0.76, 0.94], [0.01, 0.04]), ([0.84, 0.94], [0.01, 0.06])\}$	$\{([0.76, 0.91], [0.01, 0.04]), ([0.84, 0.91], [0.01, 0.06])\}$
x_3	$\{([0.8, 0.91], [0.02, 0.09])\}$	$\{([0.8, 0.94], [0.02, 0.06]), ([0.85, 0.94], [0, 0.03])\}$
	(e_1, e_3)	(e_2, e_1)
x_1	$\{([0.8, 0.92], [0, 0.02])\}$	$\{([0.88, 0.94], [0.02, 0.06]), ([0.88, 0.98], [0, 0.02])\}$
x_2	$\{([0.76, 0.94], [0, 0.02]), ([0.84, 0.94], [0, 0.03])\}$	$\{([0.76, 0.92], [0.02, 0.06])\}$
x_3	$\{([0.85, 0.97], [0, 0.03]), ([0.9, 0.97], [0, 0.03])\}$	$\{([0.88, 0.97], [0, 0.03])\}$
	(e_2, e_2)	(e_2, e_3)
x_1	$\{([0.8, 0.91], [0.02, 0.06]), ([0.8, 0.94], [0, 0.02])\}$	$\{([0.84, 0.94], [0, 0.03]), ([0.84, 0.96], [0, 0.01])\}$
x_2	$\{([0.76, 0.88], [0.02, 0.06])\}$	$\{([0.76, 0.92], [0, 0.03])\}$
x_3	$\{([0.88, 0.98], [0, 0.02]), ([0.91, 0.98], [0, 0.01])\}$	$\{([0.91, 0.99], [0, 0.01]), ([0.94, 0.99], [0, 0.1])\}$

TABLE 8. The results of "geometric" operation on $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$

U	(e_1, e_1)	(e_1, e_2)
x_1	$\{([0.35, 0.48], [0.19, 0.36])\}$	$\{([0.25, 0.42], [0.19, 0.36])\}$
x_2	$\{([0.24, 0.56], [0.19, 0.36]), ([0.36, 0.56], [0.19, 0.44])\}$	$\{([0.24, 0.49], [0.19, 0.36]), ([0.36, 0.49], [0.19, 0.44])\}$
x_3	$\{([0.3, 0.49], [0.28, 0.51])\}$	$\{([0.3, 0.56], [0.28, 0.44]), ([0.35, 0.56], [0.2, 0.3])\}$
	(e_1, e_3)	(e_2, e_1)
x_1	$\{([0.3, 0.48], [0.1, 0.28])\}$	$\{([0.42, 0.56], [0.02, 0.06]), ([0.42, 0.64], [0.1, 0.28])\}$
x_2	$\{([0.24, 0.56], [0.1, 0.28]), ([0.36, 0.56], [0.1, 0.37])\}$	$\{([0.24, 0.48], [0.28, 0.44])\}$
x_3	$\{([0.35, 0.63], [0.2, 0.37]), ([0.4, 0.63], [0.2, 0.37])\}$	$\{([0.42, 0.63], [0.1, 0.37])\}$
	(e_2, e_2)	(e_2, e_3)
x_1	$\{([0.3, 0.49], [0.28, 0.44]), ([0.3, 0.56], [0.1, 0.28])\}$	$\{([0.36, 0.56], [0.2, 0.37]), ([0.36, 0.64], [0, 0.19])\}$
x_2	$\{([0.24, 0.42], [0.28, 0.44])\}$	$\{([0.24, 0.48], [0.2, 0.37])\}$
x_3	$\{([0.42, 0.72], [0.1, 0.28]), ([0.49, 0.72], [0, 0.19])\}$	$\{([0.49, 0.81], [0, 0.19]), ([0.56, 0.81], [0, 0.19])\}$

Theorem 3.38. *(De Morgan's Law) Let $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ be two interval-valued intuitionistic hesitant fuzzy soft sets over U. Then,*
(1) $((\widetilde{F}, A) \oplus (\widetilde{G}, B))^c = (\widetilde{F}, A)^c \otimes (\widetilde{G}, B)^c$.
(2) $((\widetilde{F}, A) \otimes (\widetilde{G}, B))^c = (\widetilde{F}, A)^c \oplus (\widetilde{G}, B)^c$.

Proof. In the following, we shall prove (1); (2) is proved analogously.
(1)Suppose that $(\widetilde{F}, A) \oplus (\widetilde{G}, B) = (\widetilde{H}, A \times B)$. Since $((\widetilde{F}, A) \oplus (\widetilde{G}, B))^c = (\widetilde{H}^c, A \times B)$ and $(\widetilde{F}, A)^c \otimes (\widetilde{G}, B)^c = (\widetilde{F}^c, A) \otimes (\widetilde{G}^c, B) = (\widetilde{O}, A \times B)$. Now take $(\alpha, \beta) \in A \times B$, therefore, according to Theorem 2.12, $\widetilde{H}^c(\alpha, \beta) = (\widetilde{H}(\alpha, \beta))^c = (\widetilde{F}(\alpha) \oplus \widetilde{G}(\beta))^c = \widetilde{F}^c(\alpha) \otimes \widetilde{G}^c(\beta)$. Again, $\widetilde{O}(\alpha, \beta) = \widetilde{F}^c(\alpha) \otimes \widetilde{G}^c(\beta)$. Hence, $\widetilde{H}^c(\alpha, \beta) = \widetilde{O}(\alpha, \beta)$. Proved. □

Theorem 3.39. *If $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ are two interval-valued intuitionistic hesitant fuzzy soft sets over U, $\lambda > 0$. Then,*

(1) $\lambda((\widetilde{F}, A) \oplus \ (\widetilde{G}, B)) = \lambda(\widetilde{F}, A) \oplus \lambda(\widetilde{G}, B)$.
(2) $(\widetilde{F}, A)^{\lambda} \otimes (\widetilde{G}, B)^{\lambda} = ((\widetilde{F}, A) \otimes (\widetilde{G}, B))^{\lambda}$.

Proof. In the following, we shall prove (1); (2) is proved analogously.
(1) Suppose that $(\widetilde{F}, A) \oplus (\widetilde{G}, B) = (\widetilde{H}, A \times B)$. Since $\lambda((\widetilde{F}, A) \oplus (\widetilde{G}, B)) = \lambda(\widetilde{H}, A \times B) = (\widetilde{X}, A \times B)$ and $\lambda(\widetilde{F}, A) \oplus \lambda(\widetilde{G}, B) = (\widetilde{Y}, A \times B)$. Now take $(\alpha, \beta) \in A \times B$, therefore, according to Theorem 2.12, $\widetilde{X}(\alpha, \beta) = \lambda(\widetilde{F}(\alpha) \oplus \widetilde{G}(\beta)) = \lambda\widetilde{F}(\alpha) \oplus \lambda\widetilde{G}(\beta)$, $\widetilde{Y}(\alpha, \beta) = \lambda\widetilde{F}(\alpha) \oplus \lambda\widetilde{G}(\beta)$. Hence, $\widetilde{X}(\alpha, \beta) = \widetilde{Y}(\alpha, \beta)$. Proved. □

4. Application to decision making with interval-valued intuitionistic hesitant fuzzy soft information

Roy and Maji [19] proposed an algorithm for recognition of an object according to the comparison of different objects. Later, Kong et al. [12] modified the algorithm in [19], and presented a novel algorithm which was based on the comparison of choice values of different objects, that is to say, the higher the choice value, the better the object is. Zhang et al. [31] studied the interval-valued intuitionistic fuzzy soft set based decision making problems by using level soft sets.

Zhang [30] has shown an intensive study on interval-valued intuitionistic hesitant fuzzy information aggregation and application in group decision making. A series of operators, such as IVIHFWG, GIVIHFWG, IVIHFOWG, GIVIHFOWG , IVIHFHG, and GIVIHFHG, have been introduced to aggregate the interval-valued intuitionistic hesitant fuzzy information.

In the following, we will apply the level soft sets and aggregation operators methods to interval-valued intuitionistic hesitant fuzzy soft sets.

4.1. The method of level soft sets.

Let $U = \{x_1, x_2, \cdots, x_m\}, A = \{e_1, e_2, \cdots, e_n\}$, and $(\widetilde{F}, A)$ be an interval-valued intuitionistic hesitant fuzzy soft set over U. For $\forall e \in A, \widetilde{F}(e) = \{< x_1, \widetilde{h}(e)(x_1) >, < x_2, \widetilde{h}(e)(x_2) >, \cdots, < x_n, \widetilde{h}(e)(x_n) >\}$. According to Definition 2.14, we can compute the average of each interval-valued intuitionistic hesitant fuzzy element. Then, we define the induced interval-valued intuitionistic fuzzy set $\widetilde{\Gamma}(e) = \{< x_1,\ A(\widetilde{h}(e)(x_1)) >,\ < x_2,\ A(\widetilde{h}(e)(x_2)) >,\ \cdots,\ < x_n,$ $A(\widetilde{h}(e)(x_n)) >\}$. Once the induced interval-valued intuitionistic fuzzy soft set has been arrived at, we can determine the optimal alternative according to the algorithm in [31]. Therefore, the algorithm involves the following steps:

Algorithm 1.
Step 1. Input the interval-valued intuitionistic hesitant fuzzy soft set $(\widetilde{F}, A)$.
Step 2. Compute the induced interval-valued intuitionistic fuzzy soft set $\triangle_{\widetilde{F}} = (\widetilde{\Gamma}, A)$.
Step 3. Input a threshold interval-valued intuitionistic fuzzy set $\lambda : A \to L$ (or give a threshold value pair $(\alpha, \beta) \in L$; or choose the mid-level decision rule;or choose the top-bottom-level decision rule;or choose the bottom-bottom-level decision rule) for decision making.
Step 4. Compute the level soft set $L(\triangle_{\widetilde{F}}; \lambda)$ of $\triangle_{\widetilde{F}}$ with respect to the threshold interval-valued intuitionstic fuzzy set λ (or the $(\alpha, \beta)-$level soft set $L(\triangle_{\widetilde{F}}; \alpha, \beta)$; or the mid-level soft set $L(\triangle_{\widetilde{F}}; mid)$; or the top-bottom-level soft set(top-bottom);or

the bottom-bottom-level soft set(bottom-bottom)).
Step 5. Present the level soft set $L(\triangle_{\widetilde{F}};\lambda)$(or $L(\triangle_{\widetilde{F}};\alpha,\beta)$; or $L(\triangle_{\widetilde{F}};\text{mid})$; or $L(\triangle_{\widetilde{F}};\text{top-bottom})$; or $L(\triangle_{\widetilde{F}};\text{bottom-bottom})$) in tabular form. For any $x_i \in U$, compute the choice value c_i of x_i.
Step 6. The optimal decision is to select x_k if $c_k = \max_{x_i \in U}\{c_i\}$.
Step 7. If k has more than one value, then any one of x_k may be chosen.

Remark 4.1. If there are too many optimal choices in Step 7, we may go back to the Step 3 and change the threshold(or decision rule) such that only one optimal choice remains in the end.

4.2. **The method of aggregation techniques.**

4.2.1. *Computing the completely unknown weights: the maximizing deviation method.*

The maximizing deviation method was proposed by Wang [22] to determine the attribute weights for solving MADM problems with numerical information. For a MADM problem, the attribute with a larger deviation value among alternatives should be assigned a larger weight or vice versa.

For the attribute $x_i \in X$, the deviation of the alternative A_i to all other alternatives can be defined as follows:

$$D_{ij}(w_j) = \sum_{k=1}^{m} w_j d(\widetilde{h}_{ij},\ \widetilde{h}_{kj})$$

where $i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n$ and the $d(\widetilde{h}_{ij},\ \widetilde{h}_{kj})$ denotes the Hamming distance between the $\widetilde{h}_{ij}$ and $\widetilde{h}_{kj}$ defined as in Eq. (2.1).

$$D_j(w_j) = \sum_{i=1}^{m} D_{ij}(w_j) = \sum_{i=1}^{m}\sum_{k=1}^{m} w_j d(\widetilde{h}_{ij},\ \widetilde{h}_{kj})$$

where $j = 1, 2, \cdots, n,\ D_j(w_j)$ represents the deviation value of all alternatives to other alternatives for the attribute $x_i \in X$.

$$D(w_j) = \sum_{j=1}^{n} D_j(w_j) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} w_j d(\widetilde{h}_{ij},\ \widetilde{h}_{kj})$$

represents the total deviation value of all attributes to all alternatives, where w_j represents the weight of the attribute $x_j \in X$.

Based on the above analysis, we can construct a non-linear programming model to select the weight vector w_j which maximizes all deviation values for all the attributes which are shown as follows:

$$\begin{cases} \max\ D(w_j) = \sum_{j=1}^{n} D_j(w_j) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} w_j d(\widetilde{h}_{ij},\ \widetilde{h}_{kj}) \\ s.t.\ \sum_{j=1}^{n} w_j^2 = 1 \end{cases} \tag{4.1}$$

Constructed the Lagrange function as follows:

$$L(w_j,\ \lambda) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} w_j d(\widetilde{h}_{ij}, \widetilde{h}_{kj}) + \frac{\lambda}{2}(\sum_{j=1}^{n} w_j^2 - 1).$$

Then the partial derivatives of $L(w_j,\ \lambda)$ with respect to w_j and λ are computed as follows:

$$(4.2)\qquad \begin{cases} \frac{\partial L(w_j,\ \lambda)}{\partial w_j} = \sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m} d(\widetilde{h}_{ij}, \widetilde{h}_{kj}) + \lambda w_j = 0 \\ \frac{\partial L(w_j,\ \lambda)}{\partial \lambda} = \frac{1}{2}(\sum\limits_{j=1}^{n} w_j^2 - 1) = 0 \end{cases}$$

Therefore, w_j and λ are computed as follows:

$$(4.3)\qquad \begin{cases} \lambda = -\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m} d(\widetilde{h}_{ij},\ \widetilde{h}_{kj})\right)^2} \\ \\ w_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} d(\widetilde{h}_{ij},\ \widetilde{h}_{kj})}{\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m} d(\widetilde{h}_{ij},\ \widetilde{h}_{kj})\right)^2}} \end{cases}$$

By normalizing $w_j (j = 1, 2, \cdots, n)$, we make their sum into a unit, and get as follows:

$$(4.4)\qquad w_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} d(\widetilde{h}_{ij},\ \widetilde{h}_{kj})}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d(\widetilde{h}_{ij},\ \widetilde{h}_{kj})}$$

Algorithm 2.
Step 1. Input the interval-valued intuitionistic hesitant fuzzy soft set $(\widetilde{F}, A)$.
Step 2. Compute relative weight w_j of parameter e_j.
Step 3. Utilize the GIVIHFWG operator proposed by Zhang to obtain the interval-valued intuitionistic hesitant fuzzy elements $\widetilde{h}_i$ for the alternatives $x_i (i = 1, 2, \cdots, m)$, i.e.,
$\widetilde{h}_i = \text{GIVIHFWG}(\widetilde{h}_{i1}, \widetilde{h}_{i2}, \cdots, \widetilde{h}_{in}) = \{([(1 - \prod\limits_{j=1}^{n}(1 - (\mu^-_{\alpha_{ij}})^\lambda)^{w_j})^{1/\lambda}, (1 - \prod\limits_{j=1}^{n}(1 - (\mu^+_{\alpha_{ij}})^\lambda)^{w_j})^{1/\lambda}], [(1 - (1 - \prod\limits_{j=1}^{n}(1 - (1 - \nu^-_{\alpha_{ij}})^\lambda)^{w_j})^{1/\lambda}, (1 - (1 - \prod\limits_{j=1}^{n}(1 - (1 - \nu^+_{\alpha_{ij}})^\lambda)^{w_j})^{1/\lambda}]) \mid \widetilde{\alpha}_{i1} \in \widetilde{h}_{i1}, \widetilde{\alpha}_{i2} \in \widetilde{h}_{i2}, \cdots, \widetilde{\alpha}_{in} \in \widetilde{h}_{in}\}$.
Step 4. Computer the score values $S(\widetilde{h}_i)$ of $\widetilde{h}_i$.
Step 5. The optimal decision is to select x_j if $S(\widetilde{h}_j) = \max_i S(\widetilde{h}_i)$.
Step 6. If j has more than one value, then any one of x_j may be chosen.

Remark 4.2. If there are too many optimal choices in Step 6, we may go back to the Step 3 and change the value of λ such that only one optimal choice remains in the end.

Example 4.3. Consider an Internet company planning to select a software development project to invest. Suppose that $U = \{x_1, x_2, x_3, x_4\}$ is the set of software development projects under consideration, $A = \{e_1, e_2, e_3\}$ is the set of parameters, $A = \{e_1 = \text{economic feasibility},\ e_2 = \text{technological feasibility},\ e_3 = \text{staff feasibility}\}$. The tabular is presented in TABLE 9.

Table 9. The tabular representation of the interval-valued intuitionistic hesitant fuzzy soft set $(\widetilde{F}, A)$

U	e_1	e_2
x_1	$\{([0.3,0.4],[0.2,0.3])\}$	$\{([0.5,0.7],[0.1,0.2]),([0.7,0.7],[0.1,0.2])\}$
x_2	$\{([0.7,0.9],[0,0.1]),([0.9,0.9],[0,0.1])\}$	$\{([0.4,0.5],[0.1,0.2])\}$
x_3	$\{([0.4,0.6],[0.1,0.2])\}$	$\{([0.4,0.5],[0.2,0.3])\}$
x_4	$\{([0.5,0.7],[0,0.1])\}$	$\{([0.3,0.5],[0.1,0.2])\}$
	e_3	
x_1	$\{([0.2,0.3],[0.1,0.3])\}$	
x_2	$\{([0.5,0.8],[0,0.2])\}$	
x_3	$\{([0.2,0.5],[0.1,0.3]),([0.4,0.5],[0.1,0.3])\}$	
x_4	$\{([0.3,0.4],[0.2,0.3]),([0.3,0.6],[0.2,0.3])\}$	

According to Algorithm 1, we can compute the average of each interval-valued intuitionistic hesitant fuzzy element and obtained the induced interval-valued intuitionistic fuzzy soft set $\triangle_{\widetilde{F}} = (\widetilde{\Gamma}, A)$, which is shown in TABLE 10.

Table 10. The induced interval-valued intuitionistic fuzzy soft set $(\widetilde{F}, A)$

U	e_1	e_2	e_3
x_1	$([0.3,0.4],[0.2,0.3])$	$([0.6,0.7],[0.1,0.2])$	$([0.2,0.3],[0.1,0.3])$
x_2	$([0.8,0.9],[0,0.1])$	$([0.4,0.5],[0.1,0.2])$	$([0.5,0.8],[0,0.2])$
x_3	$([0.4,0.6],[0.1,0.2])$	$([0.4,0.5],[0.2,0.3])$	$([0.3,0.5],[0.1,0.3])$
x_4	$([0.5,0.7],[0,0.1])$	$([0.3,0.5],[0.1,0.2])$	$([0.3,0.5],[0.2,0.3])$

As an adjustable approach, we can utilize different rules to obtain different decision information. In this paper, we use the top-bottom-level decision rule. The top-bottom-threshold of $\triangle_{\widetilde{F}} = (\widetilde{\Gamma}, A)$ is an interval-valued intuitionistic fuzzy set and can be calculated as top-bottom$_{\triangle_{\widetilde{F}}} = \{< e_1, [0.8, 0.9], [0, 0.1] >, < e_2, [0.6, 0.7], [0.1, 0.2] >, < e_3, [0.5, 0.8], [0, 0.2] >\}$. According to Definition 2.14, we can obtain the choice values of each software development project which are shown in TABLE 11. From the TABLE 11, we can see that the maximum choice value is 2, so the optimal decision is to select x_2. Therefore, the internet company should select x_2 as the best software development project to invest.

Table 11. Tabular representation of $L((\widetilde{F}, A);\text{top-bottom})$ with choice values

U	e_1	e_2	e_3	Choice value(c_i)
x_1	0	1	0	$c_1 = 1$
x_2	1	0	1	$c_2 = 2$
x_3	0	0	0	$c_3 = 0$
x_4	0	0	0	$c_4 = 0$

According to Algorithm 2, we can compute the weight vector of parameter e is $w = (0.45, 0.21, 0.34)^T$ by Eq.(4.4), then we use GIVIHFWG operator and let $\lambda = 1$, we have the score values

$$S(\widetilde{h}_1) = 0.1415, \quad S(\widetilde{h}_2) = 0.5871, \quad S(\widetilde{h}_3) = 0.2611, \quad S(\widetilde{h}_4) = 0.3347.$$

Obviously, the maximum score value is 0.5871 and the optimal decision is to select x_2. Therefore, the internet company should select x_2 as the best software development project to invest.

Based on the results of Algorithm 1 and Algorithm 2, we can know that the methods proposed in this paper are practical and effective.

5. Conclusions

In this paper, we propose the concept of interval-valued intuitionistic hesitant fuzzy soft sets, which are a combination of interval-valued intuitionistic hesitant fuzzy sets and soft sets. Then we define the complement, AND, OR, union, intersection, restricted union, extended intersection, difference, necessity, possibility, power-λ, λ-multiply, average, and geometric operations on interval-valued intuitionistic hesitant fuzzy soft sets and some basic properties are discussed in detailed. Finally, we apply two propose algorithms to a decision making problems with the help of level soft sets and aggregation operators. We hope that our research can be developed deeply in the future.

Acknowledgements. The author would like to thank the anonymous referee for valuable suggestions regarding the exposition of this paper.

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