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Definition of fuzzy metric spaces with t-conorm

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Abstract. In this paper, we proposed a new definition of fuzzy metric space with t-conorm and defined open ball in this fuzzy metric space. Also, we define some property such as Cauchy sequence, completeness, continuous and contractive map and also establish fixed point theorem for the new complete fuzzy metric spaces.

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1. Introduction

The theory of fuzzy sets was introduced by L. Zadeh in 1965 [11]. Since the inception of fuzzy set theory, many authors introduced the notion of fuzzy metric spaces in different ways. One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space. This problem has been investigated by many authors from different points of view. In particular, Kramosil and Michalekin [5] generalized the concept of probabilistic metric space. Later on, George and Veeramani [3] have introduced and studied a notion of fuzzy metric space. Many researchers have been working in this field such as in [1, 4, 6, 7, 8, 9, 10]. A background of fuzzy concepts is presented in section 2. In section 3, we proposed the new definition of fuzzy metric space and open ball with some examples. Finally, conclusions are presented in Section 4.

2. Preliminaries

Definition 2.1 ([2]). A binary operation $* : [0,1] \times [0,1] \mapsto [0,1]$ is a continuous t-norm if it satisfies the following conditions.
(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Some of the basic t-norm are as follows:
(1) $*(a, b) = min\{a, b\}$,
(2) $*(a, b) = max\{a + b - 1, 0\}$,
(3) $*(a, b) = ab$,
(4) $*(a, b) = \begin{cases} min\{a, b\}, & if \quad max\{a, b\} = 1, \\ 0, & otherwise \end{cases}$

Definition 2.2 ([2]). A binary operation $* : [0, 1] \times [0, 1] \mapsto [0, 1]$ is a continuous t-conorm if it satisfies the following conditions.
(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 0 = a$ for all $a \in [0, 1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Some of the basic t-conorm are as follows:
(1) $*(a, b) = max\{a, b\}$,
(2) $*(a, b) = min\{a + b, 1\}$,
(3) $*(a, b) = a + b - ab$,
(4) $*(a, b) = \begin{cases} max\{a, b\}, & if \quad min\{a, b\} = 0, \\ 1, & otherwise \end{cases}$

Remark 2.3. If T is a t-norm then equality $S(a, b) := 1 - T(1 - a, 1 - b)$ defines a t-conorm.

Kramosil and Michalek [5] have defined fuzzy metric space as follows.

Definition 2.4. A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set , $*$ is a continuous t-norm , and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,
(KM1) $M(x, y, 0) = 0$,
(KM2) $M(x, y, t) = 1$ for $t > 0$ if and only if $x = y$,
(KM3) $M(x, y, t) = M(y, x, t)$,
(KM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(KM5) $M(x, y, .) : (0, \infty) \longmapsto [0, 1]$ is continuous.

The concept of fuzzy metric space is defined by George and Veeramani [3] as follows.

Definition 2.5. A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set , $*$ is a continuous t-norm , and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,
(GV1) $M(x, y, t) > 0$,
(GV2) $M(x, y, t) = 1$ if and only if $x = y$,
(GV3) $M(x, y, t) = M(y, x, t)$,
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(GV5) $M(x, y, .) : (0, \infty) \longmapsto [0, 1]$ is continuous.

3. New fuzzy metric space

In this section, we initially propose the new fuzzy metric space with t-conorm and present two examples for this definition. Then, we define open ball according to new fuzzy metric space. Also, we define some property such as Cauchy sequence, completeness, continuous, contractive map and fixed point theorem in new complete fuzzy metric spaces. Now we present our definition of fuzzy metric space.

Definition 3.1. A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non- empty) set, $*$ is a continuous t-conorm, and M is a fuzzy set on $X^2 \times [0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s \geq 0$,
(AF1) $0 \leq M(x, y, t) \leq 1$,
(AF2) $M(x, y, 0) = 1(M(x, y, t) = 0, \quad \forall t > 0)$ if and only if $x = y$,
(AF3) $M(x, y, t) = M(y, x, t)$,
(AF4) $M(x, y, t) * M(y, z, s) \geq M(x, z, t + s)$,
(AF5) $M(x, y, .) : [0, \infty) \longmapsto [0, 1]$ is continuous.

According to above new definition, we can interpret distance of two point x and y as follows:
$M(x, y, t) = 1$ means that $d(x, y) = t$. In other word, $M(x, y, t) = r$ means with r degree of accuracy $d(x, y) = t$.

Proposition 3.2. *If* $*(a, b) = max\{a, b\}$ *is t-conorm and* X *is an arbitrary (non-empty) set and* $M(x, y, t) = \frac{d(x,y)}{t+d(x,y)}$ *where* (X, d) *is metric space, then 3-tuple* $(X, M, *)$ *is a fuzzy metric space.*

Proof. It sufficient that prove the following property:
(AF1) $0 \leq M(x, y, t) \leq 1$,
(AF2) $M(x, y, 0) = 1(M(x, y, t) = 0, \quad \forall t > 0)$ if and only if $x = y$,
(AF3) $M(x, y, t) = M(y, x, t)$,
(AF4) $M(x, y, t) * M(y, z, s) \geq M(x, z, t + s)$,
(AF5) $M(x, y, .) : [0, \infty) \longmapsto [0, 1]$ is continuous.
(AF1), (AF2), (AF3) and (AF5) is evident but, for (AF4) it should be proved that

$$\frac{d(x,y)}{t+d(x,y)} * \frac{d(y,z)}{s+d(y,z)} \geq \frac{d(x,z)}{t+s+d(x,z)} \tag{3.1}$$

or

$$max\left\{\frac{d(x,y)}{t+d(x,y)}, \frac{d(y,z)}{s+d(y,z)}\right\} \geq \frac{d(x,z)}{t+s+d(x,z)}. \tag{3.2}$$

Let

$$max\left\{\frac{d(x,y)}{t+d(x,y)}, \frac{d(y,z)}{s+d(y,z)}\right\} = \frac{d(x,y)}{t+d(x,y)}. \tag{3.3}$$

We prove that $\frac{d(x,y)}{t+d(x,y)} \geq \frac{d(x,z)}{t+s+d(x,z)}$ or $d(x, z) \leq \frac{t+s}{t} d(x, y)$.
Because
$\frac{d(y,z)}{s+d(y,z)} \leq \frac{d(x,y)}{t+d(x,y)}$ hence $d(y, z) \leq \frac{s}{t} d(x, y)$.

Regarding to metric property of d,
$d(x,z) \leq d(x,y) + d(y,z) \leq d(x,y) + \frac{s}{t}d(x,y) = \frac{t+s}{t}d(x,y)$.
Therefore proof is complete. □

Proposition 3.3. *If* $*(a,b) = max\{a,b\}$ *and* (X,d) *is (non-empty) metric space and*

$$M(x,y,t) = e^{-(t-d(x,y))^2} \tag{3.4}$$

then 3-tuple $(X,M,*)$ *is a fuzzy metric space.*

Proof. It sufficient that prove the following property:
(AF1) $0 \leq M(x,y,t) \leq 1$,
(AF2) $M(x,y,0) = 1 (M(x,y,t) = 0, \quad \forall t > 0)$ if and only if $x = y$,
(AF3) $M(x,y,t) = M(y,x,t)$,
(AF4) $M(x,y,t) * M(y,z,s) \geq M(x,z,t+s)$,
(AF5) $M(x,y,.) : [0,\infty) \longmapsto [0,1]$ is continuous.
(AF1), (AF2), (AF3) and (AF5) is evident but, for (AF4) it should be proved that

$$e^{-(t-d(x,y))^2} * e^{-(s-d(y,z))^2} \geq e^{-(t+s-d(x,z))^2} \tag{3.5}$$

or

$$max\left\{e^{-(t-d(x,y))^2}, e^{-(s-d(y,z))^2}\right\} \geq e^{-(t+s-d(x,z))^2}. \tag{3.6}$$

Let

$$max\left\{e^{-(t-d(x,y))^2}, e^{-(s-d(y,z))^2}\right\} = e^{-(t-d(x,y))^2}. \tag{3.7}$$

Then $(s-d(y,z))^2 \geq (t-d(x,y))^2$ and then $(s-d(y,z))^2 - (t-d(x,y))^2 \geq 0$. So $(s-d(y,z)-t+d(x,y))(s-d(y,z)+t-d(x,y)) \geq 0$. It is possible that when,

$$(s-d(y,z)-t+d(x,y)) \geq 0, \quad (s-d(y,z)+t-d(x,y)) \geq 0 \tag{3.8}$$

or

$$(s-d(y,z)-t+d(x,y)) \leq 0, \quad (s-d(y,z)+t-d(x,y)) \leq 0. \tag{3.9}$$

With (3.8)

$$s-d(y,z) \geq 0. \tag{3.10}$$

And with (3.9)

$$s-d(y,z) \leq 0. \tag{3.11}$$

Now we prove that $e^{-(t-d(x,y))^2} \geq e^{-(t+s-d(x,z))^2}$ or $(t+s-d(x,z))^2 \geq (t-d(x,y))^2$ or $(t+s-d(x,z))^2 - (t-d(x,y))^2 \geq 0$ and this is possible when the,

$$(s-d(x,z)+d(x,y)) \geq 0, \quad (2t+s-d(x,z)-d(x,y)) \geq 0 \tag{3.12}$$

or

$$(s-d(x,z)+d(x,y)) \leq 0, \quad (2t+s-d(x,z)-d(x,y)) \leq 0. \tag{3.13}$$

To prove (3.12), We assume that

$$(s-d(x,z)+d(x,y)) \geq 0 \tag{3.14}$$

and we prove that $(2t + s - d(x, z) - d(x, y)) \geq 0$.
With (3.10) and (3.14)
$(2t + s - d(x, z) - d(x, y)) = (s - d(x, z) + d(x, y)) + 2(s - d(y, z)) \geq 0$.
And similarity, to prove (3.13), We assume that

$$(s - d(x, z) + d(x, y)) \leq 0 \tag{3.15}$$

and we prove that $(2t + s - d(x, z) - d(x, y)) \leq 0$.
With (3.11) and (3.15)
$(2t + s - d(x, z) - d(x, y)) = (s - d(x, z) + d(x, y)) + 2(s - d(y, z)) \leq 0$. □

Definition 3.4. Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r \leq 1$ is defined by

$$B(x, r, t) = \{y \in X \mid M(x, y, s) \geq r \ ; \ for \ \ 0 \leq s < t\}. \tag{3.16}$$

Example 3.5. If $X = R$ and $M(x, y, t) = \frac{|x-y|}{t+|x-y|}$ then,

$$\begin{aligned} B(4, 0.7, t) &= \{y \in R \mid \tfrac{|4-y|}{s+|4-y|} \geq 0.7 \ ; \ for \ \ 0 \leq s < t\} \\ &= (4 - \tfrac{7}{3}t, 4 + \tfrac{7}{3}t). \end{aligned} \tag{3.17}$$

Example 3.6. If $X = R$ and $M(x, y, t) = e^{-(t-|x-y|)^2}$ then,

$$\begin{aligned} B(4, 0.7, t) &= \{y \in R \mid e^{-(s-|4-y|)^2} \geq 0.7 \ ; \ for \ \ 0 \leq s < t\} \\ &= \{y \in R \mid (s- \mid 4 - y \mid)^2 \leq 0.3566749 \ ; \ for \ \ 0 \leq s < t\} \\ &= (4.59722266 - t, 4.59722266 + t). \end{aligned} \tag{3.18}$$

Definition 3.7. Let $(X, M, *)$ be a fuzzy metric space. A sequence $(x_n)_{n \in N} \subset X$ is called an $M-$Cauchy sequence if the following condition is satisfied:

$$\forall \ \epsilon \in (0, 1) \ \ \forall \ t > 0 \ \ \exists \ n_0 \in N \ \ \forall \ m, n \geq n_0; \quad M(x_m, x_n, t) < \epsilon \tag{3.19}$$

or

$$\forall \ \epsilon \in (0, 1) \ \ \exists \ n_0 \in N \ \ \forall \ m, n \geq n_0; \quad 1 - M(x_m, x_n, 0) < \epsilon \tag{3.20}$$

Definition 3.8. An $M-$complete fuzzy metric space is a fuzzy metric space in which every $M-$Cauchy sequence is convergent.

Definition 3.9. Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \mapsto X$ is called uniformly continuous if the following condition holds:

(3.21)
$$\forall \ t > 0 \ \ \forall \ r \in (0, 1) \ \ \exists \ s \in (0, 1) \ \ \forall \ x, y \in X \ \ \{M(x, y, t) < s \Rightarrow M(Tx, Ty, t) < r\}$$

or
(3.22)
$$\forall \ r \in (0, 1) \ \ \exists \ s \in (0, 1) \ \ \forall \ x, y \in X \ \ \{1 - M(x, y, 0) < s \Rightarrow 1 - M(Tx, Ty, 0) < r\}$$

Definition 3.10. Let $(X, M, *)$ be a fuzzy metric space. $T : X \mapsto X$ is called a fuzzy contractive mapping if the following holds:

$$\forall \ t > 0 \ \ \exists \ k \in (0, 1) \ \ \forall \ x, y \in \ X \ \ M(Tx, Ty, t) \leq kM(x, y, t) \tag{3.23}$$

or

(3.24) $\exists\ k \in (0,1)\ \ \forall\ x, y \in\ X\ \ (1 - M(Tx, Ty, 0)) \leq k(1 - M(x, y, 0))$

k is called the contractive constant of T.

Theorem 3.11. *Let $(X, M, *)$ be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy. Let $T : X \mapsto X$ be a fuzzy contractive mapping being k the contractive constant. Then T has a unique fixed point.*

Proof. Fix $x_0 \in X$. Let $x_n = T(x_{n-1}),\ n \in N$. We have for $t \geq 0$;
if $t > 0$ then
$M(x_{n+1}, x_{n+2}, t) = M(Tx_n, Tx_{n+1}, t) \leq kM(x_n, x_{n+1}, t)$
$M(x_n, x_{n+1}, t) = M(Tx_{n-1}, Tx_n, t) \leq kM(x_{n-1}, x_n, t)$,...,
$M(x_1, x_2, t) = M(Tx_0, Tx_1, t) \leq kM(x_0, x_1, t)$
$M(x_n, x_{n+1}, t) \leq k^{n+1}M(x_0, x_1, t)$
Then $\{x_n\}$ is a fuzzy contractive sequence, so it is a Cauchy sequence and, hence, $\{x_n\}$ converges to y, for some $y \in X$.

(3.25) $$\lim_{n \mapsto \infty} M(x_n, x_{n+1}, t) = \lim_{n \mapsto \infty} M(x_n, Tx_n, t) = M(y, Ty, t) = 0$$

therefore $M(y, Ty, t) = 0$ for $\forall\ t > 0$.
And if $t = 0$ then
$(1 - M(x_{n+1}, x_{n+2}, 0)) = (1 - M(Tx_n, Tx_{n+1}, 0)) \leq k(1 - M(x_n, x_{n+1}, 0))$
$(1 - M(x_n, x_{n+1}, 0)) = (1 - M(Tx_{n-1}, Tx_n, 0)) \leq k(1 - M(x_{n-1}, x_n, 0))$,...,
$(1 - M(x_1, x_2, 0)) = (1 - M(Tx_0, Tx_1, 0)) \leq k(1 - M(x_0, x_1, 0))$
$(1 - M(x_n, x_{n+1}, 0)) \leq k^{n+1}(1 - M(x_0, x_1, 0))$
Then $\{x_n\}$ is a fuzzy contractive sequence, so it is a Cauchy sequence and, hence, $\{x_n\}$ converges to y, for some $y \in X$.

(3.26)
$$\lim_{n \mapsto \infty} 1 - M(x_n, x_{n+1}, 0) = \lim_{n \mapsto \infty} 1 - M(x_n, Tx_n, 0) = 1 - M(y, Ty, 0) = 0$$

therefor $M(y, Ty, 0) = 1$ and we will see y is a fixed point for T. □

4. Conclusion

In this paper, we introduced a new fuzzy metric space. Next, we derived the open ball in this fuzzy metric space. Also, we define some property such as Cauchy sequence, completeness, continuous and contractive map. In addition, examples are given for each of the concepts presented. Finally, establish fixed point theorem for the new complete fuzzy metric spaces.

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