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On fuzzy topology and fuzzy norm

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Abstract. In the present paper, considering the fuzzy normed linear space (X, N) defined by Bag and Samanta. We construct a fuzzy topology on this space and show that fuzzy normed linear space (X, N) equipped with this fuzzy topology is not topological vector space. So, we define another fuzzy topology on fuzzy normed linear spaces and provide that (X, N) equipped with this topology is a locally convex topological vector space.

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1. Introduction

The notion of fuzzy norm on a linear space was first introduced by Katrasas [12]. Feblin [7] gave an idea of a fuzzy norm on a linear space whose associated metric is Kalva type [10]. Cheng and Menderson [4] considered a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [11]. Felbin [7] definition of a fuzzy norm of a linear operator between two fuzzy normed spaces was generalized by Xiao and Zhu [13]. Bag and Samanta [3] introduced a notion of boundedness of a linear operator between fuzzy normed spaces, and studied the relation between fuzzy continuity and fuzzy boundedness. They also considered fuzzy bounded linear functionals, the concept of fuzzy dual spaces, and established some fundamental theorems in the area of fuzzy functional analysis. Furthermore, Golet [8] generalized the definition of fuzzy normed linear space and studied some properties of this space. In [5], Das and Das define a fuzzy topology on the fuzzy normed linear space defined by Felbin and study some basic properties of this fuzzy topology. After that, Fang [6] show that X with this topology is not topological vector space and modifies the fuzzy topology and proved some results. Also, Xu and Fang define another fuzzy topological space and study these spaces [14]. This concept has been used in developing fuzzy functional analysis and its applications and a large number of

papers by different authors have been published (see [2, 9]).
In this paper, we define a fuzzy topology on fuzzy normed linear space defined by Bag and Samanta and study some properties of this fuzzy topology. It is shown that (X, τ_N) is not topological vector space. So, we define a fuzzy topology τ_N^* coarser than a fuzzy topology τ_N and show that (X, τ_N^*) is a locally convex topological vector space.

2. Preliminaries

We give below some basic preliminaries required for this paper.

Definition 2.1 ([1]). Let X be a linear space over R (real number). Let N be A fuzzy subset of $X \times R$ such that for all $x, u \in X$ and $c \in R$:
$(N1)$ $N(x,t) = 0$ for all $t \leq 0$,
$(N2)$ $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$,
$(N3)$ If $c \neq 0$ then $N(cx,t) = N(x, t/|c|)$ for all $t \in R$,
$(N4)$ $N(x+u, s+t) \geq min\{N(x,s), N(u,t)\}$ for all $s, t \in R$,
$(N5)$ $N(x,.)$ is a nondecreasing function of R and $\lim\limits_{t\to\infty} N(x,t) = 1$.
Then N is called a fuzzy norm on X.

We assume that
$(N6)$ $N(x,t) > 0$ for all $t > 0$ implies $x = 0$,
$(N7)$ For $x \neq 0$, $N(x,.)$ is a continuous function of R and strictly increasing on the subset $\{t : 0 < N(x,t) < 1\}$ of R.

Definition 2.2 ([3]). Let (X, N, N^*) be a fuzzy normed linear space.
i) A sequence $\{x_n\} \subseteq X$ is said to converge to $x \in X$, if $\lim\limits_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$.
ii) A sequence $\{x_n\} \subseteq X$ is called Cauchy, if $\lim\limits_{n,m\to\infty} N(x_n - x_m, t) = 1$, for all $t > 0$.

Definition 2.3 ([3]). Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces. A function $f : X \longrightarrow Y$ is said to be fuzzy continuous at $x \in X$, if for given $\epsilon > 0$ and $\alpha \in (0,1)$ there exist $\delta > 0$ and $\beta \in (0,1)$ such that $N_1(x - y, \delta) > \beta$ implies that $N_2(f(x) - f(y), \epsilon) > \alpha$, for all $y \in X$.

Definition 2.4 ([5]). A fuzzy subset μ of a vector space X is said to be convex if

$$\mu(kx + (1-k)y) \geq \min(\mu(x), \mu(y)), \text{ for all } x, y \in X \text{ and } k \in [0,1].$$

Definition 2.5 ([6]). Let X be a vector space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $A, B \in I^X$ and $t \in \mathbb{K}$ where $I^X = \{F : F : X \longrightarrow [0,1] \text{ is a function}\}$. Then $A + B$ and tA are defined by

$$(A+B)(x) = \sup_{u+v=x} (A(u) \wedge B(v))$$

and

$$(tA)(x) = A(x/t), \qquad t \neq 0,$$
$$(0A)(x) = \begin{cases} \sup\limits_{y\in X} A(y) & , \quad x = 0 \\ 0 & , \quad x \neq 0. \end{cases}$$

Definition 2.6 ([5]). A fuzzy topology on a set X is a family τ of fuzzy subsets of X satisfying the following:
(i) The fuzzy subsets 1 and 0 are in τ,
(ii) τ is closed under finite intersection of fuzzy subsets,
(iii) τ is closed under arbitrary union of fuzzy subsets.
The pair (X, τ) is called a fuzzy topological space.

Definition 2.7 ([5]). A fuzzy set μ in a fuzzy topological space (X, τ) is called a neighborhood of a point $x \in X$ if and only if there is ρ in τ such that $\rho \subseteq \mu$ and $\mu(x) = \rho(x) > 0$.

Definition 2.8 ([5]). Let (X, τ) be a fuzzy topological space. A sequence $\{x_n\}$ in X is said to converge to a point x and is denoted by $\lim\limits_{n\to\infty} x_n = x$ if for every open neighborhood μ of x, there exists $N \in \mathbb{N}$ such that $\mu(x_n) > 0$ for all $n > N$.

Definition 2.9 ([5]). Let (X, τ_1) and (Y, τ_2) be fuzzy topological spaces. A function $f : X \longrightarrow Y$ is called fuzzy continuous at some point $x \in X$ if and only if $f^{-1}(\mu)$ is a neighborhood of x for each neighborhood μ of $f(x)$. f is called fuzzy continuous if f is fuzzy continuous at every $x \in X$. This is equivalent to inverse of every fuzzy open subset of Y is fuzzy open in X.

Definition 2.10 ([5]). A fuzzy topological space (X, τ) is said to be fuzzy Hausdorff if for $x, y \in X$ and $x \neq y$ there exist $\eta, \mu \in \tau$ with $\mu(x) = \eta(y) = 1$ and $\eta \cap \mu = \varnothing$.

Definition 2.11 ([6]). A stratified fuzzy topology τ on a vector space X is said to be an fuzzy vector topology, if the following two mappings

$$f : X \times X \longrightarrow X,\ (x, y) \longrightarrow x + y \text{ and } g : \mathbb{K} \times X \longrightarrow X,\ (t, x) \longrightarrow tx,$$

are continuous, where $\mathbb{K}$ is equipped with the fuzzy topology induced by the usual topology and $X \times X$ and $\mathbb{K} \times X$ are equipped with the corresponding product fuzzy topologies. A vector space X with an fuzzy vector topology τ, denoted by (X, τ) is called an fuzzy topological vector space (FTVS).

Definition 2.12 ([6]). Let (X, τ) be an fuzzy topological space and $x_\alpha \in Pt(I^X)$.
(i) A fuzzy set U on X is called Q-neighborhood of x_α iff there exists $G \in \tau$ such that $x_\alpha \widetilde{\in} G \subseteq U$.
(ii) A family $\mathfrak{U}_{x_\alpha}$ of Q-neighborhoods of x_α is called a Q-neighborhood base of x_α iff for every Q-neighborhood A of x_α, there exists $U \in \mathfrak{U}_{x_\alpha}$ such that $U \subseteq A$.

Definition 2.13 ([6]). An fuzzy topological vector space (X, τ) is said to be of QL-type, if there exists a family $\mathfrak{U}$ of fuzzy sets on X such that for each $\alpha \in (0, 1]$,

$$\mathfrak{U}_\alpha = \{U \cap \underline{r} : U \in \mathfrak{U},\ r \in (1 - \alpha, 1]\}$$

is a Q-neighborhood base of 0_α in (X, τ). The family $\mathfrak{U}$ is called a Q-prebase for τ.

Theorem 2.14 ([6]). *Let (X, τ) be a fuzzy topological space, $U \in I^X$ and $x \in X$. Then U is a neighborhood of x if and only if U is a Q-neighborhood of x_α for each $\alpha \in (1 - U(x), 1]$.*

Theorem 2.15 ([6]). *Let (X, τ) be a fuzzy topological vector spaces. Then*
(i) U is an (open) Q-neighborhood of O_α iff $x + U$ is an (open) Q-neighborhood of

x_α, where $x \in X$.
(ii) U is an (open) Q-neighborhood of x_α iff tU is an (open) Q-neighborhood of $(tx)_\alpha$, where $t \in \mathbb{K}$, $t \neq 0$.

Lemma 2.16 ([6])**.** *Let τ be a stratified fuzzy topology on a vector space X. Then*
(i) the mapping f (addition) is continuous iff for every fuzzy point $(x, y)_\alpha$ in $X \times X$ and every Q-neighborhood W of $(x + y)_\alpha$, there exist a Q-neighborhood U of x_α and a Q-neighborhood V of y_α such that $U + V \subseteq W$,
(ii) the mapping g (scalar multiplication) is continuous iff for every fuzzy point $(t, x)_\alpha$ in $\mathbb{K} \times X$ and every Q-neighborhood W of $(tx)_\alpha$, there exist a Q-neighborhood U of x_α and $\delta > 0$ such that $sU \subseteq W$ for all $s \in \mathbb{K}$ with $|s - t| < \delta$.

Lemma 2.17 ([6])**.** *Let (X, τ) be an FTVS and $\mathfrak{U}_\alpha$ a Q-neighborhood base of 0_α in X, $\alpha \in (0, 1]$. Then the following conclusions hold*
(i) If $U \in \mathfrak{U}_\alpha$ or $U = \underline{r}$, where $r \in (1 - \alpha, 1]$, then there exists $\alpha_0 \in (0, \alpha)$ such that for each $\mu \in [\alpha_0, 1]$ there is a $V \in \mathfrak{U}_\mu$ such that $V \subseteq U$,
(ii) If $U, V \in \mathfrak{U}_\alpha$, then there exists $W \in \mathfrak{U}_\alpha$ such that $W \subseteq U \cap V$,
(iii) If $U \in \mathfrak{U}_\alpha$, then there exists $V \in \mathfrak{U}_\alpha$ such that $V + V \subseteq U$,
(iv) If $U \in \mathfrak{U}_\alpha$, then there exists $V \in \mathfrak{U}_\alpha$ such that $tV \subseteq U$ for all $t \in \mathbb{K}$ with $|t| \leq 1$,
(v) If $U \in \mathfrak{U}_\alpha$ and $x \in X$, there exists $\lambda > 0$ such that $x_\alpha \widetilde{\in} \lambda U$.
Conversely, let X be a vector space over $\mathbb{K}$ such that every $\alpha \in (0, 1]$ has a family $\mathfrak{U}_\alpha$ of fuzzy sets on X satisfying the conditions (i)-(v), then there exists a unique fuzzy topology τ on X such that (X, τ) is an FTVS and $\mathfrak{U}_\alpha$ is a Q-neighborhood base of 0_α.

Definition 2.18 ([6])**.** An fuzzy topological vector space (X, τ) is said to be locally convex, if for each $\alpha \in (0, 1]$, there is a base of Q-neighborhoods of 0_α consisting of convex fuzzy sets.

3. Fuzzy Topology

Definition 3.1. Let (X, N) be a fuzzy normed linear space and let $x \in X$, $\alpha \in (0, 1)$ and $\epsilon > 0$ the fuzzy set $\mu_\alpha(x, \epsilon)$ defined in X by

$$\mu_\alpha(x, \epsilon)(y) = \begin{cases} 1 - \alpha & , \quad N(x - y, \epsilon) > \alpha \\ 0 & , \quad o.w. \end{cases}$$

is said to be an α-open sphere in X.

Example 3.2. Let $(X, \|.\|)$ be a normed space. We define

$$N(x, t) = \begin{cases} t/(t + \|x\|) & , \quad t > 0, \ x \in X \\ 0 & , \quad t \leq 0, \ x \in X. \end{cases}$$

Then (X, N) is a fuzzy normed linear space such that N satisfying $(N7)$. By Definition 3.1, we have

$$\mu_\alpha(x, \epsilon)(y) = \begin{cases} 1 - \alpha & , \quad \|x - y\| < ((1 - \alpha)/a)\epsilon \\ 0 & , \quad o.w. \end{cases}$$

is a α-open sphere in X.

Definition 3.3. Fuzzy set $\mu \in I^X$ is called N-open if for $\mu(x) > 0$, there exists $\epsilon > 0$ such that $\mu_\alpha(x, \epsilon) \subseteq \mu$, for some $\alpha \in (0, 1)$.

Theorem 3.4. *Let (X, N) be a fuzzy normed linear space. Then a family*

$$\tau_N = \{\mu \in I^X : \mu \textit{ is } N - open\}$$

is a fuzzy topology on X.

Proof. (i) We have $1(x) = 1$, for all $x \in X$. So $\mu_\alpha(x, \epsilon) \subseteq 1$, for all $\epsilon > 0$ and $\alpha \in (0, 1)$. Hence $1 \in \tau$. Since $0(x) = 0$, for all $x \in X$. Thus $0 \in \tau$.
(ii) Let $\mu_1, \mu_2 \in \tau$ and $(\mu_1 \cap \mu_2)(x) > 0$. We have $\mu_1(x) > 0$ and $\mu_2(x) > 0$. So there exist $\alpha_1, \alpha_2 \in (0, 1)$ and $\epsilon_1, \epsilon_2 > 0$ such that $\mu_{\alpha_1}(x, \epsilon_1) \subseteq \mu_1$ and $\mu_{\alpha_2}(x, \epsilon_2) \subseteq \mu_2$. Suppose that $\alpha = \max\{\alpha_1, \alpha_2\}$ and $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. If $N(x - y, \epsilon) > \alpha$ then $N(x - y, \epsilon_1) \geq N(x - y, \epsilon) > \alpha \geq \alpha_1$ and $N(x - y, \epsilon_2) \geq N(x - y, \epsilon) > \alpha \geq \alpha_2$. Thus $\mu_\alpha(x, \epsilon)(y) = 1 - \alpha \leq 1 - \alpha_1 = \mu_{\alpha_1}(x, \epsilon_1)(y)$ and $\mu_\alpha(x, \epsilon)(y) = 1 - \alpha \leq 1 - \alpha_2 = \mu_{\alpha_2}(x, \epsilon_2)(y)$. Therefore $\mu_\alpha(x, \epsilon) \subseteq \mu_{\alpha_1}(x, \epsilon_1)$ and $\mu_\alpha(x, \epsilon) \subseteq \mu_{\alpha_2}(x, \epsilon_2)$. Hence $\mu_\alpha(x, \epsilon) \subseteq \mu_{\alpha_1}(x, \epsilon_1) \cap \mu_{\alpha_2}(x, \epsilon_2) \subseteq \mu_1 \cap \mu_2$. So $\mu_1 \cap \mu_2 \in \tau$.
(iii) Let $\{\mu_i\} \in \tau$ and $\bigcup_i \mu_i(x) > 0$. Then there exists i_0 such that $\mu_{i_0}(x) > 0$. So there exist $\epsilon > 0$ and $\alpha \in (0, 1)$ such that $\mu_\alpha(x, \epsilon) \subseteq \mu_{i_0}$. Therefore $\mu_\alpha(x, \epsilon) \subseteq \bigcup_i \mu_i$. Hence $\bigcup_i \mu_i \in \tau$.
Thus τ is a fuzzy topology on X. □

Theorem 3.5. *Let (X, N) be a fuzzy normed linear space such that N satisfying (N7). Then α-open sphere is a N-open set, for all $\alpha \in (0, 1)$.*

Proof. Let $x \in X$, $\epsilon > 0$ and $\alpha \in (0, 1)$. Suppose that $\mu_\alpha(x, \epsilon)(y) > 0$. Therefore $N(x - y, \epsilon) > \alpha$. Assume that $0 < \alpha < \alpha_1 < N(x - y, \epsilon)$. By $(N7)$, we obtain that there exists $0 < t < \epsilon$ such that $N(x-y, t) > \alpha_1$. Let $\delta = \epsilon - t > 0$ and $\mu_{\alpha_1}(y, \delta)(z) > 0$. Hence $N(y-z, \delta) > \alpha_1$. Thus $N(x-z, \epsilon) \geq \min\{N(y-z, \delta), N(x-y, t)\} > \alpha_1 > \alpha$. So $\mu_{\alpha_1}(y, \delta)(z) = 1-\alpha_1 \leq 1-\alpha = \mu_\alpha(x, \epsilon)(z)$. Therefore $\mu_{\alpha_1}(y, \delta) \subseteq \mu_\alpha(x, \epsilon)$. Hence $\mu_\alpha(x, \epsilon)$ is a fuzzy open set in (X, τ_N). □

Theorem 3.6. *Let (X, N) be a fuzzy normed linear space. Then α-open sphere is a fuzzy convex set, for all $\alpha \in (0, 1)$.*

Proof. Let $x \in X$, $\epsilon > 0$ and $\alpha \in (0, 1)$. Assume that $y, z \in X$ and $k \in [0, 1]$. We have

$$\begin{aligned} N(x - (ky + (1 - k)z), \epsilon) &\geq \min\{N(k(x - y), k\epsilon), N((1 - k)(x - z), (1 - k)\epsilon)\} \\ &= \min\{N(x - y, \epsilon), N((x - z), \epsilon)\}. \end{aligned}$$

Hence $\mu_\alpha(x, \epsilon)(ky + (1 - k)z) \geq \min(\mu_\alpha(x, \epsilon)(y), \mu_\alpha(x, \epsilon)(z))$. Thus $\mu_\alpha(x, \epsilon)$ is a fuzzy convex set. □

Theorem 3.7. *Let (X, N) be a fuzzy normed linear space such that N satisfying (N7). Also, let $\{x_n\} \subseteq X$ and $x \in X$. Then $\{x_n\}$ converges to x in (X, N) if and only if $\{x_n\}$ converges to x in (X, τ_N).*

Proof. let $x_n \longrightarrow x$ in fuzzy normed linear space (X, N). Hence $\lim_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$. Suppose that μ is a fuzzy open subset of (X, τ_N) and $\mu(x) > 0$. Then there exist $\epsilon > 0$ and $\alpha \in (0, 1)$ such that $\mu_\alpha(x, \epsilon) \subseteq \mu$. We have $\lim_{n\to\infty} N(x_n - x, \epsilon) = 1$. Thus there exists $N > 0$ such that $N(x_n - x, \epsilon) > \alpha$, for all $n > N$. So $\mu_\alpha(x, \epsilon)(x_n) = 1 - \alpha > 0$, for all $n > N$. By Theorem 3.5, $x_n \longrightarrow x$ in

(X, τ_N).
let $x_n \longrightarrow x$ in fuzzy topological space (X, τ_N). Assume that $t > 0$ and $0 < \epsilon < 1$. We have $\mu_{1-\epsilon}(x,t)(x) = \epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\mu_{1-\epsilon}(x,t)(x_n) > 0$, for all $n > N$. Therefore $N(x_n - x, t) > 1 - \epsilon$, for all $n > N$. Hence $\lim_{n\to\infty} N(x_n - x, t) = 1$. Thus $x_n \longrightarrow x$ in (X, N). □

Theorem 3.8. *Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces such that N_1 and N_2 satisfying (N7). Then $f : (X, \tau_{N_1}) \longrightarrow (Y, \tau_{N_2})$ is fuzzy continuous at a point if and only if $f : (X, N_1) \longrightarrow (Y, N_2)$ is fuzzy continuous at that point.*

Proof. Let $f : (X, N_1) \longrightarrow (Y, N_2)$ be fuzzy continuous at $x \in X$. Suppose that μ is a fuzzy open subset of (Y, τ_{N_2}) and $\mu(f(x)) > 0$. Then there exist $\epsilon > 0$ and $\alpha \in (0, 1)$ such that $\mu_\alpha(f(x), \epsilon) \subseteq \mu$. Since $f : (X, N_1) \longrightarrow (Y, N_2)$ is fuzzy continuous at $x \in X$, there exist $\beta_0 \in (0, 1)$ and $\delta > 0$ such that $N_1(x - y, \delta) > \beta_0$ implies that $N_2(f(x) - f(y), \epsilon) > \alpha$, for all $y \in X$. Let $\beta = \max\{\beta_0, \alpha\}$ and $\mu_\beta(x, \epsilon)(y) > 0$. Hence $N_1(x - y, \delta) > \beta \geq \beta_0$, this implies that $N_2(f(x) - f(y), \epsilon) > \alpha$. Therefore $\mu_\alpha(f(x), \epsilon)(f(y)) = 1 - \alpha$. Hence $\mu_\beta(x, \delta)(y) = 1 - \beta \leq 1 - \alpha = \mu_\alpha(f(x), \epsilon)(f(y)) \leq \mu(f(y)) = f^{-1}(\mu)(y)$. Thus $\mu_\beta(x, \delta) \subseteq f^{-1}(\mu)$. So $f^{-1}(\mu)$ is a neighborhood of x. Hence $f : (X, \tau_{N_1}) \longrightarrow (Y, \tau_{N_2})$ is fuzzy continuous at x.
Conversely, let $f : (X, \tau_{N_1}) \longrightarrow (Y, \tau_{N_2})$ is fuzzy continuous at x. Suppose that $\epsilon > 0$ and $\alpha \in (0, 1)$. We have $\mu_\alpha(f(x), \epsilon)(f(x)) > 0$. Since $f : (X, \tau_{N_1}) \longrightarrow (Y, \tau_{N_2})$ is fuzzy continuous at x it follows that $f^{-1}(\mu_\alpha(f(x), \epsilon))$ is a neighborhood of x. So there is ρ in τ_{N_1} such that $\rho \subseteq f^{-1}(\mu_\alpha(f(x), \epsilon))$ and $\rho(x) = f^{-1}(\mu_\alpha(f(x), \epsilon))(x) > 0$. Hence there exist $\delta > 0$ and $\beta \in (0, 1)$ such that $\mu_\beta(x, \delta) \subseteq \rho$. Assume that $N_1(x - y, \delta) > \beta$. Therefore $0 < \mu_\beta(x, \delta)(y) = 1 - \beta \leq \rho(y) \leq f^{-1}(\mu_\alpha(f(x), \epsilon))(y) = \mu_\alpha(f(x), \epsilon)(f(y))$. Hence $N_2(f(x) - f(y), \epsilon) > \alpha$. Thus $f : (X, N_1) \longrightarrow (Y, N_2)$ is fuzzy continuous at x. □

Theorem 3.9. *Let (X, N) be fuzzy normed linear space such that N satisfying (N7). Then the fuzzy topological space (X, τ_N) is fuzzy Hausdorff.*

Proof. Let $x, y \in X$ and $x \neq y$. By $(N2)$, there exists $t_0 > 0$ such that $N(x-y, t_0) < 1$. Assume that $\alpha \in (0, 1)$ and $N(x - y, t_0) < \alpha$. We define the fuzzy sets μ and ρ on X as follows:

$$\mu(z) = \begin{cases} 1 & , \quad z = x \\ \mu_\alpha(x, t_0/2)(z) & , \quad o.w., \end{cases}$$

and

$$\rho(z) = \begin{cases} 1 & , \quad z = y \\ \mu_\alpha(y, t_0/2)(z) & , \quad o.w., \end{cases}$$

It is claer that μ and ρ are open and $\mu(x) = 1 = \rho(y)$. If $\mu \cap \rho \neq \varnothing$. Then there exists $x_0 \in X$ such that $(\mu \cap \rho)(x_0) > 0$. Hence $\mu_\alpha(x, t_0/2)(x_0) > 0$ and $\mu_\alpha(y, t_0/2)(x_0) > 0$. Thus $N(x - x_0, t_0/2) > \alpha$ and $N(y - x_0, t_0/2) > \alpha$. Therefore $N(x - y, t_0) \geq \min\{N(x - x_0, t_0/2), N(y - x_0, t_0/2)\} \geq \alpha$, this is a contradiction. Hence $\mu_1(x, t_0/2) \cap \mu_1(y, t_0/2) = \varnothing$. So (X, τ_N) is fuzzy Hausdorff. □

Lemma 3.10. *Let (X, N) be fuzzy normed linear space and $x \in X$, $\alpha \in (0, 1)$ and $\epsilon > 0$. Then*

$$\mu_\alpha(x, \epsilon) = (x + U_{\alpha,\epsilon}) \cap (\underline{1 - \alpha}),$$

where $U_{\alpha,\epsilon} = \{z \in X : N(z, \varepsilon) > \alpha\}$.

Proof. Let $x \in X$, $\alpha \in (0,1)$ and $\epsilon > 0$. Suppose that $\mu_\alpha(x,\epsilon)(y) = 1-\alpha$. Hence $N(y-x,\epsilon) > \alpha$. Thus $y - x \in U_{\alpha,\epsilon}$. So $y \in (x + U_{\alpha,\epsilon})$. Therefore $((x + U_{\alpha,\epsilon}) \cap (\underline{1-\alpha}))(y) = 1-\alpha$. Similarly, if $\mu_\alpha(x,\epsilon)(y) = 0$ then $((x+U_{\alpha,\epsilon}) \cap (\underline{1-\alpha}))(y) = 0$. Hence $\mu_\alpha(x,\epsilon) = (x + U_{\alpha,\epsilon}) \cap (\underline{1-\alpha})$. □

Theorem 3.11. *Let (X, N) be fuzzy normed linear space such that N satisfying (N7). Then*
(i) the mapping $f : (X, \tau_N) \times (X, \tau_N) \longrightarrow (X, \tau_N)$, $(x, y) \longrightarrow x + y$, is continuous,
(ii) the mapping $g : \mathbb{R} \times (X, \tau_N) \longrightarrow (X, \tau_N)$, $(t, x) \longrightarrow tx$, is not continuous.

Proof. (i) Let $x_0, y_0 \in X$ and $\alpha_0 \in (0, 1)$. Suppose that W is a Q-neighborhood of $(x_0 + y_0)_{\alpha_0}$, then there exists $\mu \in \tau_N$ such that $(x_0 + y_0)_{\alpha_0} \widetilde{\in} \mu \subseteq W$. Hence there exist $\epsilon > 0$ and $\alpha \in (0, 1)$ such that $\mu_\alpha(x_0 + y_0, \epsilon) \subseteq \mu \subseteq W$. We have $N(x + y, \epsilon) \geq \min\{N(x, \epsilon/2), N(y, \epsilon/2)\}$, for all $x, y \in X$. Thus $U_{\alpha,\epsilon/2} + U_{\alpha,\epsilon/2} \subseteq U_{\alpha,\epsilon}$. Therefore $(x_0 + U_{\alpha,\epsilon/2}) + (y_0 + U_{\alpha,\epsilon/2}) \subseteq (x_0 + y_0) + U_{\alpha,\epsilon}$. Hence $(x_0 + U_{\alpha,\epsilon/2}) \cap (\underline{1-\alpha}) + (y_0 + U_{\alpha,\epsilon/2}) \cap (\underline{1-\alpha}) \subseteq ((x_0 + y_0) + U_{\alpha,\epsilon}) \cap (\underline{1-\alpha})$. By Lemma 3.10, we obtain that $\mu_\alpha(x_0, \epsilon/2) + \mu_\alpha(y_0, \epsilon/2) \subseteq \mu_\alpha(x_0 + y_0, \epsilon) \subseteq W$.
If $\alpha < \alpha_0$, then it is easy to see that $\mu_\alpha(x_0, \epsilon/2)$, $\mu_\alpha(y, \epsilon/2)$ are the open Q-neighborhoods of $x_{0_{\alpha_0}}$ and $y_{0_{\alpha_0}}$, respectively.
If $\alpha \geq \alpha_0$, we define the fuzzy sets A and B on X as follows:

$$A(z) = \begin{cases} \beta & , \quad z = x \\ \mu_\alpha(x_0, \epsilon/2)(z) & , \quad o.w., \end{cases}$$

and

$$B(z) = \begin{cases} \beta & , \quad z = y \\ \mu_\alpha(x_0, \epsilon/2)(z) & , \quad o.w., \end{cases}$$

where $\beta = \mu(x_0 + y_0) > 1 - \alpha_0$. It is claer that A and B are open Q-neighborhoods of $x_{0_{\alpha_0}}$ and $y_{0_{\alpha_0}}$, respectively, and $A + B \subseteq \mu \subseteq W$. By Lemma 2.16, the addition is continuous.
(ii) Let $(t_0, x_0) \in \mathbb{R} \times X$, $x \neq 0$ and $\alpha_0 \in (0, 1)$. Assume that $\alpha, \beta \in (0, 1)$ and $\alpha < 1 - \alpha_0 < \beta$. We define a fuzzy set μ on X by

$$\mu(z) = \begin{cases} \beta & , \quad z = t_0 x_0 \\ \alpha & , \quad o.w.. \end{cases}$$

Hence μ is a open Q-neighborhood of $(t_0x_0)_{\alpha_0}$. If U is a Q-neighborhood $x_{0_{\alpha_0}}$, then $(t_0 + \delta)U \nsubseteq \mu$, for all $\delta > 0$. By Lemma 2.16, the scalar multiplication is not continuous. □

Definition 3.12. Let (X, N) be fuzzy normed linear space. A fuzzy set μ on X is said to be N-linearly open if for every $x \in \sup \mu$ and $\alpha \in (1 - \mu(x), 1)$, there exists $\epsilon > 0$ such that $\mu_\alpha(x, \epsilon) \subseteq \mu$.

Theorem 3.13. *Let (X, N) be a fuzzy normed linear space. Then a family*

$$\tau_N^* = \{\mu \in I^X : \mu \textit{ is } N - \textit{linearly open}\}$$

is a fuzzy topology on X.

Proof. The proof is similar to proof of Theorem 3.4. □

Theorem 3.14. *Let (X, N) be a fuzzy normed linear space such that N satisfying (N7). Then α-open sphere is a N-linearly open, for all $\alpha \in (0, 1)$.*

Proof. The proof is similar to proof of Theorem 3.5. □

Theorem 3.15. *Let (X, N) be a fuzzy normed linear space such that N satisfying (N7) and (X, τ_N) be a fuzzy topological space generated by the fuzzy norm N. Also, let $\{x_n\} \subseteq X$ and $x \in X$. Then $\{x_n\}$ converges to x in (X, N) if and only if $\{x_n\}$ converges to x in (X, τ_N^*).*

Proof. The proof is similar to proof of Theorem 3.7. □

Theorem 3.16. *Let (X, N_1) and (Y, N_2) be fuzzy normed linear spaces such that N_1 and N_2 satisfying (N7) and (X, τ_{N_1}), (y, τ_{N_2}) be fuzzy topological spaces generated by the fuzzy norms N_1 and N_2 respectively. Then $f : (X, \tau_{N_1}^*) \longrightarrow (Y, \tau_{N_2}^*)$ is fuzzy continuous at a point if and only if $f : (X, N_1) \longrightarrow (Y, N_2)$ is fuzzy continuous at that point.*

Proof. The proof is similar to proof of Theorem 3.8. □

Theorem 3.17. *Let (X, N) be a fuzzy normed linear space such that N satisfying (N7). Then (X, τ_N^*) is a locally convex FTVS and for every $\alpha \in (0, 1)$,*

$$\mathfrak{U}_\alpha = \{U_{\beta,\epsilon} \cap \underline{(1-\beta)} : \epsilon > 0, \beta \in (0, \alpha)\} = \{\mu_\beta(0, \epsilon) : \epsilon > 0, \beta \in (0, \alpha)\}$$

is a Q-neighborhood base of 0_α, where $U_{\alpha,\epsilon} = \{z \in X : N(z, \varepsilon) > \alpha\}$.

Proof. First, we show that $\mathfrak{U}_\alpha$ satisfies conditions (i)ÔÇô(v) of Lemma 2.17, for all $\alpha \in (0, 1)$.
(i) Let $U = U_{\beta,\epsilon} \cap \underline{(1-\beta)} \in \mathfrak{U}_\alpha$. We have $0 < \beta < \alpha$. So there exists $\alpha_0 \in (0, \alpha)$ such that $\beta < \alpha_0$. Hence $V = U_{\beta,\epsilon} \cap \underline{(1-\beta)} \in \mathfrak{U}_\mu$ and $V \subseteq U$, for all $\mu \in [\alpha_0, 1]$.
Let $U = \underline{r}$ with $r \in (1-\alpha, 1)$. Then there exists $\alpha_0 \in (0, \alpha)$ such that $r > 1 - \alpha_0$. So $V = U_{1-r,\epsilon} \cap \underline{(r)} \in \mathfrak{U}_\mu$ and $V \subseteq U$, for all $\mu \in [\alpha_0, 1]$.
(ii) Let $U_{\beta_1,\epsilon_1} \cap \underline{(1-\beta_1)}, U_{\beta_2,\epsilon_2} \cap \underline{(1-\beta_2)} \in \mathfrak{U}_\alpha$. Suppose that $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and $\beta = \max\{\beta_1, \beta_2\}$. Hence $U_{\beta,\epsilon} \cap \underline{(1-\beta)} \in \mathfrak{U}_\alpha$ and $U_{\beta,\epsilon} \cap \underline{(1-\beta)} \subseteq (U_{\beta_1,\epsilon_1} \cap \underline{(1-\beta_1)}) \cap (U_{\beta_2,\epsilon_2} \cap \underline{(1-\beta_2)})$.
(iii) Let Let $U_{\beta,\epsilon} \cap \underline{(1-\beta)} \in \mathfrak{U}_\alpha$. Since $N(x+y, \epsilon) \geq \min\{N(x, \epsilon/2), N(y, \epsilon/2)\}$, for all $x, y \in X$, it follows that $U_{\beta,\epsilon/2} \cap \underline{(1-\beta)} \in \mathfrak{U}_\alpha$ and $U_{\beta,\epsilon/2} \cap \underline{(1-\beta)} + U_{\beta,\epsilon/2} \cap \underline{(1-\beta)} \subseteq U_{\beta,\epsilon} \cap \underline{(1-\beta)}$.
(iv) Let $U_{\beta,\epsilon} \cap \underline{(1-\beta)} \in \mathfrak{U}_\alpha$. We have $N(x/t, \epsilon) = N(x, |t|\epsilon) \leq N(x, \epsilon)$, for all $x \in X$ and all $t \in \mathbb{R}$ with $0 < |t| \leq 1$. Therefore $t(U_{\beta,\epsilon} \cap \underline{(1-\beta)}) \subseteq U_{\beta,\epsilon} \cap \underline{(1-\beta)}$, for all $t \in \mathbb{R}$ with $|t| \leq 1$.
(v) Let $U_{\beta,\epsilon} \cap \underline{(1-\beta)} \in \mathfrak{U}_\alpha$ and $x \in X$. By $(N5)$, we have $\lim_{t \longrightarrow \infty} N(x, t) = 1$. Thus there exists $t > 0$ such that $N(x, t\epsilon) > \beta$. So $(U_{\beta,\epsilon} \cap \underline{(1-\beta)})(x/t) = \mu_\beta(0, \epsilon)(x/t) = 1 - \beta > 1 - \alpha$. Hence $x_\alpha \widetilde{\in} t(U_{\beta,\epsilon} \cap \underline{(1-\beta)})$.
By Lemma 2.17, there exists a unique fuzzy topology τ on X such that (X, τ) is an FTVS and $\mathfrak{U}_\alpha$ is a Q-neighborhood base of 0_α. By Lemma 3.6, $U_{\beta,\epsilon} \cap \underline{(1-\beta)} = \mu_\beta(0, \epsilon)$ is a fuzzy convex set. Therefore (X, τ) is locally convex.
Now we prove $\tau = \tau_N^*$. Let $\mu \in \tau_N^*$ and $\alpha > 1 - \mu(x)$. Then there exist $\epsilon > 0$ and $\alpha >$

$\beta > 1-\mu(x)$ such that $\mu_\beta(x,\epsilon) \subseteq \mu$. By Theorem 2.15, $\mu_\beta(x,\epsilon) = x + U_{\beta,\epsilon} \cap \underline{(1-\beta)}$ is a Q-neighborhood of x_α for τ. Hence μ is is a Q-neighborhood of x_α for τ. By Theorem 2.14, μ is is a neighborhood of x for τ. Thus $\mu \in \tau$. So $\tau_N^* \subseteq \tau$.
On the other hand, let $\mu \in \tau$ and $x \in supp\mu$. Assume that $\alpha(1-\mu(x),1)$. Then we have $x_\alpha \widetilde{\in} \mu$. Thus there exists $\epsilon > 0$ and $\beta \in (0,\alpha)$ such that $x + U_{\beta,\epsilon} \cap \underline{(1-\beta)} \subseteq \mu$. Therefore $\mu_\beta(x,\epsilon) \subseteq \mu$. This shows that $\mu \in \tau_N^*$. So $\tau \subseteq \tau_N^*$. Thus $\tau = \tau_N^*$. □

Theorem 3.18. *Let (X,N) be fuzzy normed linear space such that N satisfying (N7). Then the fuzzy topological space (X,τ_N^*) is fuzzy Hausdorff.*

Proof. The proof is similar to proof of Theorem 3.9. □

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