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IVF topologies based on IVF relations

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Abstract. In this paper, " interval-valued fuzzy " denote briefly by " IVF ". We defined new level sets of interval-valued fuzzy sets, discuss propositions and decomposition theorem in interval-valued fuzzy approximation spaces and list respectively some equivalent conditions of interval-valued fuzzy relation R which is reflexive, symmetric or transitive. Then we introduce relationships between interval-valued fuzzy approximation spaces and interval-valued fuzzy topological spaces. First, the conditions that they may change into each other are given. Second, questions how to transformation are discussed. Third, relationships between the transformation and itself are studied. Specially, internal selectivity and structure of the interval-valued fuzzy topology induced by approximation space are presented. A necessary conditions and sufficient conditions on two topologies induced equal with each other are list. Finally, An interval-valued fuzzy pseudo-closure operator is illustrated.

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1. Introduction

Rough set theory, proposed by Pawlak [8], is a mathematical tool for dealing with incomplete and vague information. It may be seen as an extension of classical set theory and has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [9, 10, 11, 12].

Zadeh's fuzzy set theory [17] addresses the problem of how to understand and manipulate imperfect knowledge. Recent investigations have shown that these two theories can be combined into a more flexible and expressive framework for modeling and processing incomplete information in information systems. Various notions that combine rough sets and fuzzy sets are introduced, such as rough fuzzy sets, fuzzy rough sets, and intuitionistic fuzzy rough sets (see [1, 2, 5, 7, 13, 16, 19]).

As a generalization of Zadeh's fuzzy set, Interval-valued fuzzy (IVF, for short) sets were introduced by Gorzalczany [3] and Turksen [15], and they were applied to the fields of approximate inference, signal transmission and controller, etc. Mondal et al. [6] defined topology of IVF sets and studied their properties.

By integrating Pawlak rough set theory with IVF set theory, Sun et al. [14] introduced IVF rough sets based on an IVF approximation space, defined IVF information systems and discussed their attribute reduction. Gong et al. [4] presented IVF rough sets based on approximation spaces, studied the knowledge discovery in IVF information systems. Zhang et al. [20] discussed general IVF rough sets based on an IVF approximation space on two universes of discourse. However, topological structures of IVF rough sets based on an an IVF approximation space have not been studied.

To improve and develop the applications of topology and rough sets on IVF uncertain information, topological properties of IVF rough sets need to be studied. The purpose of this paper is to investigate IVF topologies based on IVF relations.

2. **Preliminaries**

Throughout this paper, U denotes a non-empty finite universe. I denotes $[0,1]$ and $[I]$ denotes $\{[a,b] : a,b \in I \text{ } and \text{ } a \leq b\}$. $\mathscr{P}(U)$ denotes the family of all subsets of U. $F(U)$ denotes the family of all fuzzy sets in U. $\bar{a}$ denotes $[a,a]$ for any $a \in [0,1]$. "interval-valued fuzzy" is briefly "IVF".

For any $[a_i, b_i] \in [I](\ i = 1, 2)$, we define

$$[a_1, b_1] = [a_2, b_2] \iff a_1 = a_2, b_1 = b_2;$$

$$[a_1, b_1] \leq [a_2, b_2] \iff a_1 \leq a_2, b_1 \leq b_2,$$

$$[a_1, b_1] < [a_2, b_2] \iff [a_1, b_1] \leq [a_2, b_2] \text{ } and \text{ } [a_1, b_1] \neq [a_2, b_2];$$

$$\bar{1} - [a,b] \text{ } or \text{ } [a,b]^c = [1-b, 1-a].$$

Definition 2.1 ([3, 15])**.** For any $\{[a_i, b_i] : i \in J\} \subseteq [I]$, we define

$$\bigvee_{i\in J}[a_i, b_i] = [\bigvee_{i\in J} a_i, \bigvee_{i\in J} b_i] \text{ } and \text{ } \bigwedge_{i\in J}[a_i, b_i] = [\bigwedge_{i\in J} a_i, \bigwedge_{i\in J} b_i],$$

where $\bigvee\limits_{i\in J} a_i = sup\{a_i : i \in J\}$ and $\bigwedge\limits_{i\in J} a_i = inf\{a_i : i \in J\}$.

Definition 2.2 ([3, 15])**.** An IVF set A in U is defined by a mapping $A : U \mapsto [I]$. Denote

$$A(x) = [A^-(x), A^+(x)] \quad (x \in U).$$

Then $A^-(x)$ (resp. $A^+(x)$) is called the lower (resp. upper) degree to which x belongs to A. A^- and A^+ are called the lower fuzzy set and the upper fuzzy set of A, respectively.

The set of all IVF sets in U is denoted by $F^{(i)}(U)$.

Let $a, b \in I$. $\widetilde{[a,b]}$ represents the IVF set which satisfies $\widetilde{[a,b]}(x) = [a,b]$ for each $x \in U$. We denoted $\widetilde{[a,a]}$ by $\tilde{a}$.

We recall some basic operations on $F^{(i)}(U)$ as follows ([3, 15]): for any $A, B \in F^{(i)}(U)$ and $[a, b] \in [I]$,

(1) $A = B \iff A(x) = B(x)$ for any $x \in U$.

(2) $A \subseteq B \iff A(x) \leq B(x)$ for any $x \in U$.

(3) $A = B^c \iff A(x) = B(x)^c$ for any $x \in U$.

(4) $(A \cap B)(x)=A(x) \wedge B(x)$ for any $x \in U$.

(5) $(A \cup B)(x)=A(x) \vee B(x)$ for any $x \in U$.

Moreover,

$$(\bigcup_{i\in J} A)(x) = \bigvee_{i\in J} A(x) \quad and \quad (\bigcap_{i\in J} A)(x) = \bigwedge_{i\in J} A(x),$$

where $\{A_i : i \in J\} \subseteq F^{(i)}(U)$.

(6) $([a, b]A)(x) = [a, b] \wedge [A^-(x), A^+(x)]$ for any $x \in U$.

Obviously, $A = B \iff A^- = B^-$ and $A^+ = B^+$.

For any $A \in F^{(i)}(U)$, we denote

$$R_A = \{(x, y) \in U \times U : A(x) \neq A(y)\}, R_{A^-} = \{(x, y) \in U \times U : A^-(x) > A^-(y)\}$$
$$and \ \ R_{A^+} = \{(x, y) \in U \times U : A^+(x) > A^+(y)\}.$$

Remark 2.3. Let $A \in F^{(i)}(U)$. Then

(1) $R_A = \varnothing \iff$ There exist $[a, b] \in [I]$ such that $A = \widetilde{[a, b]}$.

(2) *a)* $R_{A^-} = \varnothing \iff$ There exist $a \in I$ such that $A^- = \bar{a}$.

b) $R_{A^+} = \varnothing \iff$ There exist $a \in I$ such that $A^+ = \bar{a}$.

(3) $R_A = \varnothing \iff R_{A^-} = \varnothing$ and $R_{A^+} = \varnothing$.

Definition 2.4 ([6])**.** $A \in F^{(i)}(U)$ is called an IVF point in U, if there exist $[a, b] \in [I] - \{\bar{0}\}$ and $x \in U$ such that

$$A(y) = \begin{cases} [a, b], & y = x, \\ \bar{0}, & y \neq x. \end{cases}$$

We denote A by $x_{[a,b]}$.

If $[a, b] = \bar{1}$, then

$$x_{\bar{1}}(y) = \begin{cases} \bar{1}, & y = x, \\ \bar{0}, & y \neq x. \end{cases}$$

Remark 2.5. Let $A \in F^{(i)}(U)$. Then

$$A = \bigcup_{y\in U} (A(y)y_{\bar{1}}) \quad and \quad A = \bigcap_{y\in U} (\widetilde{A(y)} \cup (y_{\bar{1}})^c).$$

Definition 2.6 ([6])**.** $\tau \subseteq F^{(i)}(U)$ is called an IVF topology on U, if

(1) $\tilde{0}, \tilde{1} \in \tau$,

(2) $A, B \in \tau \implies A \cap B \in \tau$,

(3) $\{A_i, i \in J\} \subseteq \tau \implies \bigcup_{i\in J} A_i \in \tau$.

The pair (U, τ) is called an IVF topological space. Every member of τ is called an IVF open set in U. B is called an IVF closed set in U if $B \in \tau^c$ with $\tau^c = \{A : A^c \in \tau\}$

Let $A \in F^{(i)}(U)$. Then interior and closure of A denoted respectively by $int(A)$ and $cl(A)$, are defined as follows:

$$int(A) \text{ or } int_\tau(A) = \bigcup\{B \in \tau : B \subseteq A\}$$

and

$$cl(A)) \text{ or } cl_\tau(A) = \bigcap\{B \in \tau^c : B \supseteq A\}.$$

Proposition 2.7 ([6]). *Let (U, τ) be an IVF topological space and $A, B \in F^{(i)}(U)$. Then, the following properties hold:*
(1) $int(\tilde{1}) = \tilde{1}$, $cl(\tilde{0}) = \tilde{0}$;
(2) $int(A) \subseteq A \subseteq cl(A)$;
(3) $A \subseteq B \Longrightarrow int(A) \subseteq int(B)$, $cl(A) \subseteq cl(B)$;
(4) $int(A^c) = (cl(A))^c$, $cl(A^c) = (int(A))^c$;
(5) $int(A \cap B) = int(A) \cap int(B)$, $cl(A \cup B) = cl(A) \cup cl(B)$;
(6) $int(int(A)) = int(A)$, $cl(cl(A)) = cl(B)$.

3. IVF rough approximation operators

Recall that R is called an IVF relation on U if $R \in F^{(i)}(U \times U)$.

Definition 3.1 ([14]). Let R be an IVF relation on U. Then R is called
(1) reflexive if $R(x, x) = \bar{1}$ for any $x \in U$.
(2) symmetric if $R(x, y) = R(y, x)$ for any $x, y \in U$.
(3) transitive if $R(x, z) \geq R(x, y) \wedge R(y, z)$ for any $x, y, z \in U$.

If R is a reflexive, symmetric and transitive IVF relation on U, then R is called an equivalence IVF relation on U.

Definition 3.2 ([14]). Let R be an IVF relation on U. The pair (U, R) is called an IVF approximation space. For any $A \in F^{(i)}(U)$, the IVF lower and the IVF upper approximation of A with regard to (U, R), denoted by $\underline{R}(A)$ and $\overline{R}(A)$ are respectively, defined as follows:

$$\underline{R}(A)(x) = \bigwedge_{y \in U} (A(y) \vee (\bar{1} - R(x, y))) \quad (x \in U)$$

and

$$\overline{R}(A)(x) = \bigvee_{y \in U} (A(y) \wedge R(x, y)) \qquad (x \in U).$$

$\underline{R} : F^{(i)}(U) \mapsto F^{(i)}(U)$ and $\overline{R} : F^{(i)}(U) \mapsto F^{(i)}(U)$ are called the IVF lower approximation operator and the IVF upper approximation operator, respectively.

Remark 3.3 ([14]). Let (U, R) be an IVF approximation space. Then

$$\overline{R}(x_{\bar{1}})(y) = R(y, x) \text{ and } \underline{R}((x_{\bar{1}})^c)(y) = \bar{1} - R(y, x) \quad (x, y \in U).$$

Proposition 3.4. *Let (U, R) be an IVF approximation space. Then*

$$\underline{R}(A)^- = \underline{R^+}(A^-),\ (\underline{R}(A))^+ = \underline{R^-}(A^+),$$

$$(\overline{R}(A))^- = \overline{R^-}(A^-) \text{ and } (\overline{R}(A))^+ = \overline{R^+}(A^+) \qquad (A \in F^{(i)}(U)).$$

Proposition 3.5 ([18]). *Let (U, R) be an IVF approximation space. Then for any $A, B \in F^{(i)}(U)$ and $[a, b] \in [I]$,*

(1) $\underline{R}(\tilde{1}) = \tilde{1}$, $\overline{R}(\tilde{0}) = \tilde{0}$;
(2) $A \subseteq B \Longrightarrow \underline{R}(A) \subseteq \underline{R}(B)$, $\overline{R}(A) \subseteq \overline{R}(B)$;
(3) $\underline{R}(A^c) = (\overline{R}(A))^c$, $\overline{R}(A^c) = (\underline{R}(A))^c$;
(4) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$, $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$;
(5) $\overline{R}([a, b]A) = [a, b]\overline{R}(A)$, $\underline{R}(\widetilde{[a, b]} \cup A) = \widetilde{[a, b]} \cup \underline{R}(A)$

Theorem 3.6 ([18]). *Let (U, R) be an IVF approximation space. Then*

$$\begin{aligned}
&(1)\ R \text{ is reflexive} && \Longleftrightarrow && (ILR)\ \forall A \in F^{(i)}(U), \underline{R}(A) \subseteq A.\\
& && \Longleftrightarrow && (IUR)\ \forall A \in F^{(i)}(U), A \subseteq \overline{R}(A).\\
&(2)\ R \text{ is symmetric} && \Longleftrightarrow && (ILS)\ \forall (x, y) \in U \times U, \underline{R}((x_{\bar{1}})^c)(y) = \underline{R}((y_{\bar{1}})^c)(x).\\
& && \Longleftrightarrow && (IUS)\ \forall (x, y) \in U \times U, \overline{R}(x_{\bar{1}})(y) = \overline{R}(y_{\bar{1}})(x).\\
&(3)\ R \text{ is transitive} && \Longleftrightarrow && (ILT)\ \forall A \in F^{(i)}(U), \underline{R}(A) \subseteq \underline{R}(\underline{R}(A)).\\
& && \Longleftrightarrow && (IUT)\ \forall A \in F^{(i)}(U), \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A).
\end{aligned}$$

Corollary 3.7. *Let (U, R) be an IVF approximation space. If R is reflexive and transitive, then*

$$\underline{R}(\underline{R}(A)) = \underline{R}(A) \text{ for any } A \in F^{(i)}(U).$$
$$\overline{R}(\overline{R}(A)) = \overline{R}(A) \text{ for any } A \in F^{(i)}(U).$$

Proposition 3.8. *Let (U, R) be an IVF approximation space. If R is reflexive, then for any $[a, b] \in [I]$,*

$$\underline{R}(\widetilde{[a, b]}) = \widetilde{[a, b]} = \overline{R}(\widetilde{[a, b]}).$$

Proof. (1) For any $[a, b] \in [I]$ and $x \in U$, since

$$\underline{R}(\widetilde{[a, b]})(x) = \bigwedge_{y \in U}([a, b] \vee (\bar{1} - R(x, y))) = [a, b] \vee (\bigwedge_{y \in U}(\bar{1} - R(x, y)) \geq [a, b],$$

$\underline{R}(\widetilde{[a, b]}) \supseteq \widetilde{[a, b]}$. Note that $\underline{R}(\widetilde{[a, b]}) \subseteq \widetilde{[a, b]}$ by the reflexivity of R and Theorem 3.5(1). Then $\underline{R}(\widetilde{[a, b]}) = \widetilde{[a, b]}$.

By Theorem 3.5(3),

$$\overline{R}(\widetilde{[a, b]}) = (\underline{R}(\widetilde{[a, b]^c}))^c = (\widetilde{[a, b]^c})^c = \widetilde{[a, b]}.$$

□

4. IVF topologies based on IVF relations

Let R be an IVF relation on U. Denote

$$\sigma_R = \{A \in F^{(i)}(U) : A \subseteq \underline{R}(A)\},$$

$$\tau_R = \{A \in F^{(i)}(U) : \underline{R}(A) = A\} \text{ and } \theta_R = \{\underline{R}(A) : A \in F^{(i)}(U)\}.$$

Theorem 4.1. *Let R be an IVF relation on U.*
(1) $\tau_R \subseteq \sigma_R$, $\tau_R \subseteq \theta_R$.
(2) *If R is transitive, then* $\tau_R \subseteq \theta_R \subseteq \sigma_R$.
(3) *If R is reflexive, then* $\tau_R = \sigma_R$.
(4) *If R is reflexive and transitive, then* $\tau_R = \theta_R = \sigma_R$.

Proof. This is obvious. □

4.1. **The IVF topology induced by an IVF relation.**

Theorem 4.2. *Let R be an IVF relation on U. Then*
(1) σ_R *is an IVF topology on* U.
(2) $int_{\sigma_R}(A) \subseteq \underline{R}(A)$ *and* $cl_{\sigma_R}(A) \supseteq \overline{R}(A)$ $(A \in F^{(i)}(U))$.
(3) (U, σ_R) *is not connected.*

Proof. (1) This is obvious.
(2) For any $A \in F^{(i)}(U)$, by Proposition 3.5(2),

$$int_{\sigma_R}(A) = \bigcup\{B \subseteq \underline{R}(B) : B \subseteq A\} \subseteq \bigcup\{B \subseteq \underline{R}(B) : \underline{R}(B) \subseteq \underline{R}(A)\} \subseteq \underline{R}(A).$$

By Proposition 2.12 (4) and Proposition 3.5(3),

$$cl_{\sigma_R}(A) = (int_{\sigma_R}(A^c))^c \supseteq (\underline{R}(A^c))^c = \overline{R}(A) \quad (A \in F^{(i)}(U)).$$

(3) Pick $\bar{0} < [a, b] < \bar{1}$, by (1), $\widetilde{[a, b]}$ is closed and open. Then (U, σ_R) is not connected. □

Definition 4.3. Let R be an IVF relation on U. σ_R is called the IVF topology induced by R on U.

Theorem 4.4. *Let R_1 and R_2 be two IVF relations on U. Let σ_{R_1} and σ_{R_2} be the IVF topologies induced by R_1 and R_2 on U, respectively. Then*
(1) *If $R_1 \subseteq R_2$, then* $\sigma_{R_2} \subseteq \sigma_{R_1}$.
(2) $\sigma_{R_1 \cup R_2} = \sigma_{R_1} \cap \sigma_{R_2}$.

Proof. (1) For any $A \in \sigma_{R_2}$, we have $A \subseteq \underline{R_2}(A)$. By $R_1 \subseteq R_2$, we can easily prove that $\underline{R_2}(A) \subseteq \underline{R_1}(A)$. Then $A \subseteq \underline{R_1}(A)$. Hence $A \in \sigma_{R_1}$.
Thus, $\sigma_{R_2} \subseteq \sigma_{R_1}$.
(2) Put $R = R_1 \cup R_2$.
By (1), $\sigma_R \subseteq \sigma_{R_1}$ and $\sigma_R \subseteq \sigma_{R_2}$. Thus

$$\sigma_{R_1} \cap \sigma_{R_2} \supseteq \sigma_R.$$

For any $A \in \tau_{R_1} \cap \tau_{R_2}$, by $\underline{R_1}(A) \supseteq A$ and $\underline{R_2}(A) \supseteq A$,

$$\begin{aligned}(\underline{R})(A)(x) &= \bigwedge_{y\in U}(A(y) \vee (\bar{1} - (R_1 \cup R_2)(x,y)))\\ &= \bigwedge_{y\in U}(A(y) \vee (\bar{1} - (R_1(x,y) \vee R_2(x,y))))\\ &= \bigwedge_{y\in U}(A(y) \vee (\bar{1} - R_1(x,y)) \wedge (\bar{1} - R_2(x,y))))\\ &= \bigwedge_{y\in U}((A(y) \vee (\bar{1} - R_1(x,y))) \wedge (A(y) \vee (\bar{1} - R_2(x,y))))\\ &= (\bigwedge_{y\in U}(A(y) \vee (\bar{1} - R_1(x,y)))) \wedge (\bigwedge_{y\in U}(A(y) \vee (\bar{1} - R_2(x,y))))\\ &= \underline{R_1}(A)(x) \vee \underline{R_2}(A)(x) = (\underline{R_1}(A) \cup \underline{R_2}(A))(x)\end{aligned}$$

Then

$$\underline{R}(A) = (\underline{R_1 \cup R_2})(A) = \underline{R_1}(A) \cup \underline{R_2}(A) \supseteq A \cup A = A.$$

Hence $A \in \sigma_R$. Thus, $\sigma_{R_1} \cap \sigma_{R_2} \subseteq \sigma_R$. Hence $\sigma_{R_1 \cup R_2} = \sigma_{R_1} \cap \sigma_{R_2}$. □

4.2. **The IVF topology induced by a reflexive IVF relation.**

Theorem 4.5. *Let R be a reflexive IVF relation on U. Then*
(1) τ_R is an IVF topology on U.
(2) $int_{\tau_R}(A) \subseteq \underline{R}(A)$ and $cl_{\tau_R}(A) \supseteq \overline{R}(A)$ $(A \in F^{(i)}(U))$.

Proof. The proof is similar to Theorem 4.2. □

Definition 4.6. Let R be a reflexive IVF relation on U. τ_R is called the IVF topology induced by R on U.

Theorem 4.7. *Let R be reflexive IVF relation on U and let τ_R be the IVF topology induced by R on U. Then*
(1) $\tau_R \subseteq \theta_R$.
(2) *If R is transitive, then*
 a) $\underline{R}$ is an interior operator of τ_R;
 b) $\overline{R}$ is a closure operator of τ_R.
(3) *If $R_A = \varnothing$, then $A \in \tau_R$*
(4) *(U, τ_R) is not connected.*

Proof. (1) This is obvious.
(2) a) It suffices to show

$$\underline{R}(A) = int(A)$$

for any $A \in F^{(i)}(U)\}$, where $int(A) = \bigcup\{B \in \tau_R : B \subseteq A\}$.
Let $A \in F^{(i)}(U)\}$. By (1), $\underline{R}(A) \in \tau_R$. Note that $\underline{R}(A) \subseteq A$ by the reflexivity of R. Then $\underline{R}(A) \subseteq int(A)$.

Conversely, by $A \supseteq \bigcup\{B \in \tau_R : B \subseteq A\}$, $\underline{R}(A) \supseteq \underline{R}(\bigcup\{B \in \tau_R : B \subseteq A\})$. By Proposition 3.5(1),

$$\begin{aligned} \underline{R}(\bigcup\{B \in \tau_R : B \subseteq A\}) &= \underline{R}(\bigcup\{\underline{R}(B) : \underline{R}(B) \subseteq A \text{ and } B \in \tau_R\}) \\ &= \bigcup\{\underline{R}(B) : \underline{R}(B) \subseteq A \text{ and } B \in \tau_R\} \\ &= \bigcup\{B \in \tau_R : B \subseteq A\} = int(A) \end{aligned}$$

Then $\underline{R}(A) \supseteq int(A)$.

Hence

$$\underline{R}(A) = int(A).$$

b) This holds by (2) and Proposition 3.5(3).

(3) If $R_{\underline{A}} = \varnothing$, then there exist $[a,b] \in [I]$ such that $A = \widetilde{[a,b]}$. By Theorem 3.6(1), $\underline{R}(\widetilde{[a,b]}) = \widetilde{[a,b]}$. So $A = \underline{R}(A) \in \tau_R$.

(4) Pick $[a,b] \in [I]$ such that $\bar{0} < [a,b] < \bar{1}$. By (3) and Theorem 3.6(1),

$$\widetilde{[a,b]} = \overline{R}(\widetilde{[a,b]}) = cl(\widetilde{[a,b]}) \in (\tau_R)'.$$

Then $\widetilde{[a,b]}$ is an IVF closed set in U.

By the proof of (3), $\widetilde{[a,b]}$ is an IVF open set.

Hence (U, τ_R) is not connected. □

Example 4.8. Let $U = \{x, y, z\}$ and let R be a transitive IVF relation on U.

$$R(x,y) = R(x,z) = R(y,z) = R(z,x) = \bar{0}, R(y,x) = \bar{1}, R(z,y) = [0.1, 0.2]$$

Then R is not reflexive.

Let $A = \frac{\bar{0}}{x} + \frac{[0.3,0.5]}{y} + \frac{\bar{1}}{z}$. Then

$$\underline{R}(\underline{R}(A)) = \underline{R}(z_{\bar{1}}) = z_{[0.8,0.9]} \neq z_{\bar{1}} = \underline{R}(A).$$

Then $\underline{R}(A) \notin \tau$. Thus

$$\theta_R \neq \theta_R \ \ \text{and} \ \ \underline{R}(A) \neq int(A).$$

Example 4.9. Let $U = \{x, y, z\}$ and let R be a reflexive IVF relation on U. R is defined as follows:

$R(x,y) = R(x,z) = R(z,x) = \bar{0}$, $R(y,x) = [0.2, 0.7]$, $R(y,z) = \bar{1}, R(z,y) = [0.3, 0.8]$.

Pick

$$A = \frac{\bar{0}}{x} + \frac{[0.4, 0.5]}{y} + \frac{\bar{1}}{z} \ \ \text{and} \ \ B = \frac{\bar{1}}{x} + \frac{[0.5, 0.6]}{y} + \frac{\bar{0}}{z}.$$

(1) We have

$$R(z,y) \wedge R(y,x) = [0.2, 0.7] \not\leq \bar{0} = R(z,x).$$

Then R is not transitive.

(2) Since

$$\underline{R}(A) = \frac{\bar{0}}{x} + \frac{[0.3, 0.5]}{y} + \frac{[0.4, 0.7]}{z} \ \ \text{and} \ \ \underline{R}(\underline{R}(A)) = \frac{\bar{0}}{x} + \frac{[0.3, 0.5]}{y} + \frac{[0.3, 0.7]}{z}$$

we have $\underline{R}(\underline{R}(A)) \neq \underline{R}(A)$, Then $\underline{R}(A) \notin \tau_R$. Thus,

$$\tau_R \neq \{\underline{R}(A) : A \in F^{(i)}(U)\} \;\; and \;\; int_{\tau_R}(A) \neq \underline{R}(A).$$

Obviously, $B^c = A$. By Proposition 3.7(3),

$$(\overline{R}(B))^c = \underline{R}(B^c) = \underline{R}(A) \notin \tau_R.$$

Then $\overline{R}(B) \notin \tau_R^c$. Thus $cl_{\tau_R}(B) \neq \overline{R}(B)$.

Theorem 4.10. *Let R_1 and R_2 be two IVF relations on U. Let σ_{R_1} and σ_{R_2} be the IVF topologies induced by R_1 and R_2 on U, respectively. Denote $R = R_1 \cup R_2$.*

(1) *If R_1 and R_2 are reflexive, then*

(*a*) *If $R_1 \subseteq R_2$, then $\theta_{R_2} \subseteq \theta_{R_1}$.*

(*b*) *$R_1 \cup R_2$ is reflexive.*

(*c*) *$\theta_{R_1 \cup R_2} = \theta_{R_1} \cap \theta_{R_2}$.*

(3) *If R_1 and R_2 are reflexive and transitive, then*

(*a*) *If $R_1 \subseteq R_2$, then $\tau_{R_2} \subseteq \tau_{R_1}$.*

(*b*) *$\tau_{R_2} = \tau_{R_1} \Longleftrightarrow R_1 = R_2$.*

(*c*) *$R_1 \cup R_2$ is reflexive and transitive.*

(*d*) *$\tau_{R_1 \cup R_2} = \tau_{R_1} \cap \tau_{R_2}$.*

Proof. The proof is similar to Theorem 4.4. □

5. Conclusions

Topology and rough set theory are widely used in the research fields of machine learning and cybernetics. In this paper, we have explored the topological structures of IVF rough sets. We hope that the analysis offered in this paper will facilitate further research in uncertain reasoning under fuzziness. In future work we will study uncertain measures of IVF rough sets with application to data analysis.

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