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α-Fuzzy new ideal of PU-algebra

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Abstract. In this paper, the notion α-fuzzy new-ideal of a PU-algebra are defined and discussed. The homomorphic images (pre images) of α-fuzzy *new*-ideal under homomorphism of a PU-algebras has been obtained. Some related result have been derived.

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1. Introduction

In 1966, Imai and Iseki [1, 2, 3] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [4, 5], Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They are shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [13], Neggers and Kim introduced the notion of d-algebras, which is a generalization of BCK-algebras and investigated a relation between d-algebras and BCK-algebras. Neggers et al.[14] introduced the notion of Q-algebras , which is a generalization of BCH/BCI/BCK-algebras. Megalai and Tamilarasi [9] introduced the notion of a TM-algebra which is a generalization of BCK/BCI/BCH-algebras and several results are presented .Mostafa et al.[12] introduced a new algebraic structure called PU-algebra, which is a dual for TM-algebra and investigated severed basic properties. Moreover they derived new view of several ideals on PU-algebra and studied some properties of them. The concept of fuzzy sets was introduced by Zadeh [17]. In 1991, Xi [16] applied the concept of fuzzy sets to BCI, BCK, MV-algebras. Since its inception, the theory of fuzzy sets , ideal theory and its fuzzification has developed in many directions and is finding applications in a wide variety of fields [6, 7, 8, 10, 11, 15]. Here in this paper, we modify the ideas of Xi [16], to introduce the notion, α-fuzzy *new*-ideal of a PU-algebra. The homomorphic image (pre image) of α-fuzzy *new*-ideal of a PU-algebra under homomorprhism of a PU-algebras are discussed. Many related results have been derived.

2. Preliminaries

Now, we will recall some known concepts related to PU-algebra from the literature, which will be helpful in further study of this article

Definition 2.1 ([12])**.** A PU-algebra is a non-empty set X with a constant $0 \in X$ and a binary operation $*$ satisfying the following conditions:
(I) $0 * x = x$,
(II) $(x * z) * (y * z) = y * x$ for any $x, y, z \in X$.
On X we can define a binary relation " $\leq$ " by: $x \leq y$ if and only if $y * x = 0$.

Example 2.2. [12] Let $X = \{0, 1, 2, 3, 4\}$ in which $*$ is defined by

$*$	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

Then $(X, *, 0)$ is a PU-algebra.

Proposition 2.3 ([12])**.** *In a PU-algebra $(X, *, 0)$ the following hold : for all $x, y, z \in X$*
(a) $x * x = 0$.
(b) $(x * z) * z = x$.
(c) $x * (y * z) = y * (x * z)$.
(d) $x * (y * x) = y * 0$.
(e) $(x * y) * 0 = y * x$.
(f) If $x \leq y$*, then* $x * 0 = y * 0$.
(g) $(x * y) * 0 = (x * z) * (y * z)$.
(h) $x * y \leq z$ *if and only if* $z * y \leq x$.
(i) $x \leq y$ *if and only if* $y * z \leq x * z$.
(j) In a PU-algebra $(X, *, 0)$, the following are equivalent:
(1) $x = y$, *(2)* $x * z = y * z$, *(3)* $z * x = z * y$.
(k) The right and the left cancellation laws hold in X.
(l) $(z * x) * (z * y) = x * y$,
(m) $(x * y) * z = (z * y) * x$.
(n) $(x * y) * (z * u) = (x * z) * (y * u)$ *for all x, y, z and $u \in X$.*

Definition 2.4 ([12])**.** A non-empty subset I of a PU-algebra $(X, *, 0)$ is called a sub-algebra of X if $x * y \in I$ whenever $x, y \in I$.

Definition 2.5 ([12])**.** A non-empty subset I of a PU-algebra $(X, *, 0)$ is called a *new*-ideal of X if,
(i) $0 \in I$,
(ii) $(a * (b * x)) * x \in I$, for all $a, b \in I$ and $x \in X$.

Example 2.6 ([12])**.** Let $X = \{0, a, b, c\}$, in which $*$ is defined by the following table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then $(X, *, 0)$ is a *PU*-algebra. It is easy to show that $I_1 = \{0, a\}$, $I_2 = \{0, b\}$, $I_3 = \{0, c\}$ are *new*-ideals of X.

Proposition 2.7. *Let $(X, *, 0)$ and $(X', *', 0')$ be PU-algebras and $f : X \to X'$ be a homomorphism, then* ker f *is a new-ideal of X.*

Proof. By the definition of *PU*-algebra and its properties, we have that
$f(0) = f(0 * 0) = f(0) *' f(0) = 0'$, then $0 \in$ker f.
Let $a, b \in$ ker f and $x \in X$, it follows by properties of *PU*-algebra that
$(a * (b * x)) * x = (x * (b * x)) * a = (b * (x * x)) * a = (b * 0) * a$. Then we have that $f((a * (b * x)) * x) = f((b * 0) * a) = f(b * 0) *' f(a) = (f(b) *' f(0)) *' f(a) = (0' *' 0') *' 0' = 0'$. Hence $((a * (b * x)) * x) \in$ker f. Therefore ker f is a *new*-ideal of X. □

3. α-Fuzzy new-ideal of *PU*-algebra

In this section, we will discuss and investigate a new notion called α-fuzzy *new*-ideal of a *PU*-algebra and study several basic properties which related to α-fuzzy *new*-ideal.

Definition 3.1 ([17])**.** Let X be a non-empty set, a fuzzy subset μ in X is a function $\mu : X \to [0, 1]$.

Definition 3.2. Let X be a *PU*-algebra. A fuzzy subset μ in X is called a fuzzy sub-algebra of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in X$.

Definition 3.3. Let μ be a fuzzy subset of a *PU*-algebra X. Let $\alpha \in [0, 1]$. Then the fuzzy set μ^α of X is called the α-fuzzy subset of X (w.r.t. fuzzy set μ) and is defined by $\mu^\alpha(x) = \min\{\mu(x), \alpha\}$, for all $x \in X$.

Remark 3.4. Clearly, $\mu^1 = \mu$ and $\mu^0 = 0$.

Lemma 3.5. *If μ is a fuzzy sub-algebra of a PU-algebra X and $\alpha \in [0, 1]$, then $\mu^\alpha(x * y) \geq \min\{\mu^\alpha(x), \mu^\alpha(y)\}$, for all $x, y \in X$.*

Proof. Let X be a *PU*-algebra and $\alpha \in [0, 1]$. Then by Definition 3.3., we have that

$$\begin{aligned} \mu^\alpha(x * y) = \min\{\mu(x * y), \alpha\} &\geq \min\{\min\{\mu(x), \mu(y)\}, \alpha\} \\ &= \min\{\min\{\mu(x), \alpha\}, \min\{\mu(y), \alpha\}\} \\ &= \min\{\mu^\alpha(x), \mu^\alpha(y)\}, \text{ for all } x, y \in X. \end{aligned}$$

□

Definition 3.6. Let X be a *PU*-algebra. A fuzzy subset $\mu^\alpha(x)$ in X is called an α-fuzzy sub-algebra of X if $\mu^\alpha(x * y) \geq \min\{\mu^\alpha(x), \mu^\alpha(y)\}$ for all $x, y \in X$.

It is clear that an α-fuzzy sub-algebra of a PU-algebra X is a generalization of a fuzzy sub-algebra of X and a fuzzy sub-algebra of X is an α-fuzzy sub-algebra of X in case of $\alpha = 1$.

Definition 3.7. Let $(X, *, 0)$ be a PU-algebra, a fuzzy subset μ in X is called a fuzzy *new*-ideal of X if it satisfies the following conditions:
$(F_1)\mu(0) \geq \mu(x)$.
$(F_2)\mu((x*(y*z))*z) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y, z \in X$.

Lemma 3.8. *If μ is a fuzzy new-ideal of a PU-algebra X and $\alpha \in [0,1]$, then*
$(F_1^\alpha)\mu^\alpha(0) \geq \mu^\alpha(x)$,
$(F_2^\alpha)\mu^\alpha((x*(y*z))*z) \geq \min\{\mu^\alpha(x), \mu^\alpha(y)\}$ *for all* $x, y, z \in X$.

Proof. Let X be a PU-algebra and $\alpha \in [0,1]$. Then by Definition 3.3and Definition 3.7, we have that: $\mu^\alpha(0) = \min\{\mu(0), \alpha\} \geq \min\{\mu(x), \alpha\} = \mu^\alpha(x)$, for all $x \in X$.

$$\begin{aligned} \mu^\alpha((x*(x*z))*z) &= \min\{\mu((x*(y*z))*z), \alpha\} \\ &\geq \min\{\min\{\mu(x), \mu(y)\}, \alpha\} \\ &= \min\{\min\{\mu(x), \alpha\}, \min\{\mu(y), \alpha\}\} \\ &= \min\{\mu^\alpha(x), \mu^\alpha(y)\}, \text{ for all } x, y, z \in X. \end{aligned}$$

□

Definition 3.9. Let $(X, *, 0)$ be a PU-algebra, an α-fuzzy subset μ^α in X is called α-fuzzy *new*-ideal of X if it satisfies the following conditions:
$(F_1^\alpha)\mu^\alpha(0) \geq \mu^\alpha(x)$,
$(F_2^\alpha)\mu^\alpha((x*(y*z))*z) \geq \min\{\mu^\alpha(x), \mu^\alpha(y)\}$, for all $x, y, z \in X$.

It is clear that an α-fuzzy *new*-ideal of a PU-algebra X is a generalization of a fuzzy *new*-ideal of X and a fuzzy *new*-ideal of X is a special case, when $\alpha = 1$.

Example 3.10. Let $X = \{0, 1, 2, 3\}$ in which $*$ is defined by the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X, *, 0)$ is a PU-algebra. Define an α-fuzzy subset $\mu^\alpha : X \to [0,1]$ by

$$\mu^\alpha(x) = \begin{cases} \min\{\alpha, 0.7\} & if\ x \in \{0, 1\} \\ \min\{\alpha, 0, 3\} & otherwise \end{cases}$$

Routine calculation gives that μ^α is an α-fuzzy *new*-ideal of X.

Lemma 3.11. *Let μ^α be an α-fuzzy new-ideal of a PU-algebra X. If the inequality $x * y \leq z$ holds in X, then $\mu^\alpha(y) \geq \min\{\mu^\alpha(x), \mu^\alpha(z)\}$.*

Proof. Assume that the inequality $x * y \leq z$ holds in X, then $z * (x * y) = 0$ and by (F_2^α), $\mu^\alpha \overbrace{((z*(x*y))*y)} \geq \min\{\mu^\alpha(x), \mu^\alpha(z)\}$. Since $\mu^\alpha(y) = \mu^\alpha(0 * y)$, then we have that $\mu^\alpha(y) \geq \min\{\mu^\alpha(x), \mu^\alpha(z)\}$. □

Corollary 3.12. *Let μ be a fuzzy new-ideal of a PU-algebra X. If the inequality $x * y \leq z$ holds in X, then $\mu(y) \geq \min\{\mu(x), \mu(z)\}$.*

Lemma 3.13. *If μ^α is an α-fuzzy subset of a PU-algebra X and if $x \leq y$, then $\mu^\alpha(x) = \mu^\alpha(y)$.*

Proof. If $x \leq y$, then $y * x = 0$. Hence by the definition of PU-algebra and its properties we have that $\mu^\alpha(x) = \min\{\mu(x), \alpha\} = \min\{\mu(0 * x), \alpha\} = \min\{\mu((y * x) * x), \alpha\} = \min\{\mu(y), \alpha\} = \mu^\alpha(y)$. □

Corollary 3.14. *If μ is a fuzzy subset of a PU-algebra X and if $x \leq y$, then $\mu(x) = \mu(y)$.*

Definition 3.15. Let μ^α be an α-fuzzy *new*-ideal of a PU-algebra X and let x be an element of X. We define $(\bigcap_{i\in I} \mu_i^\alpha)(x) = \inf(\mu_i^\alpha(x))_{i\in I}$.

Proposition 3.16. *The intersection of any set of α-fuzzy new-ideals of a PU-algebra X is also an α-fuzzy new-ideal of X.*

Proof. Let $\{\mu_i^\alpha\}_{i\in I}$ be a family of α-fuzzy *new*-ideals of a PU-algebra X, then for any $x, y, z \in X$,

$$(\bigcap_{i\in I} \mu_i^\alpha)(0) = \inf(\mu_i^\alpha(0))_{i\in I} \geq \inf(\mu_i^\alpha(x))_{i\in I} = (\bigcap_{i\in I} \mu_i^\alpha)(x)$$

and

$$\begin{aligned}(\bigcap_{i\in I} \mu_i^\alpha)((x * (y * z)) * z) &= \inf(\mu_i^\alpha((x * (y * z))) * z)_{i\in I} \\ &\geq \inf(\min\{\mu_i^\alpha(x), \mu_i^\alpha(y)\})_{i\in I} \\ &= \min\{\inf(\mu_i^\alpha(x))_{i\in I}, \inf(\mu_i^\alpha(y))_{i\in I}\} \\ &= \min\{(\bigcap_{i\in I} \mu_i^\alpha)(x), (\bigcap_{i\in I} \mu_i^\alpha)(y)\}.\end{aligned}$$

This completes the proof. □

Theorem 3.17. *Let μ^α be an α-fuzzy subset of a PU-algebra X. Then μ^α is an α-fuzzy new-ideal of X if and only if it satisfies:*
$(\forall \varepsilon \in [0, 1])(U(\mu^\alpha; \varepsilon) \neq \phi \Rightarrow U(\mu^\alpha; \varepsilon)$ is a new-ideal of X),
where, $U(\mu^\alpha; \varepsilon) = \{x \in X : \mu^\alpha(x) \geq \varepsilon\}$.

Proof. Assume that μ^α is an α-fuzzy *new*-ideal of X. Let $\varepsilon \in [0, 1]$ be such that $U(\mu^\alpha; \varepsilon) \neq \phi$. Let $x \in U(\mu^\alpha; \varepsilon)$, then $\mu^\alpha(x) \geq \varepsilon$. Since $\mu^\alpha(0) \geq \mu^\alpha(x)$ for all $x \in X$, then $\mu^\alpha(0) \geq \varepsilon$. Thus $0 \in U(\mu^\alpha; \varepsilon)$. Let $x \in X$ and $a, b \in U(\mu^\alpha; \varepsilon)$, then $\mu^\alpha(a) \geq \varepsilon$ and $\mu^\alpha(b) \geq \varepsilon$. It follows by the definition of α-fuzzy *new*-ideal that $\mu^\alpha((a * (b * x)) * x) \geq \min\{\mu^\alpha(a), \mu^\alpha(b)\} \geq \varepsilon$, so that $(a * (b * x)) * x \in U(\mu^\alpha; \varepsilon)$. Hence $U(\mu^\alpha; \varepsilon)$ is a *new*-ideal of X.
Conversely, suppose that $(\forall \varepsilon \in [0, 1])(U(\mu^\alpha; \varepsilon) \neq \phi \Rightarrow U(\mu^\alpha; \varepsilon)$ is a *new*-ideal of X), where $U(\mu^\alpha; \varepsilon) = \{x \in X : \mu^\alpha(x) \geq \varepsilon\}$. If $\mu^\alpha(0) < \mu^\alpha(x)$ for some $x \in X$, then $\mu^\alpha(0) < \varepsilon_0 < \mu^\alpha(x)$ by taking $\varepsilon_0 = (\mu^\alpha(0) + \mu^\alpha(x))/2$. Hence $0 \notin U(\mu^\alpha; \varepsilon_0)$, which is a contradiction.
Let $a, b, c \in X$ be such that $\mu^\alpha((a * (b * c)) * c) < \min\{\mu^\alpha(a), \mu^\alpha(b)\}$. Taking $\varepsilon_1 = (\mu^\alpha(a*(b*c)*c)+\min\{\mu^\alpha(a), \mu^\alpha(b)\})/2$, we have $\varepsilon_1 \in [0, 1]$ and $\mu^\alpha((a*(b*c))*c) < \varepsilon_1 < \min\{\mu^\alpha(a), \mu^\alpha(b)\}$. It follows that $a, b \in U(\mu^\alpha; \varepsilon_1)$ and $(a*(b*c))*c \notin U(\mu^\alpha; \varepsilon_1)$. This is a contradiction, and therefore μ^α is an α-fuzzy *new*-ideal of X. □

Corollary 3.18. *Let μ be a fuzzy subset of a PU-algebra X. Then μ is a fuzzy new-ideal of X if and only if it satisfies:*
$(\forall\varepsilon \in [0,1])(U(\mu,\varepsilon) \neq \phi \Rightarrow U(\mu;\varepsilon)$ *is a new-ideal of X),*
where, $U(\mu,\varepsilon) = \{x \in X : \mu(x) \geq \varepsilon\}$.

Definition 3.19. Let f be a mapping from X to Y. If μ^α is an α-fuzzy subset of X, then the α-fuzzy subset β^α of Y defined by

$$f(\mu^\alpha)(y) = \beta^\alpha(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \mu^\alpha(x) & if \;\; f^{-1}(y) \neq \varphi \\ 0 & otherwise \end{cases}$$

is said to be the image of μ^α under f.
Similarly if β^α is an α-fuzzy subset of Y, then the α-fuzzy subset $\mu^\alpha = (\beta^\alpha \circ f)$ of X (i.e. the α-fuzzy subset defined by $\mu^\alpha(x) = \beta^\alpha(f(x))$ for all $x \in X$) is called the pre-image of β^α under f.

Theorem 3.20. *Let $(X,*,0)$ and $(X',*',0')$ be PU-algebras and $f : X \to X'$ be a homomorphism. If β^α is an α-fuzzy new-ideal of X' and μ^α is the pre-image of β^α under f, then μ^α is an α-fuzzy new-ideal of X.*

Proof. Since μ^α is the pre-image of β^α under f, then $\mu^\alpha(x) = \beta^\alpha(f(x))$ for all $x \in X$. Let $x \in X$, then $\mu^\alpha(0) = \beta^\alpha(f(0)) \geq \beta^\alpha(f(x)) = \mu^\alpha(x)$. Now let $x, y, z \in X$ then

$$\begin{aligned} \mu^\alpha((x*(y*z))*z) &= \beta^\alpha(f((x*(y*z))*z)) \\ &= \beta^\alpha(f(x*(y*z))*' f(z)) \\ &= \beta^\alpha((f(x)*' f(y*z))*' f(z)) \\ &= \beta^\alpha((f(x)*' (f(y)*' f(z)))*' f(z)) \\ &\geq \min\{\beta^\alpha(f(x)), \beta^\alpha(f(y))\} \\ &= \min\{\mu^\alpha(x), \mu^\alpha(y)\}, \text{and the proof is completed.} \end{aligned}$$

□

Theorem 3.21. *Let $(X,*,0)$ and $(Y,*',0')$ be PU-algebras, $f : X \to Y$ be a homomorphism, μ^α be an α-fuzzy subset of X, β^α be the image of μ^α under f and $\mu^\alpha(x) = \beta^\alpha(f(x))$ for all $x \in X$. If μ^α is an α-fuzzy new-ideal of X, then β^α is an α-fuzzy new-ideal of Y.*

Proof. Since $0 \in f^{-1}(0')$, then $f^{-1}(0') \neq \phi$. It follows that
$\beta^\alpha(0') = \sup\limits_{t\in f^{-1}(0')} \mu^\alpha(t) = \mu^\alpha(0) \geq \mu^\alpha(x)$, for all $x \in X$. Thus $\beta^\alpha(0') = \sup\limits_{t\in f^{-1}(x')} \mu^\alpha(t)$ for all $x' \in Y$ Hence $\beta^\alpha(0') \geq \beta^\alpha(x')$ for all $x' \in Y$.
For any $x', y', z' \in Y$. If $f^{-1}(x') = \phi$ or $f^{-1}(y') = \phi$, then $\beta^\alpha(x') = 0$ or $\beta^\alpha(y') = 0$, it follows that $\min\{\beta^\alpha(x'), \beta^\alpha(y')\} = 0$ and hence

$$\beta^\alpha((x' *' (y' *' z')) *' z') \geq \min\{\beta^\alpha(x'), \beta^\alpha(y')\}.$$

If $f^{-1}(x') \neq \phi$ and $f^{-1}(y') \neq \phi$, let $x_0 \in f^{-1}(x'), y_0 \in f^{-1}(y')$, be such that $\mu^\alpha(x_0) = \sup\limits_{t\in f^{-1}(x')} \mu^\alpha(t)$ and $\mu^\alpha(y_0) = \sup\limits_{t\in f^{-1}(y')} \mu^\alpha(t)$. It follows by given and properties of PU-algebra that $\beta^\alpha((x'*'(y'*'z'))*'z') = \beta^\alpha((z'*'(y'*'z'))*'x') = \beta^\alpha((y'*'(z'*'z'))*'x') = \beta^\alpha((y'*'0')*'x') = \beta^\alpha((f(y_0)*'f(0))*'f(x_0)) = \beta^\alpha(f(y_0*0)*x_0)) = \mu^\alpha((y_0*0)*x_0) = \mu^\alpha((y_0*(z_0*z_0))*x_0) = \mu^\alpha((z_0*((z_0*z_0))*x_0) = \mu^\alpha((x_0*$

$(y_0 * z_0)) * z_0) \geq \min\{\mu^\alpha(x_0), \mu^\alpha(y_0)\} = \min\{\sup_{t\in f^{-1}(x')} \mu^\alpha(t), \sup_{t\in f^{-1}(y')} \mu^\alpha(t)\} = \min\{\beta^\alpha(x'), \beta^\alpha(y')\}$. Hence β^α is an α-fuzzy *new*-ideal of Y. □

Corollary 3.22. *Let $(X, *, 0)$ and $(y, *', 0')$ be PU-algebras, $f : X \to Y$ be a homomorphism, μ be a fuzzy subset of X, β be a fuzzy subset of Y defined by*

$$\beta(y) = \begin{cases} \sup_{x\in f^{-1}(y)} \mu(x) & if\ f^{-1}(y) \neq \phi \\ 0 & otherwise \end{cases},$$

and $\mu(x) = \beta(f(x))$ for all $x \in X$.
If μ is a fuzzy new-ideal of X, then β is a fuzzy new-ideal of Y.

4. Cartesian Product of α-Fuzzy *new*-ideal of PU-algebras

In this section, we introduce the concept of Cartesian product of an α-fuzzy *new*-ideal of a PU-algebra.

Definition 4.1. An α-fuzzy relation on any set S is an α-fuzzy subset $\mu^\alpha : S \times S \to [0, 1]$.

Definition 4.2. If μ^α is an α-fuzzy relation on a set S and β^α is an α-fuzzy subset of S, then μ^α is an α-fuzzy relation on β^α if $\mu^\alpha(x, y) \leq \min\{\beta^\alpha(x), \beta^\alpha(y)\}$, for all $x, y \in S$.

Definition 4.3. If β^α is an α-fuzzy subset of a set S, the strongest α-fuzzy relation on S that is an α-fuzzy relation on β^α is $\mu^\alpha_{\beta^\alpha}$ given by $\mu^\alpha_{\beta^\alpha}(x, y) = \min\{\beta^\alpha(x), \beta^\alpha(y)\}$ for all $x, y \in S$

Definition 4.4. We define the binary operation $*$ on the Cartesian product $X \times X$ as follows: $(x_1, x_2) * (y_1, y_2) = (x_1 * y_1, x_2 * y_2)$ for all $(x_1, x_2), (y_1, y_2) \in X \times X$.

Lemma 4.5. *If $(X, *, 0)$ is a PU-algebra, then $(X \times X, *, (0, 0))$ is a PU-algebra, where $(x_1, x_2) * (y_1, y_2) = (x_1 * y_1, x_2 * y_2)$ for all $(x_1, x_2), (y_1, y_2) \in X \times X$.*

Proof. Clear. □

Theorem 4.6. *Let β^α be an α-fuzzy subset of a PU-algebra X and $\mu^\alpha_{\beta^\alpha}$ be the strongest α-fuzzy relation on X, then β^α is an α-fuzzy new-ideal of X if and only if $\mu^\alpha_{\beta^\alpha}$ is an α-fuzzy new-ideal of $X \times X$.*

Proof. ($\Rightarrow$): Assume that β^α is an α-fuzzy *new*-ideal of X, we note from (F_1^α) that: $\mu^\alpha_{\beta^\alpha}(0, 0) = \min\{\beta^\alpha(0), \beta^\alpha(0)\} \geq \min\{\beta^\alpha(x), \beta^\alpha(y)\} = \mu^\alpha_{\beta^\alpha}(x, y)$ for all $x, y \in X$.
Now, for any $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, we have from ($F_2^\alpha$):

$$\begin{aligned}
&\mu^\alpha_{\beta^\alpha}(((x_1, x_2) * ((y_1, y_2) * (z_1, z_2))) * (z_1, z_2)) \\
&= \mu^\alpha_{\beta^\alpha}(((x_1, x_2) * (y_1 * z_1, y_2 * z_2)) * (z_1, z_2)) \\
&= \mu^\alpha_{\beta^\alpha}((x_1 * (y_1 * z_1), x_2 * (y_2 * z_2)) * (z_1, z_2)) \\
&= \mu^\alpha_{\beta^\alpha}((x_1 * (y_1 * z_1)) * z_1, (x_2 * (y_2 * z_2)) * z_2) \\
&= \min\{\beta^\alpha((x_1 * (y_1 * z_1)) * z_1), \beta^\alpha((x_2 * (y_2 * z_2)) * z_2)\} \\
&\geq \min\{\min\{\beta^\alpha(x_1), \beta^\alpha(y_1)\}, \min\{\beta^\alpha(x_2), \beta^\alpha(y_2)\}\} \\
&= \min\{\min\{\beta^\alpha(x_1), \beta^\alpha(x_2)\}, \min\{\beta^\alpha(y_1), \beta^\alpha(y_2)\}\} \\
&= \min\{\mu^\alpha_{\beta^\alpha}(x_1, x_2), \mu^\alpha_{\beta^\alpha}(y_1, y_2)\}.
\end{aligned}$$

Hence $\mu^{\alpha}_{\beta^{\alpha}}$ is an α-fuzzy *new*-ideal of$X \times X$.
($\Leftarrow$): For all $(x, x) \in X \times X$, we have $\mu^{\alpha}_{\beta^{\alpha}}(0, 0) = \min\{\beta^{\alpha}(0), \beta^{\alpha}(0)\} \geq \mu^{\alpha}_{\beta^{\alpha}}(x, x)$.
Then $\beta^{\alpha}(0) = \min\{\beta^{\alpha}(0), \beta^{\alpha}(0)\} \geq \min\{\beta^{\alpha}(x), \beta^{\alpha}(x)\} = \beta^{\alpha}(x)$ for all $x \in X$.
Now, for all $x, y, z \in X$, we have

$$\begin{aligned}
\beta^{\alpha}((x*(y*z))*z) &= \min\{\beta^{\alpha}((x*(y*z))*z), \beta^{\alpha}((x*(y*z))*z)\} \\
&= \mu^{\alpha}_{\beta^{\alpha}}((x*(y*z))*z, (x*(y*z))*z) \\
&= \mu^{\alpha}_{\beta^{\alpha}}((x*(y*z), x*(y*z))*(z, z)) \\
&= \mu^{\alpha}_{\beta^{\alpha}}(((x, x)((y*z), (y*z)))*(z, z)) \\
&= \mu^{\alpha}_{\beta^{\alpha}}(((x, x)*((y, y)*(z, z)))*(z, z)) \\
&\geq \min\{\mu^{\alpha}_{\beta^{\alpha}}(x, x), \mu^{\alpha}_{\beta^{\alpha}}(y, y)\} \\
&= \min\{\min\{\beta^{\alpha}(x), \beta^{\alpha}(x)\}, \min\{\beta^{\alpha}(y), \beta^{\alpha}(y)\}\} \\
&= \min\{\beta^{\alpha}(x), \beta^{\alpha}(y)\}.
\end{aligned}$$

Hence β^{α} is an α-fuzzy *new*-ideal of X. □

Definition 4.7. Let μ and δ be the fuzzy subsets in X. The Cartesian product $\mu \times \delta : X \times X \to [0, 1]$ is defined by $(\mu \times \delta)(x, y) = \min\{\mu(x), \delta(y)\}$ for all $x, y \in X$.

Definition 4.8. Let μ^{α} and δ^{α} be the α-fuzzy subsets in X. The Cartesian product $\mu^{\alpha} \times \delta^{\alpha} : X \times X \to [0, 1]$ is defined by $(\mu^{\alpha} \times \delta^{\alpha})(x, y) = \min\{\mu^{\alpha}(x), \delta^{\alpha}(y)\}$, for all $x, y \in X$.

Theorem 4.9. *If μ^{α} and δ^{α} are α-fuzzy new-ideals in a PU-algebra X, then $\mu^{\alpha} \times \delta^{\alpha}$ is an α-fuzzy new-ideal in $X \times X$.*

Proof. $(\mu^{\alpha} \times \delta^{\alpha})(0, 0) = \min\{\mu^{\alpha}(0), \delta^{\alpha}(0)\} \geq \min\{\mu^{\alpha}(x_1), \delta^{\alpha}(x_2)\} = (\mu^{\alpha} \times \delta^{\alpha})(x_1, x_2)$ for all $(x_1, x_2) \in X \times X$. Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$. then we have that

$$\begin{aligned}
&(\mu^{\alpha} \times \delta^{\alpha})(((x_1, x_2)*((y_1, y_2)*(z_1, z_2)))*(z_1, z_2)) \\
&= (\mu^{\alpha} \times \delta^{\alpha})(((x_1, x_2)*(y_1*z_1, y_2*z_2))*(z_1, z_2)) \\
&= (\mu^{\alpha} \times \delta^{\alpha})((x_1*(y_1*z_1), x_2*(y_2*z_2))*(z_1, z_2)) \\
&= (\mu^{\alpha} \times \delta^{\alpha})((x_1*(y_1*z_1))*z_1, (x_2*(y_2*z_2))*z_2) \\
&= \min\{\mu^{\alpha}(x_1*(y_1*z_1))*z_1), \delta^{\alpha}(x_2*(y_2*z_2))*z_2)\} \\
&\geq \min\{\min\{\mu^{\alpha}(x_1), \mu^{\alpha}(y_1)\}, \min\{\delta^{\alpha}(x_2), \delta^{\alpha}(y_2)\}\} \\
&= \min\{(\mu^{\alpha} \times \delta^{\alpha})(x_1, x_2), (\mu^{\alpha} \times \delta^{\alpha})(y_1, y_2)\}.
\end{aligned}$$

Therefore $\mu^{\alpha} \times \delta^{\alpha}$ is an α-fuzzy *new*-ideal in $X \times X$. □

5. Conclusions

In the present paper, we have introduced the concept of α-fuzzy *new*-ideal of PU-algebras and investigated some of their useful properties. We believe that these results are very useful in developing algebraic structures also these definitions and main results can be similarly extended to some other algebraic structure such as PS-algebras , Q-algebras, SU-algebras , IS-algebras, β algebras and semirings. It is our hope that this work would other foundations for further study of the theory of BCI-algebras. In our future study of fuzzy structure of PU-algebras, may be the following topics should be considered:
(1) To establish the interval valued, bipolar and intuitionistic α-fuzzy *new*-ideal in PU-algebras.

(2) To establish $\tilde{\tau}$-interval-valued α-fuzzy *new*-ideal of PU-algebras.
(3) To consider the structure of $(\tilde{\tau}, \tilde{\rho})$-interval-valued α-fuzzy *new*-ideal of PU-algebras.
(4) To get more results in $\tilde{\tau}$-cubic α-fuzzy *new*-ideal of PU-algebras and it's application.

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Algorithms for PU-algebra

Input (X: set with 0 element, $*$: Binary operation)
Output ("X is a PU-algebra or not")
If $X = \phi$ then;
Go to (1.)
End if
If $0 \notin X$ then go to (1.);
End If
Stop: = false
$i = 1$;
While $i \leq |X|$ and not (Stop) do
If $0 * x_i \neq x_i$, then
Stop: = true
End if
$j = 1$;
While $j \leq |X|$, and not (Stop) do
$k = 1$;
While $k \leq |X|$ and not (stop) do
If $(x_i * x_k) * (x_j * x_k) \neq x_j * x_i$, then
Stop: = true
End if
End while
End if
End while
If stop then
Output ("X is a PU-algebra")
Else
(1.) Output ("X is not a PU-algebra")
End if
End.

Algorithms for PU-ideal in PU-algebra

Input (X: PU-algebra, I: subset of X)
Output ("I is a PU-ideal of X or not")
If $I = \phi$ then
Go to (1.);

End if
If $0 \notin I$ then
Go to (1.);
End if
Stop: = false
$i = 1$;
While $i \leq |X|$ and not (stop) do
$j = 1$
While $j \leq |X|$ and not (stop)
$k = 1$
While $k \leq |X|$ and not (stop) do
If $x_j * x_i \in I$, and $x_i * x_k \in I$ then
If $x_j * x_k \notin I$ then
Stop: = false
End if
End while
End while
End while
If stop then
Output ("I is a PU-ideal of X")
Else
(1.) Output ("I is not ("I is a PU-ideal of X")
End if
End.
Algorithm for fuzzy subsets
Input (X: PU-algebra, $A : X \to [0, 1]$);
Output ("A is a fuzzy subset of X or not")
Begin
Stop: =false;
$i := 1$;
While $i \leq |X|$ and not (Stop) do
If $(A(x_i) < 0)$ or $(A(x_i) > 1$ then
Stop: = true;
End If
End While
If Stop then
Output ("A is a fuzzy subset of X")

Else
Output ("A is not a fuzzy subset of X")

End If
End

Algorithm for fuzzy subsetsAlgorithm fuzzy new-ideal
Input (X: PU-algebra, I: subset of X);

Output ("I is an *new*-ideal of X or not");
Begin
If $I = \phi$ then go to (1.);
End If
If $0 \notin I$ then go to (1.);
End If
Stop: =false;
$i := 1$;
While $i \leq |X|$ and not (Stop) do
$j := 1$
While $j \leq |X|$ and not (Stop) do
$k := 1$
While $k \leq |X|$ and not (Stop) do
If $x_i, x_j \in I$ and $x_k \in X$, then
If $(x_i * (x_j * x_k)) * x_k \notin I$ then

Stop: = true;
End If
End If
End While
End While
End While
If Stop then
Output ("$I$is is *new*-ideal of X")
Else (1.) Output ("I is not is *new*-ideal of X")

End If
End

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