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On fuzzy GID spaces

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Abstract. In this paper, the concepts of fuzzy GID spaces are introduced and several characterizations of fuzzy GID spaces are studied. The conditions under which fuzzy topological spaces become fuzzy GID spaces are investigated.

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1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by L. A. Zadeh [21] in 1965. This concept provides a natural foundation for treating mathematically the fuzzy phenomena, which exist pervasively in our real world. The first notion of fuzzy topological spaces had been defined by C. L. Chang [4] in 1968 and this paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then, much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed.

A. K. Mishra [7] introduced the concepts of P-spaces as a generalization of ω_α-additive spaces of Sikorski [8] and L. W. Cohen and C. Goffman [5]. The concept of P-spaces in fuzzy setting was introduced by G. Balasubramanian in [3]. The concept of almost P-spaces in classical topology was introduced by A. I. Veksler [19] and was also studied further by R. Levy [6]. The concept of almost GP-spaces in classical topology was introduced by M. R. Ahmadi Zand [1] in 2010. The concept of almost GP-spaces in fuzzy setting was introduced by the authors in [10]. In this paper, the concepts of fuzzy GID spaces are introduced and several characterizations of fuzzy GID spaces are studied. The conditions under which fuzzy topological spaces become fuzzy GID spaces, are also investigated in this paper. The condition under which a fuzzy Baire space becomes a fuzzy Volterra space is also obtained in this paper.

Some results concerning functions that preserve fuzzy GID spaces in the context of images and pre-images are obtained. Several examples are given to illustrate the concepts introduced in this paper.

2. Preliminaries

Now we introduce some basic notions and results used in this sequel. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang [4]. A fuzzy set λ on a set X is a function from X to [0,1]. That is, $\lambda : X \longrightarrow [0, 1]$.

Definition 2.1. Let λ and μ be fuzzy sets in X. Then, for all $x \in X$,

(i) $\lambda = \mu \;\Leftrightarrow\; \lambda(x) = \mu(x)$
(ii) $\lambda \leq \mu \;\Leftrightarrow\; \lambda(x) \leq \mu(x)$
(iii) $\psi = \lambda \vee \mu \Leftrightarrow \psi(x) = max\{\lambda(x),\ \mu(x)\}$
(iv) $\delta = \lambda \wedge \mu \Leftrightarrow \delta(x) = min\{\lambda(x),\ \mu(x)\}$
(v) $\eta = \lambda^c \;\Leftrightarrow \eta(x) = 1 - \lambda(x)$.

For a family $\{\lambda_i / i \in I\}$ of fuzzy sets in (X, T), the union $\psi = \vee_i \lambda_i$ and intersection $\delta = \wedge_i \lambda_i$ are defined respectively as $\psi(x) = sup_i\{\lambda_i(x), x \in X\}$ and $\delta(x) = inf_i\{\lambda_i(x), x \in X\}$.

Definition 2.2. Let (X, T) be a fuzzy topological space. For a fuzzy set λ of X, the interior and the closure are defined respectively as

(i) $int(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$
(ii) $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$.

Lemma 2.3 ([2])**.** *For a fuzzy set λ of a fuzzy topological space X,*

(i) $1 - int(\lambda) = cl(1 - \lambda)$
(ii) $1 - cl(\lambda) = int(1 - \lambda)$.

Definition 2.4 ([15])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy dense if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$. That is, $cl(\lambda) = 1$.

Definition 2.5 ([15])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is, $intcl(\lambda) = 0$.

Definition 2.6 ([3])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy G_δ-set in (X, T) if $\lambda = \wedge_{i=1}^{\infty} \lambda_i \;\; where \;\; \lambda_i \in T, \;\; for \;\; i \in I$.

Definition 2.7 ([3])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy F_σ-set in (X, T) if $\lambda = \vee_{i=1}^{\infty} \lambda_i \;\; where \;\; 1 - \lambda_i \in T, \;\; for \;\; i \in I$.

Definition 2.8 ([16])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy σ-nowhere dense if λ is a fuzzy F_σ-set in (X, T) such that $int(\lambda) = 0$.

Definition 2.9 ([15])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy first category set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy second category. If λ is a fuzzy first category set in (X, T), then $1 - \lambda$ is called a fuzzy residual set in (X, T)[12].

Lemma 2.10 ([2]). *For a family $\mathscr{A} = \{\lambda_\alpha\}$ of fuzzy sets of a fuzzy space X, $\vee\big(cl(\lambda_\alpha)\big) \leq cl\big(\vee(\lambda_\alpha)\big)$. In case $\mathscr{A}$ is a finite set, $\vee\big(cl(\lambda_\alpha)\big) = cl\big(\vee(\lambda_\alpha)\big)$. Also $\vee\big(int(\lambda_\alpha)\big) \leq int\big(\vee(\lambda_\alpha)\big)$.*

Definition 2.11 ([3]). A fuzzy topological space (X, T) is called a fuzzy submaximal space if for each fuzzy set λ in (X, T) such that $cl(\lambda) = 1$, then $\lambda \in T$ in (X, T).

Definition 2.12 ([14]). A fuzzy topological space (X, T) is called a fuzzy P-space if countable intersection of fuzzy open sets in (X, T) is fuzzy open. That is, every non-zero fuzzy G_δ-set in (X, T) is fuzzy open in (X, T).

Definition 2.13 ([9]). A fuzzy topological space (X, T) is called a fuzzy almost P-space if for every non-zero fuzzy G_δ-set λ in (X, T), $int(\lambda) \neq 0$ in (X, T).

Definition 2.14 ([10]). A fuzzy topological space (X, T) is called a fuzzy almost GP-space if $int(\lambda) \neq 0$, for each non-zero fuzzy dense and fuzzy G_δ-set λ in (X, T).

Definition 2.15 ([2]). A fuzzy set λ in a fuzzy topological space (X, T) is called

(i) a fuzzy semi-open set if $\lambda \leq clint(\lambda)$;
(ii) a fuzzy semi-closed set if $intcl(\lambda) \leq \lambda$.

Lemma 2.16 ([2]). *Let $f : (X, T) \to (Y, S)$ be a mapping and $\{\lambda_\alpha\}$ be a family of fuzzy sets of Y. Then*

(a) $f^{-1}(\cup_\alpha \lambda_j) = \cup_\alpha f^{-1}(\lambda_j)$,
(b) $f^{-1}(\cap_\alpha \lambda_j) = \cap_\alpha f^{-1}(\lambda_j)$.

Lemma 2.17 ([4]). *Let $f : (X, T) \to (Y, S)$ be a mapping. For fuzzy sets λ and μ of (X, T) and (Y, S) respectively, the following statements hold:*

1. $ff^{-1}(\mu) \leq \mu$;
2. $f^{-1}f(\lambda) \geq \lambda$;
3. $f(1 - \lambda) \geq 1 - f(\lambda)$;
4. $f^{-1}(1 - \mu) = 1 - f^{-1}(\mu)$;
5. *If f is one-to-one, then $f^{-1}f(\lambda) = \lambda$;*
6. *If f is onto, then $ff^{-1}(\mu) = \mu$;*
7. *If f is one-to-one and onto, then $f(1 - \lambda) = 1 - f(\lambda)$.*

3. Fuzzy GID spaces

Motivated by the classical concept introduced in [1],we define the class of fuzzy GID spaces as follows:

Definition 3.1. A fuzzy topological space (X, T) is called a fuzzy GID space if for each fuzzy dense and fuzzy G_δ-set λ in (X, T), $clint(\lambda) = 1$ in (X, T).

Example 3.2. Let $X = \{a, b, c\}$. The fuzzy sets λ, μ and γ are defined on X as follows :

$\lambda : X \to [0, 1]$ is defined as $\lambda(a) = 0.5$; $\lambda(b) = 0.6$; $\lambda(c) = 0.4$
$\mu : X \to [0, 1]$ is defined as $\mu(a) = 0.4$; $\mu(b) = 0.7$; $\mu(c) = 0.5$
$\gamma : X \to [0, 1]$ is defined as $\gamma(a) = 0.5$; $\gamma(b) = 0.8$; $\gamma(c) = 0.6$

Let $T = \{0, \lambda, \mu, \gamma, \lambda \vee \mu, \lambda \wedge \mu, 1\}$. Clearly T is a fuzzy topology on X. Now $\lambda \wedge \mu = \lambda \wedge \gamma \wedge (\lambda \vee \mu)$ and $\mu = \mu \wedge (\lambda \vee \mu) \wedge \gamma$. Then $\lambda \wedge \mu$ and μ are fuzzy G_δ-sets in (X, T). Also $cl(\lambda \wedge \mu) = 1$ and $cl(\mu) = 1$ and $clint(\lambda \wedge \mu) = cl(\lambda \wedge \mu) = 1$ and $clint(\mu) = cl(\mu) = 1$. Hence (X, T) is a fuzzy GID space.

Proposition 3.3. *Let (X, T) be a fuzzy topological space. Then the following are equivalent :*

(i) *(X, T) is a fuzzy GID space.*
(ii) *Each fuzzy dense and fuzzy G_δ-set in (X, T), is fuzzy semi-open in (X, T).*

Proof. (**i**) $\Longrightarrow$ (**ii**). Let (X, T) be a fuzzy GID space and λ be a fuzzy dense and fuzzy G_δ-set in (X, T). Since (X, T) is a fuzzy GID space, $clint(\lambda) = 1$, for the fuzzy dense and fuzzy G_δ-set λ in (X, T). Then, we have $\lambda \leq clint(\lambda)$ and therefore λ is a fuzzy semi-open set in (X, T).

(**ii**) $\Longrightarrow$ (**i**). Let λ be a fuzzy dense and fuzzy G_δ-set in (X, T) such that $\lambda \leq clint(\lambda)$. We know that $\lambda \leq cl(\lambda)$ for any fuzzy set λ in (X, T). Then we have $cl(\lambda) \leq clint(\lambda)$. Since λ is a fuzzy dense set in (X, T), $cl(\lambda) = 1$. This implies that $1 \leq clint(\lambda)$. That is, $clint(\lambda) = 1$ and therefore (X, T) is a fuzzy GID space.

Proposition 3.4. *If λ is a fuzzy σ- nowhere dense set in a fuzzy GID space (X, T), then λ is a fuzzy nowhere dense set in (X, T).*

Proof. Let λ be a fuzzy σ-nowhere dense set in (X, T). Then, λ is a fuzzy F_σ-set in (X, T) such that $int(\lambda) = 0$. This implies that $1 - \lambda$ is a fuzzy G_δ-set in (X, T) and $cl(1 - \lambda) = 1 - int(\lambda) = 1 - 0 = 1$. Since (X, T) is a GID space, $clint(1 - \lambda) = 1$ for the fuzzy dense and fuzzy G_δ-set $1 - \lambda$ in (X, T). Then $1 - clint(1 - \lambda) = 0$. Hence $1 - (1 - intcl(\lambda)) = 0$, implies that $intcl(\lambda) = 0$ and therefore λ is a fuzzy nowhere dense set in (X, T).

Proposition 3.5. *If λ is a fuzzy σ-nowhere dense set in a fuzzy GID space (X, T), then λ is a fuzzy semi-closed set in (X, T).*

Proof. Let λ be a fuzzy σ-nowhere dense set in the fuzzy GID space (X, T). Then, by proposition 3.4, λ is a fuzzy nowhere dense set in (X, T) and hence $intcl(\lambda) = 0$. This implies that $intcl(\lambda) \leq \lambda$ and therefore λ is a fuzzy semi-closed set in (X, T).

Theorem 3.6 ([13])**.** *If λ is a fuzzy dense and fuzzy G_δ-set in a fuzzy topological space (X, T), then $1 - \lambda$ is a fuzzy first category set in (X, T).*

Proposition 3.7. *If (X, T) is a fuzzy GID space, then the fuzzy first category set formed from the fuzzy dense and fuzzy G_δ-set in (X, T), is a fuzzy nowhere dense set in (X, T).*

Proof. Let λ be a fuzzy dense and fuzzy G_δ-set in (X, T). By theorem 3.6, $1 - \lambda$ is a fuzzy first category set in (X, T). Since (X, T) is a fuzzy GID space, $clint(\lambda) = 1$, for the fuzzy dense and fuzzy G_δ-set λ in (X, T). Then, $1 - clint(\lambda) = 0$ and hence $intcl(1 - \lambda) = 0$. Hence the fuzzy first category set $1 - \lambda$ is a fuzzy nowhere dense set in (X, T).

Proposition 3.8. *If each fuzzy F_σ-set is a fuzzy nowhere dense set in a fuzzy topological space (X, T), then (X, T) is a fuzzy GID space.*

Proof Let λ be a fuzzy dense and fuzzy G_δ-set in (X, T). Then $1 - \lambda$ is a F_σ-set in (X, T). By hypothesis, $1 - \lambda$ is a fuzzy nowhere dense set in (X, T) and hence

$intcl(1-\lambda) = 0$. This implies that $1 - clint(\lambda) = 0$. Thus, for the fuzzy dense and fuzzy G_δ-set λ in (X, T), we have $clint(\lambda) = 1$. Hence (X, T) is a fuzzy GID space.

Proposition 3.9. *If $clint(\wedge_{i=1}^{\infty}(\lambda_i)) = 1$, where (λ_i)'s are fuzzy dense and fuzzy G_δ-sets in a fuzzy topological space (X, T), then (X, T) is a fuzzy GID space.*

Proof. Let (λ_i)'s ($i = 1$ *to* ∞) be fuzzy dense and fuzzy G_δ-sets in (X, T) such that $clint(\wedge_{i=1}^{\infty}(\lambda_i)) = 1$. Then, $clint(\wedge_{i=1}^{\infty}(\lambda_i)) \leq \wedge_{i=1}^{\infty} clint(\lambda_i)$ implies that $1 \leq \wedge_{i=1}^{\infty} clint(\lambda_i)$. That is, $\wedge_{i=1}^{\infty} clint(\lambda_i) = 1$. This will imply that $clint(\lambda_i) = 1$. Hence, for the fuzzy dense and fuzzy G_δ-sets λ_i in (X, T), we have $clint(\lambda_i) = 1$. Therefore (X, T) is a fuzzy GID space.

Proposition 3.10. *If $intcl(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy σ-nowhere dense sets in a fuzzy topological space (X, T), then (X, T) is a fuzzy GID space.*

Proof. Let (λ_i)'s ($i = 1$ *to* ∞) be fuzzy σ-nowhere dense sets in (X, T) such that $intcl(\vee_{i=1}^{\infty}(\lambda_i)) = 0$. Then, $1 - intcl(\vee_{i=1}^{\infty}(\lambda_i)) = 1$. This implies that $clint(\wedge_{i=1}^{\infty}(1 - \lambda_i)) = 1$. Since (λ_i)'s are fuzzy σ-nowhere dense sets in (X, T), (λ_i)'s are fuzzy F_σ-sets in (X, T) such that $int(\lambda_i) = 0$ and hence $(1 - \lambda_i)$'s are fuzzy G_δ-sets in (X, T) and $cl(1 - \lambda_i) = 1 - int(\lambda_i) = 1 - 0 = 1$. Hence we have $clint(\wedge_{i=1}^{\infty}(1 - \lambda_i)) = 1$ where $(1 - \lambda_i)$'s are fuzzy dense and fuzzy G_δ-sets in (X, T). Then, by proposition 3.9, (X, T) is a fuzzy GID space.

Proposition 3.11. *If a fuzzy topological space (X, T) is a fuzzy GID space, then $cl(1 - \lambda) \neq 1$, for a fuzzy dense and fuzzy G_δ-set λ in (X, T).*

Proof. Let λ be a fuzzy dense and fuzzy G_δ-set in a fuzzy GID space (X, T). Since (X, T) is a fuzzy GID space, $clint(\lambda) = 1$, in (X, T). Then, we have $int(\lambda) \neq 0$ in (X, T), [otherwise, $int(\lambda) = 0$, implies that $clint(\lambda) = cl(0) = 0 \neq 1$, a contradiction]. Now $cl(1 - \lambda) = 1 - int(\lambda) \neq 1$. Hence, for the fuzzy dense set λ in (X, T), $cl(1 - \lambda) \neq 1$.

4. Fuzzy GID spaces and other fuzzy topological spaces

Proposition 4.1. *If (X, T) is a fuzzy P-space, then (X, T) is a fuzzy GID space.*

Proof. Let λ be a fuzzy dense and fuzzy G_δ-set in a fuzzy P-space (X, T). Since (X, T) is a fuzzy P-space, the fuzzy G_δ-set λ is a fuzzy open set in (X, T) and hence λ is a fuzzy semi-open set in (X, T). Thus, the fuzzy dense and fuzzy G_δ-set λ in (X, T), is fuzzy semi-open in (X, T). Hence, by proposition 3.3, (X, T) is a fuzzy GID space.

Remark 4.2. The converse of the above proposition need not be true. That is, a fuzzy GID space need not be a fuzzy P-space. For consider the following example :

Example 4.3. Let $X = \{a, b, c, d\}$. The fuzzy sets λ, μ and γ are defined on X as follows :

$\lambda : X \to [0, 1]$ is defined as $\lambda(a) = 0.6$; $\lambda(b) = 0.5$; $\lambda(c) = 0.4$; $\lambda(d) = 0.7$
$\mu : X \to [0, 1]$ is defined as $\mu(a) = 0.5$; $\mu(b) = 0.7$; $\mu(c) = 0.6$; $\mu(d) = 0.4$
$\gamma : X \to [0, 1]$ is defined as $\gamma(a) = 0.7$; $\gamma(b) = 0.4$; $\gamma(c) = 0.7$; $\gamma(d) = 0.6$

Let $T = \{0, \lambda, \mu, \gamma, \lambda \vee \gamma, \mu \vee \gamma, \lambda \wedge \mu, \lambda \wedge \gamma, \mu \wedge \gamma, \lambda \vee \mu, \lambda \vee (\mu \wedge \gamma), \mu \vee (\lambda \wedge \gamma), \gamma \vee (\lambda \wedge \mu), \lambda \wedge (\mu \vee \gamma), \mu \wedge (\lambda \vee \gamma), \gamma \wedge (\lambda \vee \mu), \lambda \vee \mu \vee \gamma, \lambda \wedge \mu \wedge \gamma, 1\}$. Clearly T is a fuzzy topology on X. Now $\alpha = [\lambda \vee \mu] \wedge [\lambda \vee \gamma] \wedge [\mu \vee \gamma] \wedge [\lambda \vee (\mu \wedge \gamma)] \wedge [\mu \vee (\lambda \wedge \gamma)] \wedge [\gamma \vee (\lambda \wedge \mu)]$, is a fuzzy G_δ-set in (X, T). Also $cl(\alpha) = 1$ and $clint(\alpha) = cl(\gamma \wedge (\lambda \vee \mu)) = 1$. Hence (X, T) is a fuzzy GID space. But (X, T) is not a fuzzy P-space, since the fuzzy G_δ-set α is not a fuzzy open set in (X, T).

Proposition 4.4. *If the fuzzy topological space (X, T) is a fuzzy submaximal space, then (X, T) is a fuzzy GID space.*

Proof. Let λ be a fuzzy dense and fuzzy G_δ-set in a fuzzy submaximal space (X, T). Since (X, T) is a fuzzy submaximal space, the fuzzy dense set λ in (X, T), is fuzzy open in (X, T) and hence λ is a fuzzy semi-open set in (X, T). Thus, the fuzzy dense and fuzzy G_δ-set λ in (X, T), is fuzzy semi-open in (X, T). Hence, by proposition 3.3, (X, T) is a fuzzy GID space.

Remark 4.5. The converse of the above proposition need not be true. That is, a fuzzy GID space need not be a fuzzy submaximal space. For, in example 4.3, the fuzzy set α is fuzzy dense but not fuzzy open in (X, T).

Definition 4.6 ([12])**.** Let (X, T) be a fuzzy topological space. Then (X, T) is called a fuzzy Baire space if $int\big(\vee_{i=1}^{\infty} (\lambda_i)\big) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T).

Definition 4.7 ([16])**.** Let (X, T) be a fuzzy topological space. Then (X, T) is called a fuzzy σ-Baire space if $int\big(\vee_{i=1}^{\infty} (\lambda_i)\big) = 0$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X, T).

Definition 4.8 ([18])**.** A fuzzy topological space (X, T) is called a fuzzy Volterra space if $cl\big(\wedge_{i=1}^{N} (\lambda_i)\big) = 1$, where (λ_i)'s are fuzzy dense and fuzzy G_δ-sets in (X, T).

Definition 4.9 ([17])**.** A fuzzy topological space (X, T) is said to be a fuzzy strongly irresolvable space if $clint(\lambda) = 1$ for each fuzzy dense set λ in (X, T).

Theorem 4.10 ([12])**.** *Let (X, T) be a fuzzy topological space. Then the following are equivalent :*

1. *(X, T) is a fuzzy Baire space.*
2. *$int(\lambda) = 0$ for every fuzzy first category set λ in (X, T).*
3. *$cl(\mu) = 1$ for every fuzzy residual set μ in (X, T).*

Theorem 4.11 ([11])**.** *If λ is a fuzzy residual set in a fuzzy submaximal space (X, T), then λ is a fuzzy G_δ-set in (X, T).*

Proposition 4.12. *If (X, T) is a fuzzy GID space, then (X, T) is a fuzzy almost GP-space.*

Proof. Let λ be a fuzzy dense and fuzzy G_δ-set in a fuzzy GID space (X, T). Since (X, T) is a fuzzy GID space, $clint(\lambda) = 1$, in (X, T). Then, $int(\lambda) \neq 0$ in (X, T). Hence, for the fuzzy dense and fuzzy G_δ-set λ in (X, T), we have $int(\lambda) \neq 0$ and therefore (X, T) is a fuzzy almost GP-space.

Proposition 4.13. *If a fuzzy topological space (X, T) is a fuzzy GID space, then (X, T) is a fuzzy Baire space.*

Proof. Let λ be a fuzzy dense and fuzzy G_δ-set in a fuzzy GID space (X, T). Then, we have $clint(\lambda) = 1$ in (X, T). This implies that $1 - clint(\lambda) = 0$ and hence $intcl(1 - \lambda) = 0$. But $int(1 - \lambda) \leq intcl(1 - \lambda)$. This implies that $int(1 - \lambda) \leq 0$. That is, $int(1 - \lambda) = 0$. Since λ is a fuzzy dense and fuzzy G_δ-set in (X, T), by theorem 3.6, $1 - \lambda$ is a fuzzy first category set in (X, T) and hence $1 - \lambda = \vee_{j=1}^{\infty}(\mu_j)$, where (μ_j)'s are fuzzy nowhere dense sets in (X, T). Now $int(1 - \lambda) = 0$, implies that $int\big(\vee_{j=1}^{\infty}(\mu_j)\big) = 0$. Therefore (X, T) is a fuzzy Baire space.

Proposition 4.14. *If the fuzzy topological space (X, T) is a fuzzy Baire and fuzzy submaximal space, then (X, T) is a fuzzy GID space.*

Proof. Let (X, T) be a fuzzy Baire and fuzzy submaximal space and λ be a fuzzy residual set in (X, T). Since (X, T) is a fuzzy Baire space, by theorem 4.10, $cl(\lambda) = 1$, in (X, T). Also, since λ is a fuzzy residual set in the fuzzy submaximal space (X, T), by theorem 4.11, λ is a fuzzy G_δ-set in (X, T). Also, since (X, T) is a fuzzy submaximal space, the fuzzy dense set λ in (X, T), is fuzzy open in (X, T) and hence λ is a fuzzy semi-open set in (X, T). Thus, the fuzzy dense and fuzzy G_δ-set λ in (X, T), is fuzzy semi-open in (X, T). Hence, by proposition 3.3, (X, T) is a fuzzy GID space.

Proposition 4.15. *If a fuzzy topological space (X, T) is a fuzzy strongly irresolvable space, then (X, T) is a fuzzy GID space.*

Proof. Let λ be a non-zero fuzzy dense and fuzzy G_δ-set in (X, T). Since (X, T) is a fuzzy strongly irresolvable space, for the fuzzy dense set λ in (X, T), we have $clint(\lambda) = 1$, in (X, T). Thus, for the fuzzy dense and fuzzy G_δ-set λ in (X, T), we have $clint(\lambda) = 1$. Hence (X, T) is a fuzzy GID space.

Proposition 4.16. *If each fuzzy G_δ-set is fuzzy dense in a fuzzy GID space, then (X, T) is a fuzzy almost P-space.*

Proof. Let λ be a fuzzy G_δ-set in a fuzzy GID space such that $cl(\lambda) = 1$. Then the fuzzy set λ is a fuzzy dense and fuzzy G_δ-set in (X, T). Since (X, T) is a fuzzy GID space, $clint(\lambda) = 1$, in (X, T). Then, $int(\lambda) \neq 0$ in (X, T), [otherwise, $int(\lambda) = 0$, implies that $clint(\lambda) = cl(0) \neq 1$, a contradiction]. Hence, for the fuzzy G_δ-set λ in (X, T), $int(\lambda) \neq 0$ and therefore (X, T) is a fuzzy almost P-space.

Proposition 4.17. *If a fuzzy topological space (X, T) is a fuzzy Baire and fuzzy P-space, then (X, T) is a fuzzy GID space.*

Proof. Let λ be a fuzzy dense and fuzzy G_δ-set in (X, T). Then, by theorem 3.6, $1 - \lambda$ is a fuzzy first category set in (X, T). Since (X, T) is a fuzzy Baire space, $int(1 - \lambda) = 0$. Then we have $1 - cl(\lambda) = 0$ and hence $cl(\lambda) = 1$. Since (X, T) is a fuzzy P-space, the fuzzy G_δ-set λ is fuzzy open in (X, T). Hence $int(\lambda) = \lambda$. Then $cl(\lambda) = 1$ implies that $clint(\lambda) = 1$. Therefore (X, T) is a fuzzy GID space.

Proposition 4.18. *If the fuzzy topological space (X, T) is a fuzzy Baire and fuzzy GID space, then (X, T) is a fuzzy Volterra space.*

Proof. Let (X, T) be a fuzzy Baire and fuzzy GID space and (λ_i)'s $(i = 1\ to\ N)$ be fuzzy dense and fuzzy G_δ-sets in (X, T). Since (X, T) is a fuzzy GID space, we have

$clint(\lambda_i) = 1$, in (X, T). Then, $1 - clint(\lambda_i) = 0$. This implies that $intcl(1 - \lambda_i) = 0$ and hence $(1 - \lambda_i)$'s are fuzzy nowhere dense sets in (X, T). Let (μ_i)'s $(i = 1\ to\ \infty)$ be fuzzy nowhere dense sets in (X, T) in which the first N fuzzy nowhere dense sets be $(1 - \lambda_i)$'s. Now $int(\vee_{i=1}^{N}(1 - \lambda_i)) \leq int(\vee_{i=1}^{\infty}(\mu_i))$. Since (X, T) is a fuzzy Baire space, $int(\vee_{i=1}^{\infty}(\mu_i)) = 0$. Then, we have $int(\vee_{i=1}^{N}(1 - \lambda_i)) \leq 0$. That is, $int(\vee_{i=1}^{N}(1 - \lambda_i)) = 0$. This implies that $int(1 - \wedge_{i=1}^{N}(\lambda_i)) = 0$ and hence we have $cl(\wedge_{i=1}^{N}(\lambda_i)) = 1$, where (λ_i)'s are fuzzy dense and fuzzy G_δ-sets in (X, T). Therefore (X, T) is a fuzzy Volterra space.

Proposition 4.19. *If the fuzzy topological space (X, T) is a fuzzy σ-Baire and fuzzy GID space, then (X, T) is a fuzzy Baire space.*

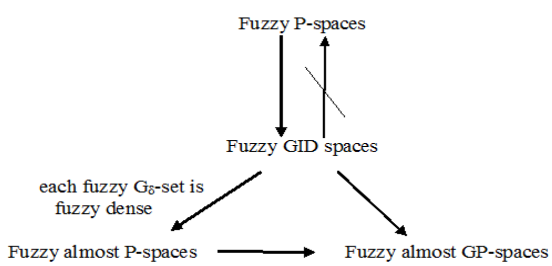
Proof. Let (X, T) be a fuzzy σ-Baire and fuzzy GID space. Since (X, T) is a fuzzy σ-Baire space, $int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where the fuzzy sets (λ_i)'s are fuzzy σ-nowhere dense sets in (X, T). Since (X, T) is a fuzzy GID space, by proposition 3.4, the fuzzy σ-nowhere dense sets (λ_i)'s are fuzzy nowhere dense sets in (X, T). Hence we have $int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Therefore (X, T) is a fuzzy Baire space.

Proposition 4.20. *If the fuzzy topological space (X, T) is a fuzzy σ-Baire and fuzzy GID space, then (X, T) is a fuzzy Volterra space.*

Proof. The proof follows from propositions 4.19 and 4.18.

Theorem 4.21. [10] *If the fuzzy topological space (X, T) is a fuzzy almost P-space, then (X, T) is a fuzzy almost GP-space.*

Remark 4.22. From theorem 4.21, propositions 4.1, 4.12 and 4.16, the relationship among the classes of fuzzy P-spaces, fuzzy almost P-spaces and fuzzy almost GP-spaces and fuzzy GID spaces are summarized as follows :



5. Fuzzy GID spaces and functions

Theorem 5.1 ([20])**.** *Let $f : (X, T) \to (Y, S)$ be a fuzzy open function. Then for every fuzzy set β in (Y, S), $f^{-1}(cl(\beta)) \leq cl(f^{-1}(\beta))$.*

Theorem 5.2 ([17])**.** *Let the function $f : (X, T) \to (Y, S)$, from a fuzzy topological space (X, T) into another fuzzy topological space (Y, S), be fuzzy continuous, fuzzy open, 1-1 and onto function. Then, for any fuzzy set λ in (X, T), λ is a fuzzy nowhere dense set in (X, T) if and only if $f(\lambda)$ is a fuzzy nowhere dense set in (Y, S).*

Let f be a function from the fuzzy topological space (X, T) to the fuzzy topological space (Y, S). Under what conditions on "f"may we assert that if (X, T) is a fuzzy GID space, then (Y, S) is a fuzzy GID space. The following propositions establish the desired conditions.

Proposition 5.3. *If the function $f : (X, T) \to (Y, S)$ from a fuzzy topological space (X, T) onto another fuzzy topological space (Y, S) is a fuzzy continuous and fuzzy open function, 1-1 and if (X, T) is a fuzzy GID space, then (Y, S) is a fuzzy GID space.*

Proof. Let λ be a fuzzy dense and fuzzy G_δ-set in (Y, S). Then $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$ where $\lambda_i \in S$ and $cl(\lambda) = 1$ in (Y, S). Since f is an fuzzy open function, by theorem, 5.1, we have $f^{-1}(cl(\lambda)) \leq cl(f^{-1}(\lambda))$. Then, $f^{-1}(1) \leq cl(f^{-1}(\lambda))$ and hence $1 \leq cl(f^{-1}(\lambda))$. That is, $cl(f^{-1}(\lambda)) = 1$. Therefore $f^{-1}(\lambda)$ is a fuzzy dense set in (X, T).

Now $f^{-1}(\lambda) = f^{-1}(\wedge_{i=1}^{\infty}(\lambda_i)) = \wedge_{i=1}^{\infty}(f^{-1}(\lambda_i))$, (by lemma 2.16). Since f is a fuzzy continuous function from (X, T) onto (Y, S) and (λ_i)'s are fuzzy open sets in (Y, S), $[f^{-1}(\lambda_i)]$'s are fuzzy open sets in (X, T). Then $f^{-1}(\lambda)$ is a fuzzy G_δ-set in (X, T). Thus $f^{-1}(\lambda)$ is a fuzzy dense and fuzzy G_δ-set in (X, T). Since (X, T) is a fuzzy GID space, $clint[f^{-1}(\lambda)] = 1$. Now $1 - clint[f^{-1}(\lambda)] = 0$ implies that $intcl[f^{-1}(1 - \lambda)] = 0$. Therefore $f^{-1}(1 - \lambda)$ is a fuzzy nowhere dense set in (X, T). Since f is a fuzzy continuous and fuzzy open function, 1-1 and onto function, by theorem 5.2, $f[f^{-1}(1-\lambda)]$ is a fuzzy nowhere dense set in (Y, S). Since the function f is onto, by lemma 2.17, $f[f^{-1}(1 - \lambda)] = 1 - \lambda$. Hence $1 - \lambda$ is a fuzzy nowhere dense set in (Y, S). Then, $intcl(1 - \lambda) = 0$. This implies that $clint(\lambda) = 1$, for the fuzzy dense and fuzzy G_δ-set λ in (Y, S). Therefore (Y, S) is a fuzzy GID space.

Definition 5.4 ([15]). A function $f : (X, T) \to (Y, S)$ from a fuzzy topological space (X, T) into another fuzzy topological space (Y, S) is called somewhat fuzzy open if for every $\lambda \in T$ and $\lambda \neq 0$, there exists a fuzzy open set μ in (Y, S) such that $\mu \neq 0$ and $\mu \leq f(\lambda)$. That is, $int[f(\lambda)] \neq 0$.

Theorem 5.5 ([15]). *Suppose that (X, T) and (Y, S) are fuzzy topological spaces. Let $f : (X, T) \to (Y, S)$ be an onto function. Then the following conditions are equivalent :*

1. *f is somewhat fuzzy open.*
2. *If λ is a fuzzy dense set in (Y, S), then $f^{-1}(\lambda)$ is a fuzzy dense set in (X, T).*

Theorem 5.6 ([17]). *If the function $f : (X, T) \to (Y, S)$ from a fuzzy topological space (X, T) into another fuzzy topological space (Y, S) is somewhat fuzzy continuous and 1-1 and onto and if $clint(\lambda) = 1$ for any non-zero fuzzy set λ in (X, T), then $clint[f(\lambda)] = 1$ in (Y, S).*

Proposition 5.7. *If the function $f : (X, T) \to (Y, S)$ from a fuzzy topological space (X, T) onto another fuzzy topological space (Y, S) is a fuzzy continuous and somewhat fuzzy open function and if (X, T) is a fuzzy GID space, then (Y, S) is a fuzzy GID space.*

Proof. Let λ be a fuzzy dense and fuzzy G_δ-set in a fuzzy topological space (Y, S). Then $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$ where $\lambda_i \in S$ and $cl(\lambda) = 1$ in (Y, S). Since f is a somewhat fuzzy open function from (X, T) onto (Y, S) and λ is a fuzzy dense set in (Y, S), by theorem 5.5, $f^{-1}(\lambda)$ is a fuzzy dense set in (X,T). Now $f^{-1}(\lambda) = f^{-1}[\wedge_{i=1}^{\infty}(\lambda_i)] = \wedge_{i=1}^{\infty} f^{-1}[(\lambda_i)]$. Since f is a fuzzy continuous function from (X,T) onto (Y,S) and (λ_i)'s are fuzzy open sets in (Y,S), $f^{-1}(\lambda_i)$'s are fuzzy open sets in (X,T). Then $f^{-1}(\lambda)$ is a fuzzy G_δ-set in (X,T). Thus $f^{-1}(\lambda)$ is a fuzzy dense and fuzzy G_δ-set in (X,T). Since (X,T) is a fuzzy GID space, $clint[f^{-1}(\lambda)] = 1$. By theorem 5.6, $clint(f[f^{-1}(\lambda)]) = 1$, in (Y,S). Since the function f is onto, by lemma 2.17, we have $f[f^{-1}(\lambda)] = \lambda$. Hence, $clint(\lambda) = 1$, for the fuzzy dense and fuzzy G_δ-set λ in (Y,S). Therefore (Y,S) is a fuzzy GID space.

6. Conclusions

The concepts of fuzzy GID spaces are introduced and some of its properties and relation with the existing spaces are studied in this paper. The condition underwhich fuzzy Baire space becomes a fuzzy Volterra space is obtained and the condition for a fuzzy σ-Baire space to be a fuzzy Baire space is also obtained in this paper. Some results concerning functions that preserve fuzzy GID spaces in the context of images and pre-images are obtained.

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