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Some characterizations of SBL-algebras with an involutive negation by their ideals

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Abstract. In the paper, we give the concept of ¬-ideals dual to that of ¬-filters by using the operation ⊛ in SBL¬-algebras, and also show some characterizations of ¬-ideals. The relationships between ¬-ideals and ¬-filter are investigated by the notion of the set of complement elements. It is obtained that for a ¬-proper ideal I of a SBL¬-algebra L, the set $I \cup N(I)$ is the least SBL¬-subalgebra of L containing I. Some properties of ¬-ideals are also studied via the notions of radicals of ¬-ideals, and then a concrete description of the radical of an ¬-ideal is given.

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1. INTRODUCTION

BL-algebras as the algebraic structures for Basic Logic were invented by Hájek [4] in order to study the basic logic frame work of fuzzy set theory. MV-algebras, Godel algebras and product algebras are three famous BL-algebras. It has been proved that BL-algebras satisfying the double negation are MV-algebras. Esteva et al. [3] introduced SBL¬-algebras as the algebraic sructures for the Strict Basic (Fuzzy) Logic SBL which is a propositional logic given by standard BL-algebras. Recently, Kondo [8] studied the property of strict residuated lattices (SRL-algebras) with a new involutive negation ¬ and also provided a prime filter theorem of SRL¬-algebras.

The ideal (filter) theory plays an important role in studying logical algebras. From the logic point of view, the sets of provable formulas in corresponding systems can be described by ideals (filters) of those algebraic semantics. In various algebraic structures, the notion of ideals is at the center, while in BL-algebras, the focus is shifted to filters owing to lack of ideals in BL-algebras. The study of BL-algebras has experienced a tremendous growth over the recent years and many important results are obtained. It has been proved that the notion of filters coincides with deductive systems in [13]. Haveshki et al. [6] extended the algebraic analysis of BL-algebras and introduced positive implicative filters of BL-algebras, which were proved to coincide with Turunen's implicative deductive systems in [9]. Recently, some new kinds of filters such as implication filters [1] and integral filters [2] were studied and some of their characterizations were presented.

In order to fulfill the ideal theory in BL-algebras, Lele and Nganou [10] introduced the notion of ideals in BL-algebras as a natural generalization of that of ideals in MV-algebras by using the notion of pseudo-addition. They also proved that their definition of fuzzy ideals [11] coincide with that given in [15, 12]. Compared with the filter theory, we observe that the notion of $\neg$-ideals in SBL$\neg$-algebras dual to that of $\neg$-filters is vacant, and then we give the notion of $\neg$-ideals in SBL$\neg$-algebras by introduce a new operation $\circledast$. It is pointed out that an $\neg$-ideal is an ideal in SBL$\neg$-algebras. Some characterizations of $\neg$-ideals are presented and the relationships between $\neg$-ideals and $\neg$-filters are investigated by the notion of the set of complement elements. The notion of radicals of $\neg$-ideals are also introduced to study the properties of $\neg$-ideals, and a concrete description of the radical of an $\neg$-ideal is given.

2. Preliminaries

In the section, we summarize some definitions and results about BL-algebras and SBL$\neg$-algebras which will be used in the sequel.

Definition 2.1 ([4])**.** An algebra structure $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a BL-algebras if it satisfies the following conditions: for any $x, y, z \in L$,

(BL-1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(BL-2) $(L, \otimes, 1)$ is a commutative monoid,
(BL-3) $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$,
(BL-4) $x \otimes (x \rightarrow y) = x \wedge y$,
(BL-5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

A BL-algebra L satisfying the double negation is called an MV-algebra,that is $x = x''$ for any $x \in L$, where $x' := x \rightarrow 0$. A BL-algebra satisfying $x \vee x' = 1$ is called a Boolean algebra. A BL-algebra L is called a SBL-algebra if it satisfies the axiom (S): $(x \otimes y)' = x' \vee y'$, for any $x, y \in L$. Esteva et al. extended SBL with a unary connective $\neg$ and introduced the notion of SBL$\neg$-algebras as follows.

Definition 2.2 ([3])**.** A SBL$\neg$-algebra is a structure $(L, \wedge, \vee, \otimes, \rightarrow, \neg, 0, 1)$ which is a SBL-algebra expended with a unary operation $\neg$ satisfying the following conditions: for any $x, y \in L$,

(SBL$\neg$1) $\neg\neg x = x$,
(SBL$\neg$2) $x' \leq \neg x$,
(SBL$\neg$3) $\triangle(x \rightarrow y) = \triangle(\neg y \rightarrow \neg x)$, where $\triangle x = (\neg x)'$,
(SBL$\neg$4) $\triangle x \vee (\triangle x)' = 1$,
(SBL$\neg$5) $\triangle(x \vee y) \leq \triangle x \vee \triangle y$,
(SBL$\neg$6) $\triangle x \otimes \triangle(x \rightarrow y) \leq \triangle y$.

It has been proved by Halaš [5] that the above axiom system is not dependent. In fact, if L is a SBL-algebra, then L is a SBL$\neg$-algebra if and only if it satisfies (SBL$\neg$1) and (SBL$\neg$3).

In the following lemma, we summarize some results related to SBL$\neg$-algebras which are required in the following sections.

Lemma 2.3 ([4, 3, 8]). *Let L be a SBL$\neg$-algebra. Then the following relations hold: for any $x, y, z \in L$,*

(1) $x \leq y$ *if and only if* $x \to y = 1$, $x \otimes y = 0$ *if and only if* $x \leq y'$;
(2) $x \to (y \to z) = (x \otimes y) \to z = y \to (x \to z)$, $\triangle\neg x = \triangle x' = x'$;
(3) $(x \to y)'' = y' \to x'$, $x' = x' \otimes x'$, $x' \otimes y' = x' \wedge y'$, $x \otimes y \leq x \wedge y$;
(4) $x \otimes x' = 0$, $x' = x'''$, $x \leq x''$, $\triangle x \leq x$, $\triangle\triangle x = \triangle x$, $\triangle x \otimes \triangle y = \triangle(x \otimes y)$;
(5) $0' = 1$, $1' = 0$, $\neg 0 = 1$, $\neg 1 = 0$, $1 \to x = x$, $x \to 1 = 1$, $y \leq x \to y$;
(6) $\neg(x \wedge y) = \neg x \vee \neg y$, $\neg(x \vee y) = \neg x \wedge \neg y$, $x \leq y$ *implies* $y \to z \leq x \to z$.

Lele and Nganou defined the pseudo-addition of a BL-algebra L: $x \oslash y := x' \to y$ for any $x, y \in L$, and then introduced the notion of ideal in BL-algebras as a natural generalization of that of ideals in MV-algebras.

Definition 2.4 ([10]). Let $(L, \wedge, \vee, \otimes, \to, 0, 1)$ be a BL-algebra and I a nonempty subset of L. We say that I is an ideal of L if it satisfies:

(1) for any $x, y \in I$, $x \oslash y \in I$,
(2) for any $x, y \in L$, if $x \leq y$ and $y \in I$, then $x \in I$.

From the above definition, it is easy to see that for any ideal I, $x \in I$ if and only if $x'' \in I$ for any $x \in L$.

Denote $x \ominus y = x \otimes y'$ for any $x, y \in L$, then a subset I of a BL-algebra L containing 0 is an ideal if and only if $x \ominus y \in I$ and $y \in I$ imply $x \in I$ for any $x, y \in L$ [10].

A nonempty subset F of a BL-algebra L is called a filter if it satisfies: for any $x, y \in L$, (i) $x \in F$ and $y \in F$ imply $x \otimes y \in F$; (ii) $x \in F$ and $x \leq y$ imply $y \in F$. A filter F is said to be a normal filter if $x'' \in F$ implies $x \in F$ for any $x \in L$ [7]. A filter F of SBL$\neg$-algebra L is called a $\neg$-filter if it satisfies (iii) $x \to y \in F$ implies $\neg y \to \neg x \in F$. It is easy to check that a nonempty subset F of a SBL$\neg$-algebra L is a $\neg$-filter if and only if it satisfies: (i) $1 \in F$; (ii) $x \in F$ and $x \to y \in F$ imply $y \in F$ for any $x, y \in L$; (iii) $x \in F$ implies $\triangle x \in F$ for any $x \in L$ [14]. For a $\neg$-filter F of L, we have that $x \in F$ if and only if $\triangle x \in F$ for any $x \in L$.

3. Some characterizations of $\neg$-ideals

In the section, we introduce the notion of $\neg$-ideals and investigate their properties. From now on, $(L, \wedge, \vee, \otimes, \to, \neg, 0, 1)$ is a SBL$\neg$-algebra unless otherwise mentioned, which will often be referred by its support set L.

we define a binary operation $\circledast$ on L by

$$x \circledast y := \neg(x' \otimes y'),$$

for any $x, y \in L$. We define $1x := x \circledast 0$ and $nx := (n-1)x \circledast x$, for any $n > 1$. In the following, $'$ has priority over $\neg$ in the operation process.

In order to discuss $\neg$-ideals, we first establish some properties of the operation $\circledast$ as follows.

Proposition 3.1. *Let L be a SBL$\neg$-algebra. Then the following statements are valid: for any $x, y, z \in L$,*

(1) $x \circledast y = \neg x' \vee \neg y'$, $(x \circledast y)' = x' \otimes y'$, $x \circledast y = y \circledast x$;
(2) $x \circledast 1 = 1$, $x \circledast 0 = x \circledast x$, $x \circledast x' = 1$;
(3) $(x \circledast y) \circledast z = x \circledast (y \circledast z)$;
(4) $x \vee y \leq x \circledast y$, $x \leq y$ *implies* $x \circledast z \leq y \circledast z$;
(5) $x \circledast (y \wedge z) = (x \circledast y) \wedge (x \circledast z)$;
(6) $x \circledast (y \vee z) = (x \circledast y) \vee (x \circledast z)$;
(7) $\triangle(x \circledast y) = x'' \vee y''$, $\triangle(x \ominus y) = \triangle x \ominus y$;
(8) $\triangle(\triangle x \circledast \triangle y) = \triangle x \circledast \triangle y$, $\triangle(\triangle x \ominus \triangle y) = \triangle x \ominus \triangle y$.

Proof. We only show the cases of (3) and (5). Concerning to the case of (3): $(x \circledast y) \circledast z = x \circledast (y \circledast z)$. Since $(x \circledast y) \circledast z = \neg((\neg x' \vee \neg y')') \vee \neg z' = \neg((\neg x')' \wedge (\neg y')') \vee \neg z' = \neg(\triangle x' \wedge \triangle y') \vee \neg z' = \neg(x' \wedge y') \vee \neg z' = \neg x' \vee \neg y' \vee \neg z'$, similarly, $x \circledast (y \circledast z) = \neg x' \vee \neg y' \vee \neg z'$, thus (3) holds.

For the case of (5): $x \circledast (y \wedge z) = (x \circledast y) \wedge (x \circledast z)$, we have $x \circledast (y \wedge z) = \neg x' \vee (\neg y' \wedge \neg z') = (\neg x' \vee \neg y') \wedge (\neg x' \vee \neg z') = (x \circledast y) \wedge (x \circledast z)$, and so (5) is valid. □

We would like to point out that for any $x \in L$, $x \circledast 0 = x$ is not true, since $x' \neq \neg x$ in general.

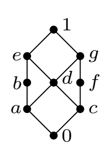
It is known that the notion dual to that of filters is that of ideals, by using the operator $\circledast$ defined on SBL$\neg$-algebras, we can introduce the notion of $\neg$-ideals as follows.

Definition 3.2. Let I be a nonempty subset of L. Then I is called an $\neg$-ideal of L if it satisfies the following conditions: for any $x, y \in L$,

(1) if $x, y \in I$, then $x \circledast y \in I$;
(2) $y \in I$ and $x \leq y$ imply $x \in I$;
(3) $\neg y \ominus \neg x \in I$ implies $x \ominus y \in I$.

For better understanding of $\neg$-ideals, we illustrate it by the following example.

Example 3.3. Let $L = \{0, a, b, c, d, e, f, g, 1\}$ be a set with a Hasse diagram and Cayley tables as follows.



$\neg$	0	a	b	c	d	e	f	g	1
	1	g	f	e	d	c	b	a	0

$\otimes$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	a	a	0	a	a	0	a	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	c	c	c	c	c	c
d	0	a	a	c	d	d	c	d	d
e	0	a	b	c	d	e	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	a	a	c	d	d	f	g	g
1	0	a	b	c	d	e	f	g	1

$\to$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	f	1	1	f	1	1	f	1	1
b	f	g	1	f	g	1	f	g	1
c	b	b	b	1	1	1	1	1	1
d	0	b	b	f	1	1	f	1	1
e	0	a	b	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	0	b	b	c	e	e	f	1	1
1	0	a	b	c	d	e	f	g	1

then L is a SBL$\neg$-algebra. Obviously, $\{0\}$ is an $\neg$-ideal.

Lemma 3.4. *Let I be an $\neg$-ideal of L. Then the following statements hold: for any $x, y \in L$,*

(1) *$x \in I$ if and only if $\triangle x \in I$;*
(2) *$x' \in I$ if and only if $\neg x \in I$.*

Proof. (1) Suppose that $x \in I$, since $\triangle x \leq x$, then $\triangle x \in I$.
Conversely, assume that $\triangle x \in I$, that is, $\neg 0 \ominus \neg x \in I$, then $x = x \ominus 0 \in I$.
(2) It is directly from $\triangle \neg x = \triangle x' = x'$ and (1). □

Theorem 3.5. *Let I be a nonempty subset of L. Then I is an $\neg$-ideal of L if and only if I satisfies the following conditions:*

(1) *$0 \in I$;*
(2) *$y \in I$ and $x \ominus y \in I$ imply $x \in I$, for any $x, y \in L$;*
(3) *$\triangle x \in I$ implies $x \in I$ for any $x \in L$.*

Proof. Suppose that I is an $\neg$–ideal of L. Since I is a nonempty set, then there exists $x \in I$. Considering that $0 \leq x$, we get that $0 \in I$. For any $x, y \in L$, if $y \in I$ and $x \ominus y \in I$, then we have $(x \ominus y) \circledast y = \neg((y' \to x') \otimes y') = \neg(y' \wedge x') \geq y'' \vee x'' \geq x$, thus $x \in I$. The proof (3) is directly from Lemma 3.4.

Conversely, let $x, y \in L$ be such that $x \leq y$ and $y \in I$. Since $x \ominus y = x \otimes y' \leq x \otimes x' = 0 \in I$, then $x \ominus y = 0 \in I$, and apply the hypothesis to conclude $x \in I$. Let $x, y \in L$ be such that $x, y \in I$. We observe that $((x' \otimes y')' \ominus x) \ominus y = ((x' \otimes y')' \otimes x') \otimes y' = 0$, it follows that $\triangle \neg(x' \otimes y') = (x' \otimes y')' \in I$. From the hypothesis, we have $x \circledast y = \neg(x' \otimes y') \in I$. Let $x, y \in L$ be such that $\neg y \ominus \neg x \in I$, that is, $\neg y \otimes \triangle x \in I$. Considering that $\triangle(y' \otimes x) = \triangle y' \otimes \triangle x = \triangle \neg y \otimes (\triangle\triangle x) = \triangle(\neg y \otimes \triangle x) \leq \neg y \otimes \triangle x \in I$, we have $\triangle(y' \otimes x) \in I$, and thus $x \ominus y = y' \otimes x \in I$. □

As an application of Theorem 3.5, we have the following result.

Corollary 3.6. *Let I be a nonempty subset of L. Then I is an $\neg$-ideal of L if and only if for any $x, y \in L$,*

(1) *if $x, y \in I$, then $x \circledast y \in I$;*
(2) *$y \in I$ and $x \leq y$ imply $x \in I$;*
(3) *$\triangle x \in I$ implies $x \in I$.*

Proposition 3.7. *Let I be an $\neg$-ideal of L. Then $x \circledast y \in I$ if and only if $x \oslash y \in I$, for any $x, y \in L$.*

Proof. Assume that $x \circledast y \in I$. Since $x \oslash y = x' \to y \leq x' \to y'' = (x' \otimes y')' \leq \neg(x' \otimes y') = x \circledast y$, we get $x \oslash y \in I$.

Conversely, suppose that $x \oslash y \in I$. Noticing that $(x' \otimes y')' \ominus (x \oslash y) = (y' \to x'') \otimes (x' \to y)' = (y' \to x'') \otimes (y' \to x'')' = 0 \in I$, we have $(x' \otimes y')' \in I$, and so $x \circledast y = \neg(x' \otimes y') \in I$ by Lemma 3.4. □

Remark 3.8. From Proposition 3.7, it is easy to see that an $\neg$-ideal is an ideal in a SBL$\neg$-algebra, however the converse is not true in general.

Example 3.9. Let $L = \{0, a, b, 1\}$ be such that $0 < a < b < 1$. The operations $\otimes$, $\to$ and $\neg$ are defined as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

$\neg$	0	a	b	1
	1	b	a	0

then L is a SBL$\neg$-algebra. It is easy to check that $I = \{0\}$ is an ideal of L, but I is not an $\neg$-ideal of L since $\triangle a = 0 \in I$, while $a \notin I$.

In the following section, we will investigate the relationships between $\neg$-ideal and $\neg$-filters, we first introduce the set of complement elements $N(X)$.

Definition 3.10 ([11]). Let X be a nonempty subset of L. The set of complement elements with respect to X is denoted by $N(X)$ and is defined by $N(X) = \{x \in L | x' \in X\}$.

Proposition 3.11. *Let F be a $\neg$-normal filter and I be an $\neg$-ideal of L. Then*

(1) *$N(F)$ is an $\neg$-ideal of L;*
(2) *$N(I)$ is a $\neg$-filter of L;*
(3) *$I = N(N(I))$, $F \subseteq N(N(I))$;*
(4) *$N(F) = N(N(N(F)))$.*

Proof. (1) Since $0' = 1 \in F$, it is clear that $0 \in N(F)$. Let $x \ominus y \in N(F)$ and $y \in N(F)$, then $(x \ominus y)' = y' \to x' \in F$ and $y' \in F$. Noticing that F is a $\neg$-filter, we have $x' \in F$, and so $x \in N(F)$. If $\triangle x \in N(F)$, that is, $(\triangle x)' = (\neg x)'' \in F$, then $\neg x \in F$, and thus $x' = \triangle \neg x \in F$. It follows that $x \in N(F)$, and so $N(F)$ is an $\neg$-ideal of L.

(2) Assume that I is an $\neg$-ideal of L, since $1' = 0 \in I$, then $1 \in N(I)$. Let $x, y \in L$ be such that $x \in N(I)$ and $x \to y \in N(I)$. Then $x' \in I$ and $(x \to y)' \in I$. As $y' \ominus x' \leq (x \to y)' \in I$, we get $y' \in I$, and so $y \in N(I)$. Let $x \in N(I)$, that is, $x' \in I$. According to Lemma 3.4, we obtain $\neg x \in I$. From Remark 3.8, it follows that $(\triangle x)' = (\neg x)'' \in I$, and so $\triangle x \in N(I)$. Thus $N(I)$ is a $\neg$-filter of L.

(3) $I = N(N(I))$ follows from the fact that $x'' \in I$ if and only if $x \in I$ for any $x \in L$. $F \subseteq N(N(I))$ follows from the fact that $x \leq x''$ for any $x \in L$.

(4) It follows from $I = N(N(I))$ by setting $I = N(F)$. □

Theorem 3.12. *Let I be a nonempty subset of L. If I is a $\neg$-proper ideal, then $I \cup N(I)$ is the least SBL$\neg$-subalgebra of L containing I.*

Proof. We will show that $I \cup N(I)$ is closed under the operations on L. For any $x, y \in L$,

(1) $x \in I \cup N(I)$ implies $x' \in I \cup N(I)$ and $\neg x \in I \cup N(I)$.

Indeed, let $x \in I \cup N(I)$. Then $x \in I$ or $x' \in I$. If $x \in I$, noticing that I is a $\neg$-proper ideal, we have $x'' \in I$, and so $x' \in N(I)$. If $x' \in I$, it is clear that $\neg x \in I$ by Lemma 3.4, and thus (1) is valid.

(2) $x, y \in I \cup N(I)$ implies $x \wedge y \in I \cup N(I)$.

If $x \in I$ or $y \in I$, for instance, $x \in I$, since $x \wedge y \leq x \in I$, we get $x \wedge y \in I$. If $x \in N(I)$ and $y \in N(I)$, then $x', y' \in I$. It follows that $(x \wedge y)' = x' \vee y' \in I$, and so $x \wedge y \in N(I)$.

(3) $x, y \in I \cup N(I)$ implies $x \vee y \in I \cup N(I)$.

It is similar to (2).

(4) $x, y \in I \cup N(I)$ implies $x \otimes y \in I \cup N(I)$.

If $x \in I$ or $y \in I$, for instance, $x \in I$, since $x \otimes y \leq x \in I$, we have $x \otimes y \in I$. If $x \in N(I)$ and $y \in N(I)$, since $N(I)$ is a $\neg$-filter of L, we have $x \otimes y \in N(I)$, and so $x \otimes y \in I \cap N(I)$.

(5) $x, y \in I \cup N(I)$ implies $x \to y \in I \cup N(I)$.

Indeed, if $y \in N(I)$, since $y \leq x \to y$ and $N(I)$ is a $\neg$-filter, then $x \to y \in N(I)$. If $y \in I$, we consider two cases.

(i) $x \in I$. We have $x'' \in I$, and so $x' \in N(I)$. Since $x \to y \geq x \to 0 = x' \in N(I)$ and $N(I)$ is a $\neg$-filter, we get $x \to y \in N(I)$.

(ii) $x \in N(I)$. Then $x' \in I$. Since $(x \to y)'' \ominus x' \ominus y = (y \to x)' \otimes x'' \otimes y' = 0$, then $(x \to y)'' \in I$, and so $x \to y \in I$.

Finally, let J be a SBL$\neg$-subalgebra containing I and $x \in I \cup N(I)$. If $x \in I$, it is clear that $x \in J$. If $x \in N(I)$, then $x' \in I$, and so $\neg x \in I \subseteq J$. Obverse that J be a SBL$\neg$-subalgebra, we have $x = \neg\neg x \in J$. Thus $\in I \cup N(I) \subseteq J$, and so $I \cup N(I)$ is the least SBL$\neg$-subalgebra of L containing I. □

Let I be an $\neg$-ideal of L. I is called prime if $x \wedge y \in I$ implies $x \in I$ or $y \in I$, for any $x, y \in L$. It has been proved that an $\neg$-ideal I is prime if and only if $(x \to y)' \in I$ or $(y \to x)' \in I$, for any $x, y \in L$. The notions of $\neg$-maximal ideals can be similarly defined.

Proposition 3.13. *Let I be a $\neg$-proper ideal of L. Then I is a $\neg$-prime ideal of L if and only if $x \ominus y \in I$ or $y \ominus x \in I$, for any $x, y \in L$.*

Proof. Suppose that I is a $\neg$-prime ideal of L, then $(x \to y)' \in I$ or $(y \to x)' \in I$, for any $x, y \in L$. If $(x \to y)' \in I$, since $(x \ominus y) \ominus (x \to y)' = 0 \in I$, then $x \ominus y \in I$. If $(y \to x)' \in I$, $y \ominus x \in I$ can be proved similarly.

Conversely, assume that $x \ominus y \in I$ or $y \ominus x \in I$, for any $x, y \in L$. Let $x \wedge y \in I$. If $x \ominus y \in I$, as $(y \ominus (y \ominus x)) \ominus (x \wedge y) = 0$, noticing that I is an $\neg$-ideal of L, we have $y \in I$. If $y \ominus x \in I$, $x \in I$ can be proved in a similar way. □

Dual to $\neg$-filters, we can obtain some similar results concerning to $\neg$-ideals.

Proposition 3.14. *Let I be an $\neg$-ideal and $a \in L \backslash I$. We have*

1. *there exists a $\neg$-prime ideal P such that $I \subseteq P$, but $a \notin P$;*
2. *every $\neg$-prime ideal of L is contained in a unique $\neg$-maximal ideal of L.*

Definition 3.15. Let I be an $\neg$-ideal of L. The intersection of all $\neg$-maximal ideals of L that containing I is called the radical of I, and it is denoted by $Rad(I)$.

It is easy to see that the radical $Rad(I)$ of an $\neg$-ideal I is an $\neg$-ideal and $I \subseteq Rad(I)$.

Proposition 3.16. *Let I be a $\neg$-proper ideal of L. Then $L\backslash Rad(I)$ is a $\neg$-proper filter of L.*

Proof. Since $0 \in Rad(I)$, then $1 \in L\backslash Rad(I)$. Let $x \in L\backslash Rad(I)$ and $x \to y \in L\backslash Rad(I)$. Then $x' \in Rad(I)$ and $(x \to y)' \in Rad(I)$. Since $(y' \ominus x') \ominus (x \to y)' = 0 \in Rad(I)$, noticing that $Rad(I)$ is an $\neg$-ideal, we have $y' \ominus x' \in Rad(I)$. Thus $y' \in Rad(I)$, and so $y \in L\backslash Rad(I)$. Assume that $x \in L\backslash Rad(I)$, then $x' \in Rad(I)$, and thus $\neg x \in Rad(I)$. Since $(\triangle x)' \ominus \neg x = 0$, we get $(\triangle x)' \in Rad(I)$, therefore $\triangle x \in L\backslash Rad(I)$, and thus $L\backslash Rad(I)$ is a $\neg$-proper filter of L. □

Proposition 3.17. *Let M be a $\neg$-proper ideal of L. If M is a $\neg$-maximal ideal, then $L\backslash M$ is a $\neg$-proper filter of L.*

Proof. The proof is similar to that of Proposition 3.16. □

Theorem 3.18. *Let I be an $\neg$-ideal of L. Then $Rad(I) = \{x \in L | x^n \ominus \neg x \in I; \forall n \geq 1\}$.*

Proof. Let $x \in Rad(I)$. Suppose that there exists $k \geq 1$ such that $x^k \ominus \neg x \notin I$. From Proposition 3.14, it follows that there exists a $\neg$-prime ideal P of L such that $I \subseteq P$ and $x^k \ominus \neg x \notin P$. Since P is a $\neg$-prime ideal, then $\neg x \ominus x^k \in P$. According to Proposition 3.14, there exists a $\neg$-maximal ideal M of L such that $P \subseteq M$, and so $\neg x \ominus x^k \in M$. Assume that $x \in M$, then $x^k \in M$. Since M is a $\neg$-maximal ideal and $\neg x \ominus x^k \in M$, then $\neg x \in M$, and so $x' \in M$. Hence $x \circledast x' = 1 \in M$, which is a contradiction, therefore $x \notin M$. Since $I \subseteq P \subseteq M$, then $x \notin I \subseteq Rad(I)$, a contradiction. Hence $x^n \ominus \neg x \in I$ for any $n \geq 1$.

Conversely, assume that $x^n \ominus \neg x \in I$ for any $n \geq 1$, but $x \notin Rad(I)$. Then there exists a $\neg$-maximal ideal M of L such that $x \notin M$, it follows that $x \in L\backslash M$. According to Proposition 3.17, $L\backslash M$ is a $\neg$-proper filter of L, thus we have $x' \in M$, and so $\neg x \in M$. Considering that $x \ominus \neg x \in I \subset M$, we obtain $x \in M$, which is a contradiction. Therefore $x \in Rad(I)$, and thus $Rad(I) = \{x \in L | x^n \ominus \neg x \in I; \forall n \geq 1\}$. □

Proposition 3.19. *Let I be a $\neg$-proper ideal of L. Then the following statements are valid:*

(1) *for any $x, y \in L$, if $x, y \in Rad(I)$, then $x \ominus y \in Rad(I)$;*
(2) *for any $x, y \in L$, if $x \in Rad(I)$ and $y \in L\backslash Rad(I)$, then $x \ominus y \in Rad(I)$, $y \ominus x \in L\backslash Rad(I)$.*

Proof. (1) Let $x, y \in Rad(I)$. Suppose that $x \ominus y \notin Rad(I)$, that is, $x \ominus y \in L\backslash Rad(I)$. Since $L\backslash Rad(I)$ is a $\neg$-proper filter of L and $x \ominus y \leq x$, then $x \in L\backslash Rad(I)$, a contradiction. Thus $x \ominus y \in Rad(I)$.

(2) Let $x \in Rad(I)$ and $y \in L\backslash Rad(I)$. Then we have $y' \in Rad(I)$. Considering that $Rad(I)$ is an $\neg$-ideal of L and $x \ominus y \leq y'$, we get $x \ominus y \in Rad(I)$. Suppose that $y \ominus x \notin L\backslash Rad(I)$, then $y \ominus x = y \to x'' \in L\backslash Rad(I)$. Noticing that $L\backslash Rad(I)$

is a $\neg$-proper filter of L, we have $x'' \in L\backslash Rad(I)$, and so $x' \in Rad(I)$. Therefore $x \circledast x' = 1 \in Rad(I)$, which is a contradiction, and thus $y \ominus x \in L\backslash Rad(I)$. ☐

Proposition 3.20. *Let I be an $\neg$-ideal of L. The relation $\sim_I$ on L is defined by: for any $x, y \in L$,*

$$x \sim_I y \text{ if and only if } x \ominus y \in I \text{ and } y \ominus x \in I.$$

Then $\sim_I$ is a congruence on L.

Proof. The proof is similar to that of Theorem 4.2 in [10]. ☐

Let I be a $\neg$-ideal of L and $x \in L$. Define the set $[x] = \{y \in L | x \sim_I y\}$, which is called a congruence class of x by $\sim_I$. The set $L/I = \{[x] | x \in L\}$ is called a quotient set by $\sim_I$. For any $x, y \in L$, we define the operations $\wedge$, $\vee$, $\otimes$, $\rightarrow$ and $\neg$ on L/I as follows:

$$[x] \wedge [y] = [x \wedge y],\ [x] \vee [y] = [x \vee y],\ [x] \otimes [y] = [x \otimes y],$$
$$[x] \rightarrow [y] = [x \rightarrow y],\ \neg[x] = [\neg x].$$

Proposition 3.21. *Let I be an $\neg$-ideal of L. Then $(L/I, \wedge, \vee, \otimes, \rightarrow, \neg, [0], [1])$ is an MV-algebra, and it is also a Boolean algebra.*

Proof. Obviously, L/I is a BL-algebra. According to Remark 3.8, we get that the $\neg$-ideal I is an ideal of L, thus $x \in I$ if and only if $x'' \in I$ for any $x \in L$. So $[x] = [x]''$, and thus L/I is an MV-algebra. By Lemma 3.4, we have $[\triangle x] = [x]$ and $[\neg x] = [x]'$ for any $x \in L$, then $[x] \vee [x]' = [\triangle x] \vee [(\triangle x)'] = [\triangle x \vee (\triangle x)'] = [1]$, thus L/I is a Boolean algebra. ☐

4. Conclusions

We have introduced the concept of $\neg$-ideals in SBL$\neg$-algebras and given some characterizations of $\neg$-ideals. The relationships between $\neg$-ideals and $\neg$-filter are investigated, in general, $N(I)$ is a $\neg$-filter, but $N(F)$ is an $\neg$-ideal with an conditions that $x'' \in F$ implying that $x \in F$. It is obtained that for a $\neg$-proper ideal I, the set $I \cup N(I)$ is a SBL$\neg$-subalgebra of L. We have discussed some properties of $\neg$-ideals via the notions of radicals of $\neg$-ideals, and given a concrete description of the radical of an $\neg$-ideal.

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