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On fuzzy rough $\mathcal{BG}$-boundary spaces

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Abstract. In this paper, the concepts of fuzzy rough topological groups and fuzzy rough $\mathcal{G}$ structure spaces are introduced and studied. In this connection, the concept of fuzzy rough $\mathcal{BG}$-boundary space is introduced. Interesting characterization is established.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [15]. Fuzzy sets have applications in many fields such as information [9] and control [10]. The theory of fuzzy topological spaces was introduced and developed by Chang [3] and since then various notions in classical topology has been extended to fuzzy topological spaces. Pawlak [8] introduced the concept of rough set. The concept of rough group and rough subgroup was introduced by R. Biswas and S. Nanda [2]. The concept of rough topological space was introduced by B. P. Mathew and S. J. John [6]. S. Nanda and S. Majumdar [7] introduced the concept of fuzzy rough set. The concept of fuzzy group and fuzzy topological group was introduced and studied by D. Foster [4]. The concept of boundary of a fuzzy set was introduced by R. H. Warren [14]. The notion of B-set in topological space were introduced and developed by J. Tong [11]. The concept of fuzzy B-set was introduced by M. K. Uma, E. Roja and G. Balasubramanian [12]. In this paper, the concepts of fuzzy rough topological groups and fuzzy rough $\mathcal{G}$ structure spaces are introduced and studied. In this connection, the concept of fuzzy rough $\mathcal{BG}$-boundary space is introduced. Interesting characterization is established.

2. Preliminaries

Definition 2.1 ([8]). Let U be a nonempty set and let $\mathscr{B}$ be a complete subalgebra of the Boolean algebra $\mathscr{P}(U)$ of subsets of U. The pair $(U, \mathscr{B})$ is called a rough universe.

Let $\mathscr{V} = (U, \mathscr{B})$ be a given fixed rough universe. Let $\mathscr{R}$ be the relation defined as follows: $A = (A_L, A_U) \in \mathscr{B}$ if and only if $A_L, A_U \in \mathscr{B}$, $A_L \subset A_U$. The elements of $\mathscr{R}$ are called rough sets and the elements of $\mathscr{B}$ are called exact sets. We identify the element $(X, X) \in \mathscr{R}$ with the element $X \in \mathscr{B}$ and hence an exact set is a rough set in the sense of the above identification. But a rough set need not be exact; for example if U is any nonempty set, then (Φ, U) is a rough set which is not exact.

Let $A = (A_L, A_U)$ and $B = (B_L, B_U)$ be any two rough sets. Then $A \cup B = (A_L \cup B_L, A_U \cup B_U)$, $A \cap B = (A_L \cap B_L, A_U \cap B_U)$, $A \subset B$ if and only if $A \cap B$=A.

Note 2.1 ([8]). We denote a rough set X with lower approximation X_L and upper approximation X_U by $X = (X_L, X_U)$.

Definition 2.2 ([7]). Let U be a set and $\mathscr{B}$ be a Boolean subalgebra of the Boolean algebra of all subsets of U. Let L be a lattice. Let X be a rough set. Then $X = (X_L, X_U) \in \mathscr{B}^2$ with $X_L \subset X_U$.

A fuzzy rough set $A = (A_L, A_U)$ in X is characterized by a pair of maps $\mu_{A_L} : X_L \to L$ and $\mu_{A_U} : X_U \to L$ with the property that $\mu_{A_L}(x) \leq \mu_{A_U}(x)$ for all $x \in X_U$. The collection of all fuzzy rough sets in X is denoted by $FRS(X)$.

Note 2.2 ([7]). In particular L could be the closed interval $[0, 1]$.

Definition 2.3 ([7]). For any two fuzzy rough sets $A = (A_L, A_U)$ and $B = (B_L, B_U)$ in X, we define

(i) $A = B$ if and only if $\mu_{A_L}(x) = \mu_{B_L}(x)$ for each $x \in X_L$ and $\mu_{A_U}(x) = \mu_{B_U}(x)$ for each $x \in X_U$.
(ii) $A \subseteq B$ if and only if $\mu_{A_L}(x) \leq \mu_{B_L}(x)$ for each $x \in X_L$ and $\mu_{A_U}(x) \leq \mu_{B_U}(x)$ for every $x \in X_U$.
(iii) $C = A \cup B$ if and only if $\mu_{C_L}(x) = max[\mu_{A_L}(x), \mu_{B_L}(x)]$ for all $x \in X_L$, $\mu_{C_U}(x) = max[\mu_{A_U}(x), \mu_{B_U}(x)]$ for all $x \in X_U$.
(iii) $D = A \cap B$ if and only if $\mu_{D_L}(x) = min[\mu_{A_L}(x), \mu_{B_L}(x)]$ for all $x \in X_L$, $\mu_{D_U}(x) = min[\mu_{A_U}(x), \mu_{B_U}(x)]$ for all $x \in X_U$.

More generally, if L then is a complete lattice, then for any index set I, if $\{A_i : i \in I\}$ is a family of fuzzy rough sets we have $E = \cup_i A_i$ if and only if $\mu_{E_L}(x) = sup_{i \in I} \mu_{A_{L_i}}(x)$ for all $x \in X_L$ and $\mu_{E_U}(x) = sup_{i \in I} \mu_{A_{U_i}}(x)$ for all $x \in X_U$. Similarly, $F = \cap_i A_i$ iff $\mu_{F_L}(x) = inf_{i \in I} \mu_{A_{L_i}}(x)$ for all $x \in X_L$ and $\mu_{F_U}(x) = inf_{i \in J} \mu_{A_{U_i}}(x)$ for all $x \in X_U$.

We define the complement A' of A by the ordered pair (A'_L, A'_U) of membership functions where $\mu_{A'_L}(x) = 1 - \mu_{A_U}(x)$ for all $x \in X_L$ and $\mu_{A'_U}(x) = 1 - \mu_{A_L}(x)$ for all $x \in X_U$.

Definition 2.4 ([5]). Let V, V_1 be any two sets and $\mathscr{B}$, $\mathscr{B}_1$ be any two Boolean subalgebra of the Boolean algebra of all subset of V, V_1. Let $(V, \mathscr{B})$ and $(V_1, \mathscr{B}_1)$ be two rough universes and $f : (V, \mathscr{B}) \to (V_1, \mathscr{B}_1)$.

Let $A = (A_L, A_U)$ be a fuzzy rough set in X. Then $Y = f(X) \in \mathcal{B}_1^2$ and $Y_L = f(X_L), Y_U = f(X_U)$. The image of A under f, denoted by $f(A) = (f(A_L), f(A_U))$ is defined by

$$f(A_L)(y) = \vee\{A_L(x) : x \in X_L \cap f^{-1}(y)\} \text{ for every } y \in Y_L, \text{ and}$$
$$f(A_U)(y) = \vee\{A_U(x) : x \in X_U \cap f^{-1}(y)\} \text{ for every } y \in Y_U.$$

Definition 2.5 ([5])**.** Let V, V_1 be any two sets and $\mathscr{B}$, $\mathscr{B}_1$ be any two Boolean subalgebra of the Boolean algebra of all subset of V, V_1. Let $(V, \mathscr{B})$ and $(V_1, \mathscr{B})$ be two rough universes and $f : (V, \mathscr{B}) \to (V_1, \mathscr{B}_1)$.

Let $B = (B_L, B_U)$ be a fuzzy rough set in Y where $Y = (Y_L, Y_U) \in \mathscr{B}_1'$ is a rough set. Then $X = f^{-1}(Y) \in \mathscr{B}_1^2$, where $X_L = f^{-1}(Y_L), X_U = f^{-1}(Y_U)$. Then the inverse image of B under f, denoted by $f^{-1}(B) = (f^{-1}(B_L), f^{-1}(B_U))$ is defined by

$$f^{-1}(B_L)(x) = B_L(f(x)) \text{ for every } x \in X_L \text{ and}$$
$$f^{-1}(B_U)(x) = B_U(f(x)) \text{ for every } x \in X_U.$$

Proposition 2.6 ([5])**.** *Let V, V_1 be any two sets and $\mathscr{B}$, $\mathscr{B}_1$ be any two Boolean subalgebra of the Boolean algebra of all subset of V, V_1. If $f : V \to V_1$ be such that $f^{-1} : (V_1, \mathscr{B}_1) \to (V, \mathscr{B})$. Then for all fuzzy rough sets $B_i, i \in J$ in Y we have*

(i) $f^{-1}(\bar{B}) \supset \overline{f^{-1}(B)}$,
(ii) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$,
(iii) *If $g : V_1 \to V_2$ be a mapping such that $g^{-1} : (V_2, \mathscr{B}_2) \to (V_1, \mathscr{B}_1)$, then $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$, for any FRS C in Z where $Z = (Z_L, Z_U) \in \mathscr{B}_2^2$ is a rough set and $g \circ f$ is the composition of g and f,*
(iv) $f^{-1}(\cup_i B_i) = \cup_i f^{-1}(B_i)$,
(v) $f^{-1}(\cap_i B_i) = \cap_i f^{-1}(B_i)$.

Proposition 2.7 ([5])**.** *If A and B are fuzzy rough sets, then*

(i) $(A \cup B)' = A' \cap B'$,
(ii) $(A \cap B)' = A' \cup B'$.

Moreover, if L is a complete lattice, for each family $A = \{A_j\}_{j\in J}$ of fuzzy rough sets,

(i) $(\cup_{j\in J} A_j)' = \cap_{j\in J}(A_j)'$,
(ii) $(\cap_{j\in J} A_j)' = \cup_{j\in J}(A_j)'$.

Theorem 2.8 ([5])**.** *If A be any fuzzy rough set in X, $\tilde{0} = (0_L, 0_U)$ be the null fuzzy rough set and $\tilde{1} = (1_L, 1_U)$ be the whole fuzzy rough set in X, then (i) $\tilde{0} \subset A \subset \tilde{1}$ and (ii) $\bar{\tilde{0}} = \tilde{1}$, $\bar{\tilde{1}} = \tilde{0}$.*

Definition 2.9 ([4])**.** Let X be a group and G be a fuzzy set in X with membership function μ_G. Then G is a fuzzy group in X iff the following conditions are satisfied:

(i) $\mu_G(xy) \geq min\{\mu_G(x), \mu_G(y)\}$, for all $x, y \in X$;
(ii) $\mu_G(x^{-1}) \geq \mu_G(x)$, for all $x \in X$.

Definition 2.10 ([14])**.** Let λ be a fuzzy set X in an fuzzy topological space. Then, the boundary of λ, is defined as $Bd(\lambda) = Cl(\lambda) \cap Cl(\lambda')$. Obviously, $Bd(\lambda)$ is a fuzzy closed set.

Definition 2.11 ([4])**.** Let A be a fuzzy set in X and $\mathscr{T}$ be a fuzzy topology on X. Then the induced fuzzy topology on A is the family of fuzzy subsets of A which are the intersections with A of $\mathscr{T}$-open fuzzy sets in X. The induced fuzzy topology is denoted by $\mathscr{T}_A$, and the pair $(A, \mathscr{T}_A)$ is called a fuzzy subspace of $(X, \mathscr{T})$.

Definition 2.12 ([4])**.** Let $(A, \mathscr{T}_A)$ and $(B, \mathscr{U}_B)$ be fuzzy subspaces of fuzzy topological spaces $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ respectively. Then a mapping f of $(A, \mathscr{T}_A) \to (B, \mathscr{U}_B)$ is relatively fuzzy continuous if and only if for each open fuzzy set V' in $\mathscr{U}_B$, the intersection $f^{-1}[V'] \cap A$ is in $\mathscr{T}_A$.

Definition 2.13 ([4])**.** Let X be a group and $\mathscr{T}$ be a fuzzy topology on X. Let G be a fuzzy group in X and let G be endowed with the induced fuzzy topology $\mathscr{T}_G$. Then G is a fuzzy topological group in X if and only if it satisfies the following two conditions:

- (i) The mapping $\alpha : (x, y) \to xy$ of $(G, \mathscr{T}_G) \times (G, \mathscr{T}_G)$ into $(G, \mathscr{T}_G)$ is relatively fuzzy continuous.
- (ii) The mapping $\beta : x \to x^{-1}$ of $(G, \mathscr{T}_G)$ into $(G, \mathscr{T}_G)$ is relatively fuzzy continuous.

Definition 2.14 ([4])**.** Let X and Y be fuzzy spaces. The fuzzy product space of X and Y is the cartesian product $X \times Y$ of sets X and Y together with the fuzzy topology $\tau X \times Y$ generated by the family $\{\ p_1^{-1}(\lambda_\alpha)\ ,\ p_2^{-1}(\mu_\beta) \mid \lambda_\alpha \ \in\ \tau X, \mu_\beta \in \tau Y$, where p_1 and p_2 are projections of $X \times Y$ onto X and Y, respectively$\}$.

Note 2.3 ([1])**.** For a mapping $f : X \to Y$, the graph $g : X \to X \times Y$ of f is defined by $g(x) = (x, f(x))$, for each $x \in X$.

Definition 2.15 ([1])**.** A fuzzy space X is product related to another fuzzy topological space Y if for any fuzzy set υ of X and ζ of Y whenever $\lambda' \not\supseteq \upsilon$ and $\mu' \not\supseteq \zeta$ implies $(\lambda' \times \tilde{1}) \cup (\tilde{1} \times \mu') \geq \upsilon \times \zeta$, where $\lambda \in \tau X$ and $\mu \in \tau Y$, there exist $\lambda_1 \in \tau X$ and $\mu_1 \in \tau Y$ such that $\lambda'_1 \supseteq \upsilon$ or $\mu'_1 \geq \zeta$ and $(\lambda'_1 \times \tilde{1}) \cup (\tilde{1} \times \mu'_1) = (\lambda'_1 \times \tilde{1}) \cup (\tilde{1} \times \mu'_1)$.

Definition 2.16 ([13])**.** A fuzzy rough topology on a rough set X is a family T of fuzzy rough sets in X which satisfies the following conditions:

- (i) $\tilde{0}, \tilde{1} \in T$.
- (ii) If $A, B \in T$, then $A \cap B \in T$.
- (iii) If $A_j \in T$ for all $j \in J$, then $\cup_{j \in J} A_j \in T$.

Then the pair (X, T) is called a fuzzy rough topological space and any fuzzy rough set in T is called a fuzzy rough open set in X. The complement A' of a fuzzy rough open set A is a fuzzy rough closed set.

Definition 2.17 ([13])**.** Let (X, T), (Y, S) be any two fuzzy rough topological spaces. A function $f : (X, T) \to (Y, S)$ is said to be fuzzy rough continuous iff for each fuzzy rough open set W in S the inverse image $f^{-1}(W)$ is fuzzy rough open in T.

3. Fuzzy rough $\mathcal{G}$ structure space and fuzzy rough $\mathcal{BG}$-boundary

Definition 3.1. Let X be a rough set. Then X is said to be a rough group if X_L and X_U are groups.

Definition 3.2. A fuzzy rough set $A = (A_L, A_U)$ in X is characterized by a pair of maps $A_L : X_L \to I$ and $A_U : X_U \to I$ with $A_L(x) \leq A_U(x)$ for every $x \in X_U$. The collection of all fuzzy rough sets in X is denoted by $FRS(X)$.

Definition 3.3. Let X be a rough group. A fuzzy rough set $G = (G_L, G_U)$ on X is said to be a fuzzy rough group if and only if it satisfies the following conditions:

(i) $G_L(\, x\, y\,) \geq min\{\, G_L(x), G_L(y\,)\}$ and $G_U(\, x\, y\,) \geq min\{G_U(x), G_U(y)\}$ for all $x, y \in X$.
(ii) $G_L(x^{-1}) \geq G_L(x)$ and $G_U(x^{-1}) \geq G_U(x)$ for all $x \in X$.

Definition 3.4. Let A be a fuzzy rough set in X and T be a fuzzy rough topology on X. Then the fuzzy rough subspace topology on A is the family of fuzzy rough subsets of A which are the intersections with A of fuzzy rough open sets in X. The fuzzy rough subspace topology is denoted by T_A, and the pair (A, T_A) is called a fuzzy rough subspace of (X, T)

Definition 3.5. Let (A, T_A) and (B, S_B) be any two fuzzy rough subspaces of fuzzy rough topological spaces (X, T), (Y, S) respectively. A function $f : (A, T_A) \to (B, S_B)$ is said to be a relatively fuzzy rough continuous function if and only if for each fuzzy rough open $\acute{V} = V \cap B$ in S_B, the intersection $f^{-1}(\acute{V}) \cap A$ is fuzzy rough open in T_A.

Definition 3.6. Let X be a rough group and T be a fuzzy rough topology on X. Let G be any fuzzy rough group in X and let G be endowed with the fuzzy rough subspace topology T_G. Then G is a fuzzy rough topological group in X if and only if it satisfies the following two conditions:

(i) The mapping $\alpha : (x, y) \to xy$ of $(G, T_G) \times (G, T_G)$ into (G, T_G) is relatively fuzzy rough continuous.
(ii) The mapping $\beta : x \to x^{-1}$ of (G, T_G) into (G, T_G) is relatively fuzzy rough continuous.

Definition 3.7. Let X be a non empty. A family $\mathcal{G}$ is a fuzzy rough topological groups in X satisfies the following conditions:

(i) $\tilde{0}, \tilde{1} \in \mathcal{G}$.
(ii) If $A, B \in \mathcal{G}$, then $A \cap B \in \mathcal{G}$.
(iii) If $A_j \in \mathcal{G}$ for all $j \in J$, then $\cup_{j \in J} A_j \in \mathcal{G}$.

Then $\mathcal{G}$ is said to be a fuzzy rough topological group structure on X and the pair $(X, \mathcal{G})$ is said to be a fuzzy rough topological group (in short, fuzzy rough $\mathcal{G}$) structure space. Any member of fuzzy rough $\mathcal{G}$ structure space is called a fuzzy rough open group. The complement of fuzzy rough open group is a fuzzy rough closed group.

Definition 3.8. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let $A = (A_L, A_U)$ be any fuzzy rough topological group. Then the fuzzy rough $\mathcal{G}$ interior of A is defined by

$$FR\mathcal{G}int(A) = \cup\{B : B \text{ is a fuzzy rough open group and } B \subseteq A\}.$$

Definition 3.9. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let $A = (A_L, A_U)$ be any fuzzy rough topological group. Then the fuzzy rough $\mathcal{G}$ closure of A is defined by

$$FR\mathcal{G}cl(A) = \cap\{B : B \text{ is a fuzzy rough closed group and } B \supseteq A\}.$$

Definition 3.10. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let A be any fuzzy rough topological group. Then A is said to be a fuzzy rough t-open group if $FR\mathcal{G}int(A) = FR\mathcal{G}int(FR\mathcal{G}cl(A))$

Definition 3.11. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let A be any fuzzy rough topological group. Then A is said to be a fuzzy rough $\mathcal{B}$-open group if $A = B \cap C$ where B is a fuzzy rough open group and C is a fuzzy rough t-open group. The complement of fuzzy rough B-open group is a fuzzy rough $\mathcal{B}$-closed group.

Definition 3.12. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let $A = (A_L, A_U)$ be any fuzzy rough topological group. Then the fuzzy rough $\mathcal{BG}$ interior of A is defined by

$$FR\mathcal{BG}int(A) = \cup\{B : B \text{ is a fuzzy rough } \mathcal{B}\text{-open group in } X \text{ and } B \subseteq A\}.$$

Definition 3.13. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let $A = (A_L, A_U)$ be any fuzzy rough topological group. Then the fuzzy rough $\mathcal{BG}$ closure of A is defined by

$$FR\mathcal{BG}cl(A) = \cap\{B : B \text{ is a fuzzy rough } \mathcal{B}\text{-closed group in } X \text{ and } B \supseteq A\}.$$

Proposition 3.14. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let A be any fuzzy rough topological group. Then the following conditions hold:

(i) $FR\mathcal{BG}int(A) \subseteq A \subseteq FR\mathcal{BG}cl(A)$.
(ii) $(FR\mathcal{BG}int(A))' = FR\mathcal{BG}cl(A')$.
(iii) $(FR\mathcal{BG}cl(A))' = FR\mathcal{BG}int(A')$.

Proof. The proof follows from Definition 3.12 and Definition 3.13. □

Definition 3.15. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let A be any fuzzy rough topological group. Then the fuzzy rough $\mathcal{G}$-boundary of A, is denoted and defined as

$$FR\mathcal{G}bd(A) = FR\mathcal{G}cl(A) \cap FR\mathcal{G}cl(A').$$

Definition 3.16. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let A be any fuzzy rough topological group. Then the fuzzy rough $\mathcal{BG}$-boundary of A, is denoted and defined as

$$FR\mathcal{BG}bd(A) = FR\mathcal{BG}cl(A) \cap FR\mathcal{BG}cl(A').$$

Proposition 3.17. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let A and B be any two fuzzy rough topological groups. Then the following conditions hold:

(i) $FR\mathcal{BG}bd(A) = FR\mathcal{BG}bd(A')$.
(ii) If A is a fuzzy rough closed group, then $FR\mathcal{BG}bd(A) \subseteq A$.
(iii) If A is a fuzzy rough open group, then $FR\mathcal{BG}bd(A) \subseteq A'$.
(iv) Let $A \subseteq B$ and B be any fuzzy rough closed group (resp., A be any fuzzy rough open group). Then $FR\mathcal{BG}bd(A) \subseteq B$ (resp., $FR\mathcal{BG}bd(A) \subseteq B'$).
(v) $(FR\mathcal{BG}bd(A))' = FR\mathcal{BG}int(A) \cup FR\mathcal{BG}int(A')$.

Proof. (i)
$$\begin{aligned} FR\mathcal{BG}bd(A) &= FR\mathcal{BG}cl(A) \cap FR\mathcal{BG}cl(A') \\ &= FR\mathcal{BG}cl(A') \cap FR\mathcal{BG}cl(A) \\ &= FR\mathcal{BG}cl(A') \cap FR\mathcal{BG}cl(A')' \\ &= FR\mathcal{BG}bd(A'). \end{aligned}$$

(ii)
$$\begin{aligned} FR\mathcal{BG}bd(A) &= FR\mathcal{BG}cl(A) \cap FR\mathcal{BG}cl(A') \\ &\subseteq FR\mathcal{BG}cl(A) \\ &\subseteq A. \end{aligned}$$

Hence, $FR\mathcal{BG}bd(A) \subseteq A$.

(iii) Let A be any fuzzy rough $\mathcal{B}$-open group. Then, A' is fuzzy rough $\mathcal{B}$-closed group. By (ii), $FR\mathcal{BG}bd(A') \subseteq A'$ and by (i), $FR\mathcal{BG}bd(A) \subseteq A'$.

(iv) Since $A \subseteq B$ implies that $FR\mathcal{BG}cl(A) \subseteq FR\mathcal{BG}cl(B)$, we have
$$\begin{aligned} FR\mathcal{BG}bd(A) &= FR\mathcal{BG}cl(A) \cap FR\mathcal{BG}cl(A') \\ &\subseteq FR\mathcal{BG}cl(B) \cap FR\mathcal{BG}cl(A') \\ &\subseteq FR\mathcal{BG}cl(B) \\ &= B, \text{ since } B \text{ is a } \mathcal{B}\text{-closed group.} \end{aligned}$$

(v)
$$\begin{aligned} (FR\mathcal{BG}bd(A))' &= (FR\mathcal{BG}cl(A) \cap FR\mathcal{BG}cl(A'))' \\ &= (FR\mathcal{BG}cl(A))' \cup (FR\mathcal{BG}cl(A'))' \\ &= FR\mathcal{BG}int(A') \cup FR\mathcal{BG}int(A). \end{aligned}$$
□

Definition 3.18. Let A and B be any two fuzzy rough topological groups. Then $A - B$ is defined by $A - B = A \cap B'$.

Proposition 3.19. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let A be any fuzzy rough topological group. Then the following conditions hold:

(i) $FR\mathcal{BG}bd(A) = FR\mathcal{BG}cl(A) - FR\mathcal{BG}int(A)$.
(ii) $FR\mathcal{BG}bd(FR\mathcal{BG}int(A)) \subseteq FR\mathcal{BG}bd(A)$.
(iii) $FR\mathcal{BG}bd(FR\mathcal{BG}cl(A)) \subseteq FR\mathcal{BG}bd(A)$.
(iv) $FR\mathcal{BG}int(A) \subset A - FR\mathcal{BG}bd(A)$.

Proof. (i) Since $(FR\mathcal{BG}cl(A'))' = FR\mathcal{BG}int(A)$. Therefore,
$$\begin{aligned} FR\mathcal{BG}bd(A) &= FR\mathcal{BG}cl(A) \cap FR\mathcal{BG}cl(A') \\ &= FR\mathcal{BG}cl(A) - (FR\mathcal{BG}cl(A'))' \\ &= FR\mathcal{BG}cl(A) - FR\mathcal{BG}int(A). \end{aligned}$$

Thus, $FR\mathcal{BG}bd(A) = FR\mathcal{BG}cl(A) - FR\mathcal{BG}int(A)$. Hence (i).

(ii)
$$\begin{aligned} FR\mathcal{BG}bd(FR\mathcal{BG}int(A)) &= FR\mathcal{BG}cl(FR\mathcal{BG}int(A)) - FR\mathcal{BG}int(FR\mathcal{BG}int(A)) \\ &= FR\mathcal{BG}cl(FR\mathcal{BG}int(A)) - FR\mathcal{BG}int(A) \\ &\subseteq FR\mathcal{BG}cl(A) - FR\mathcal{BG}int(A) \\ &= FR\mathcal{BG}bd(A). \end{aligned}$$

(iii) $FR\mathcal{BG}bd(FR\mathcal{BG}cl(A))$
$$\begin{aligned} &= FR\mathcal{BG}cl(FR\mathcal{BG}cl(A)) - FR\mathcal{BG}int(FR\mathcal{BG}cl(A)) \\ &= FR\mathcal{BG}cl(A) - FR\mathcal{BG}int(FR\mathcal{BG}cl(A)) \\ &\subseteq FR\mathcal{BG}cl(A) - FR\mathcal{BG}int(A) \\ &= FR\mathcal{BG}bd(A). \end{aligned}$$

(iv)
$$\begin{aligned} A - FR\mathcal{BG}bd(A) &= A \cap (FR\mathcal{BG}bd(A))' \\ &= A \cap (FR\mathcal{BG}cl(A) \cap FR\mathcal{BG}cl(A'))' \\ &= A \cap (FR\mathcal{BG}int(A') \cup FR\mathcal{BG}int(A)) \end{aligned}$$

$$= (A \cap FR\mathcal{BG}int(A')) \cup (A \cap FR\mathcal{BG}int(A))$$
$$= (A \cap FR\mathcal{BG}int(A')) \cup FR\mathcal{BG}int(A)$$
$$\supseteq FR\mathcal{BG}int(A).$$ □

Remark 3.20. Let $\{A_\alpha\}_{\alpha \in J}$ be the family of fuzzy rough sets and J be an indexed set. Then for $\alpha \in J$,
$\cup_\alpha FRcl(A_\alpha) \subseteq FRcl(\cup_\alpha(A_\alpha))$
$\cup_\alpha FRint(A_\alpha) \subseteq FRint(\cup_\alpha(A_\alpha))$.
Also for any finite $n \in J$, $\cup_n FRcl(A_n) = FRcl(\cup_n(A_n))$.

Remark 3.21. Let (X, T) be a fuzzy rough topological space. Let A and B be any two fuzzy rough sets. Then $\cap_{i\in J} FR\mathcal{B}cl(A_i) \supseteq FR\mathcal{B}cl(\cap_{i\in J}(A_i))$, where J is an indexed set.

Proposition 3.22. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let A and B be any two fuzzy rough topological groups. Then, $FR\mathcal{BG}bd(A \cup B) \subseteq FR\mathcal{BG}bd(A) \cup FR\mathcal{BG}bd(B)$.

Proof. $FR\mathcal{BG}bd(A \cup B) = FR\mathcal{BG}cl(A \cup B) \cap FR\mathcal{BG}cl(A \cup B)'$
$$\subseteq (FR\mathcal{BG}cl(A) \cup FR\mathcal{BG}cl(B)) \cap (FR\mathcal{BG}cl(A') \cap FR\mathcal{BG}cl(B'))$$
$$= [FR\mathcal{BG}cl(A) \cap (FR\mathcal{BG}cl(A') \cap FR\mathcal{BG}cl(B'))] \cup [FR\mathcal{BG}cl(B) \cap (FR\mathcal{BG}cl(A') \cap FR\mathcal{BG}cl(B'))]$$
$$= (FR\mathcal{BG}bd(A) \cap FR\mathcal{BG}cl(B')) \cup (FR\mathcal{BG}bd(B) \cap FR\mathcal{BG}cl(A'))$$
$$\subseteq FR\mathcal{BG}bd(A) \cup FR\mathcal{BG}bd(B).$$ □

Proposition 3.23. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let A and B be any two fuzzy rough topological groups. Then, $FR\mathcal{BG}bd(A \cap B) \subseteq FR\mathcal{BG}bd(A) \cup FR\mathcal{BG}bd(B)$.

Proof. $FR\mathcal{BG}bd(A \cap B)$
$$= FR\mathcal{BG}cl(A \cap B) \cap FR\mathcal{BG}cl(A \cap B)'$$
$$\subseteq (FR\mathcal{BG}cl(A) \cap FR\mathcal{BG}cl(B)) \cap (FR\mathcal{BG}cl(A') \cup FR\mathcal{BG}cl(B'))$$
$$= [(FR\mathcal{BG}cl(A) \cap FR\mathcal{BG}cl(B)) \cap FR\mathcal{BG}cl(A'))] \cup [(FR\mathcal{BG}cl(A) \cap FR\mathcal{BG}cl(B)) \cap FR\mathcal{BG}cl(B')]$$
$$= (FR\mathcal{BG}bd(A) \cap FR\mathcal{BG}cl(B)) \cup (FR\mathcal{BG}bd(B) \cap FR\mathcal{BG}cl(A))$$
$$\subseteq FR\mathcal{BG}bd(A) \cup FR\mathcal{BG}bd(B).$$ □

Proposition 3.24. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let A be any fuzzy rough topological group. Then the following conditions hold:

(i) $FR\mathcal{BG}bd(FR\mathcal{BG}bd(A)) \subseteq FR\mathcal{BG}bd(A)$.
(ii) $FR\mathcal{BG}bd(FR\mathcal{BG}bd(FR\mathcal{BG}bd(A))) \subseteq FR\mathcal{BG}bd(FR\mathcal{BG}bd(A))$.

Proof. (i) $FR\mathcal{BG}bd(FR\mathcal{BG}bd(A))$
$$= FR\mathcal{BG}cl(FR\mathcal{BG}bd(A)) \cap FR\mathcal{BG}cl(FR\mathcal{BG}bd(A))'$$
$$\subseteq FR\mathcal{BG}cl(FR\mathcal{BG}bd(A))$$
$$= FR\mathcal{BG}bd(A).$$

(ii) $FR\mathcal{BG}bd(FR\mathcal{BG}bd(FR\mathcal{BG}bd(A)))$
$$= FR\mathcal{BG}cl(FR\mathcal{BG}bd(FR\mathcal{BG}bd(A))) \cap FR\mathcal{BG}cl(FR\mathcal{BG}bd(FR\mathcal{BG}bd(A)))'$$
$$= FR\mathcal{BG}bd(FR\mathcal{BG}bd(A)) \cap (FR\mathcal{BG}cl(FR\mathcal{BG}bd(FR\mathcal{BG}bd(A))))'$$
$$\subseteq FR\mathcal{BG}bd(FR\mathcal{BG}bd(A)).$$ □

Definition 3.25. Let $A = (A_L, A_U)$ be a fuzzy rough topological group of X and $B = (B_L, B_U)$ be a fuzzy rough topological group of Y, then the fuzzy rough topological group $A \times B = (A_L \times B_L, A_U \times B_U)$ of $X \times Y$ is defined by

$$(A_L \times B_L)(x, y) = min\{A_L(x), B_L(y)\} \text{ for every } (x, y) \in X_L \times Y_L \text{ and}$$

$$(A_U \times B_U)(x, y) = min\{A_U(x), B_U(y)\} \text{ for every } (x, y) \in X_U \times Y_U.$$

Note 3.1. Let A and B be any two fuzzy rough topological groups in X and Y then, $(A \times B)' = (1_L - (A_L \times B_L), 1_U - (A_U \times B_U))$.

Proposition 3.26. If $A = (A_L, A_U)$ is a fuzzy rough topological group of X and $B = (B_L, B_U)$ is a fuzzy rough topological group of Y, then $(A \times B)' = A' \times \tilde{1} \cup \tilde{1} \times B'$.

Proof. Since,

$$\begin{aligned}
A_L \times B_L(x, y) &= min(A_L(x), B_L(y)), \quad \text{for every } (x, y) \in X_L \times Y_L \\
1_L - (A_L \times B_L)(x, y) &= max(1 - A_L(x), 1 - B_L(y)) \\
&= max(A'_U(x), B'_U(y)) \\
&= max((A'_U \times 1_U)(x, y), (1_U \times B'_U)(x, y)) \\
1_L - (A_L \times B_L) &= A'_U \times 1_U \cup 1_U \times B'_U
\end{aligned}$$

and similarly $1_U - (A_U \times B_U) = A'_L \times 1_L \cup 1_L \times B'_L$. This implies that,

$$(A \times B)' = A' \times \tilde{1} \cup \tilde{1} \times B'.$$

□

Note 3.2. (i) $(A \times \tilde{1}) \cap (\tilde{1} \times B) = A \times B$.
(ii) $(A \times \tilde{1}) \cap (\tilde{1} \times B) = (A' \times B')'$.

Definition 3.27. Let $(X, \mathcal{G}_1)$ and $(Y, \mathcal{G}_2)$ be any two fuzzy rough structure spaces. The fuzzy rough product $\mathcal{G}$ structure space of $(X, \mathcal{G}_1)$ and $(Y, \mathcal{G}_2)$ is the cartesian product $(X, \mathcal{G}_1) \times (Y, \mathcal{G}_2)$ of sets $(X, \mathcal{G}_1)$ and $(Y, \mathcal{G}_2)$ together with the fuzzy rough structure $\mathcal{G}_1 \times \mathcal{G}_2$ generated by the family, $\{ p_1^{-1}(A), p_2^{-1}(B) \mid A \in \mathcal{G}_1, B \in \mathcal{G}_2$, where p_1 and p_2 are projections of $(X, \mathcal{G}_1) \times (Y, \mathcal{G}_2)$ onto $(X, \mathcal{G}_1)$ and $(Y, \mathcal{G}_2)$, respectively$\}$.

Proposition 3.28. Let $A = (A_L, A_U)$ be a fuzzy rough $\mathcal{B}$-closed group of a fuzzy rough $\mathcal{G}_1$ structure space X and $B = (B_L, B_U)$ be a fuzzy rough $\mathcal{B}$-closed group of a fuzzy rough $\mathcal{G}_2$ structure space Y. Then $A \times B$ is a fuzzy rough $\mathcal{B}$-closed group of the fuzzy rough product $\mathcal{G}$ structure space $X \times Y$.

Proof. Let A and B be any fuzzy rough topological groups in X and Y. By Proposition 3.26, $\tilde{1} - (A \times B) = A' \times \tilde{1} \cup \tilde{1} \times B'$. Since $A' \times \tilde{1}$ and $\tilde{1} \times B'$ are fuzzy rough $\mathcal{B}$-open groups in X and Y respectively. $A' \times \tilde{1} \cup \tilde{1} \times B'$ is a fuzzy rough $\mathcal{B}$-open group of $X \times Y$. Hence, $\tilde{1} - (A \times B)$ is a fuzzy rough $\mathcal{B}$-open group of $X \times Y$. Consequently, $A \times B$ is a fuzzy rough $\mathcal{B}$-closed group of $X \times Y$. □

Proposition 3.29. If $A = (A_L, A_U)$ is a fuzzy rough topological group of a fuzzy rough $\mathcal{G}_1$ structure space X and $B = (B_L, B_U)$ is a fuzzy rough topological group of a fuzzy rough $\mathcal{G}_2$ structure space Y, then

(i) $FR\mathcal{B}\mathcal{G}_1cl(A) \times FR\mathcal{B}\mathcal{G}_2cl(B) \supseteq FR\mathcal{B}\mathcal{G}cl(A \times B)$.
(ii) $FR\mathcal{B}\mathcal{G}_1int(A) \times FR\mathcal{B}\mathcal{G}_2int(B) \subseteq FR\mathcal{B}\mathcal{G}int(A \times B)$.

Proof. (i) Since $A \subseteq FR\mathcal{B}\mathcal{G}_1cl(A)$ and $B \subseteq FR\mathcal{B}\mathcal{G}_2cl(B)$, $A \times B \subseteq FR\mathcal{B}\mathcal{G}_1cl(A) \times FR\mathcal{B}\mathcal{G}_2cl(B)$. Now, $FR\mathcal{B}\mathcal{G}cl(A \times B) \subseteq FR\mathcal{B}\mathcal{G}cl(FR\mathcal{B}\mathcal{G}_1cl(A) \times FR\mathcal{B}\mathcal{G}_2cl(B))$. By Proposition 3.28, $FR\mathcal{B}\mathcal{G}cl(A \times B) \subseteq FR\mathcal{B}\mathcal{G}_1cl(A) \times FR\mathcal{B}\mathcal{G}_2cl(B)$.

(ii) follows from (i) to the fact that $FR\mathcal{B}\mathcal{G}cl(A') = (FR\mathcal{B}\mathcal{G}int(A))'$ and $FR\mathcal{B}\mathcal{G}int(A') = (FR\mathcal{B}\mathcal{G}cl(A))'$ □

Definition 3.30. A fuzzy rough $\mathcal{G}_1$ structure space $(X, \mathcal{G}_1)$ is fuzzy rough $\mathcal{B}$-product related to another fuzzy rough $\mathcal{G}_2$ structure space $(Y, \mathcal{G}_2)$ if for any fuzzy rough topological group $C = (C_L, C_U)$ of X and $D = (D_L, D_U)$ of Y whenever $A' \not\supseteq C$ and $B' \not\supseteq D$ implies that $(A' \times \tilde{1}) \cup (\tilde{1} \times B') \supseteq C \times D$, where $A = (A_L, A_U)$ is a fuzzy rough $\mathcal{B}$-open group of X and $B = (B_L, B_U)$ is a fuzzy rough $\mathcal{B}$-open group of Y, there exist $A_1 \in \mathcal{G}_1$ and $B_1 \in \mathcal{G}_2$ such that $A'_1 \supseteq C$ or $B'_1 \supseteq D$ and $(A' \times \tilde{1}) \cup (\tilde{1} \times B') = (A'_1 \times \tilde{1}) \cup (\tilde{1} \times B'_1)$.

Proposition 3.31. Let $(X, \mathcal{G}_1)$ and $(Y, \mathcal{G}_2)$ be any two fuzzy rough structure spaces such that $(X, \mathcal{G}_1)$ is $\mathcal{B}$-product related to $(Y, \mathcal{G}_2)$. Then, for a fuzzy rough topological group $A = (A_L, A_U)$ of X and a fuzzy rough topological group $B = (B_L, B_U)$ of Y,

(i) $FR\mathcal{B}\mathcal{G}cl(A \times B) = FR\mathcal{B}\mathcal{G}_1cl(A) \times FR\mathcal{B}\mathcal{G}_2cl(B)$, and
(ii) $FR\mathcal{B}\mathcal{G}int(A \times B) = FR\mathcal{B}\mathcal{G}_1int(A) \times FR\mathcal{B}\mathcal{G}_2int(B)$.

Proof. (i) For fuzzy rough topological groups $A_i = (A_{L_i}, A_{U_i})$'s of X and $B_j = (B_{L_j}, B_{U_j})$'s of Y, we first note that,

(i) $inf\{A_i, B_j\} = min(inf(A_i), inf(B_j))$,
(ii) $inf\{A_i \times \tilde{1}\} = inf(A_i) \times \tilde{1}$,
(iii) $inf\{\tilde{1} \times B_j\} = \tilde{1} \times inf(B_j)$.

By Proposition 3.29, it follows that

$$FR\mathcal{B}\mathcal{G}_1cl(A) \times FR\mathcal{B}\mathcal{G}_2cl(B) \supseteq FR\mathcal{B}\mathcal{G}cl(A \times B). \tag{3.1}$$

It is sufficient to show that $FR\mathcal{B}\mathcal{G}cl(A \times B) \supseteq FR\mathcal{B}\mathcal{G}_1cl(A) \times FR\mathcal{B}\mathcal{G}_2cl(B)$. Let A_i be a fuzzy rough $\mathcal{B}$-open group in $\mathcal{G}_1$ and B_j be a fuzzy rough $\mathcal{B}$-open group in $\mathcal{G}_2$. Then,

$$\begin{aligned} FR\mathcal{B}\mathcal{G}cl(A \times B) &= inf\{(A_i \times B_j)' | (A_i \times B_j)' \supseteq A \times B\} \\ &= inf\{A'_i \times \tilde{1} \cup \tilde{1} \times B'_j | A'_i \times \tilde{1} \cup \tilde{1} \times B'_j \geq A \times B\} \\ &= inf\{A'_i \times \tilde{1} \cup \tilde{1} \times B'_j | A'_i \supseteq A \text{ or } B'_j \supseteq B\} \\ &= min(inf\{A'_i \times \tilde{1} \cup \tilde{1} \times B'_j | A'_i \supseteq A\}, inf\{A'_i \times \tilde{1} \cup \tilde{1} \times B'_j | B'_j \supseteq B\}). \end{aligned}$$

Since

$$\begin{aligned} inf\{A'_i \times \tilde{1} \cup \tilde{1} \times B'_j | A'_i \supseteq A\} &\supseteq inf\{A'_i \times \tilde{1} | A'_i \supseteq A\} \\ &= inf\{A'_i | A'_i \supseteq A\} \times \tilde{1} \\ &= FR\mathcal{B}\mathcal{G}_1cl(A) \times \tilde{1} \end{aligned}$$

and

$$\begin{aligned} inf\{A'_i \times \tilde{1} \cup \tilde{1} \times B'_j | B'_j \supseteq B\} &\supseteq inf\{\tilde{1} \times B'_j | B'_j \supseteq B\} \\ &= \tilde{1} \times inf\{B'_j | B'_j \supseteq B\} \\ &= \tilde{1} \times FR\mathcal{BG}_2 cl(B). \end{aligned}$$

We have, $FR\mathcal{BG}cl(A \times B) \supseteq min(FR\mathcal{BG}_1 cl(A') \times \tilde{1}, \tilde{1} \times FR\mathcal{BG}_2 cl(B')) = FR\mathcal{BG}_1 cl(A) \times FR\mathcal{BG}_2 cl(B)$.

$$FR\mathcal{BG}cl(A \times B) \supseteq FR\mathcal{BG}_1 cl(A) \times FR\mathcal{BG}_2 cl(B) \tag{3.2}$$

From (3.1) and (3.2),

$$FR\mathcal{BG}cl(A \times B) = FR\mathcal{BG}_1 cl(A) \times FR\mathcal{BG}_2 cl(B).$$

(ii) The proof is similar to that of (i) and Proposition 3.29. □

Proposition 3.32. Let A, B, C and D be fuzzy rough topological groups in X. Then $(A \cap B) \times (C \cap D) = (A \times D) \cap (B \times C)$.

Proof.

$$\begin{aligned} ((A_L \cap B_L) \times (C_L \cap D_L))(x, y) &= min((A_L \cap B_L)(x), (C \cap D)(y)) \\ &= min(min(A_L(x), B_L(x)), min(C_L(y), D_L(y))) \\ &= min(min(A_L(x), D_L(y)), min(B_L(x), C_L(y))) \\ &= min((A_L \times D_L)(x, y), (B_L \times C_L)(x, y)) \\ &= ((A_L \times D_L) \cap (B_L \times C_L))(x, y) \\ &\quad \text{for all } (x, y) \in X_L \times X_L. \end{aligned}$$

Similarly,

$((A_U \cap B_U) \times (C_U \cap D_U))(x, y) = ((A_U \times D_U) \cap (B_U \times C_U))(x, y)$ for all $(x, y) \in X_U \times X_U$.

Hence, $(A \cap B) \times (C \cap D) = (A \times D) \cap (B \times C)$. □

Proposition 3.33. Let $(X, \mathcal{G}_i)$ (i=1,2,...,n) be a family of fuzzy rough product related structures spaces. If each A_i is a fuzzy rough topological groups in X_i, then

$$\begin{aligned} FR\mathcal{BG}_i bd \prod_{i=1}^{n} (A_i) = &[FR\mathcal{BG}_1 bd A_1 \times FR\mathcal{BG}_2 cl(A_2) \times .... \times FR\mathcal{BG}_n cl A_n)] \\ &\cup [FR\mathcal{BG}_1 cl A_1 \times FR\mathcal{BG}_2 bd(A_2) \times .... \times FR\mathcal{BG}_n cl A_n)] \\ &\cup ... \cup [FR\mathcal{BG}_1 cl A_1 \times FR\mathcal{BG}_2 cl(A_2) \times .... \times FR\mathcal{BG}_n bd A_n)]. \end{aligned}$$

Proof. We use Propositions 3.19, 3.31 and 3.32 to prove this. It suffices to prove this for n=2. Consider

$$\begin{aligned} &FR\mathcal{BG}_n bd(A_1 \times A_2) \\ &= FR\mathcal{BG}_n cl(A_1 \times A_2) - FR\mathcal{BG}_n int(A_1 \times A_2) \\ &= (FR\mathcal{BG}_1 cl(A_1) \times FR\mathcal{BG}_2 cl(A_2)) - (FR\mathcal{BG}_1 int(A_1) \times FR\mathcal{BG}_2 int(A_2)) \\ &= (FR\mathcal{BG}_1 cl(A_1) \times FR\mathcal{BG}_2 cl(A_2)) - (FR\mathcal{BG}_1 int(A_1) \cap FR\mathcal{BG}_1 cl(A_1)) \\ &\qquad \times (FR\mathcal{BG}_2 int(A_2) \cap FR\mathcal{BG}_2 cl(A_2)) \\ &= (FR\mathcal{BG}_1 cl(A_1) \times FR\mathcal{BG}_2 cl(A_2)) - (FR\mathcal{BG}_1 int(A_1) \times FR\mathcal{BG}_2 cl(A_2)) \\ &\qquad \cap (FR\mathcal{BG}_1 cl(A_1) \times FR\mathcal{BG}_2 int(A_2)) \text{ (by Proposition 3.32)} \end{aligned}$$

$$= [(FR\mathcal{BG}_1cl(A_1)\times FR\mathcal{BG}_2cl(A_2))-(FR\mathcal{BG}_1int(A_1)\times FR\mathcal{BG}_2cl(A_2))]$$
$$\cup[(FR\mathcal{BG}_1cl(A_1)\times FR\mathcal{BG}_2cl(A_2))-(FR\mathcal{BG}_1cl(A_1)\times FR\mathcal{BG}_2int(A_2))]$$
$$= [(FR\mathcal{BG}_1cl(A_1) - FR\mathcal{BG}_1int(A_1)) \times FR\mathcal{BG}_2cl(A_2)]$$
$$\cup[FR\mathcal{BG}_1cl(A_1) \times (FR\mathcal{BG}_2cl(A_2) - FR\mathcal{BG}_2int(A_2))]$$
$$= (FR\mathcal{BG}_1bd(A_1) \times FR\mathcal{BG}_2cl(A_2)) \cup (FR\mathcal{BG}_1cl(A_1) \times FR\mathcal{BG}_2bd(A_2)).$$

□

Definition 3.34. Let $(X, \mathcal{G}_1)$ and $(Y, \mathcal{G}_2)$ be any two fuzzy rough structure spaces. A function $f : (X, \mathcal{G}_1) \to (Y, \mathcal{G}_2)$ is said to be fuzzy rough $\mathcal{BG}$-continuous if and only if for each fuzzy rough open group W in $\mathcal{G}_2$ the inverse image $f^{-1}(W)$ is a fuzzy rough $\mathcal{B}$-open group in $\mathcal{G}_1$.

Proposition 3.35. Let $(X, \mathcal{G}_1)$ and $(Y, \mathcal{G}_2)$ be any two fuzzy rough structure spaces. Let $f : (X, \mathcal{G}_1) \to (Y, \mathcal{G}_2)$ be a fuzzy rough $\mathcal{BG}$-continuous function. Then,

$$FR\mathcal{BG}bd(f^{-1}(A)) \subseteq f^{-1}(FR\mathcal{G}bd(A)).$$

Proof. Let f be a fuzzy rough $\mathcal{BG}$-continuous function. Let A be any fuzzy rough topological group in $(Y, \mathcal{G}_2)$. Then, $FR\mathcal{G}cl(A)$ is a fuzzy rough $\mathcal{G}$-closed group in $(Y, \mathcal{G}_2)$, which implies that $f^{-1}(FR\mathcal{G}cl(A))$ is a fuzzy rough $\mathcal{BG}$-closed group in $(X, \mathcal{G}_1)$. Therefore,

$$\begin{aligned}FR\mathcal{BG}bd(f^{-1}(A)) &= FR\mathcal{BG}cl(f^{-1}(A)) \cap FR\mathcal{BG}cl(f^{-1}(A))'\\ &\subseteq FR\mathcal{BG}cl(f^{-1}(FR\mathcal{G}cl(A))) \cap FR\mathcal{BG}cl(f^{-1}(FR\mathcal{G}cl(A')))\\ &= f^{-1}(FR\mathcal{G}cl(A)) \cap f^{-1}(FR\mathcal{G}cl(A'))\\ &= f^{-1}(FR\mathcal{G}cl(A) \cap FR\mathcal{G}cl(A'))\\ &= f^{-1}(FR\mathcal{G}bd(A)).\end{aligned}$$

Therefore, $FR\mathcal{BG}bd(f^{-1}(A)) \subseteq f^{-1}(FR\mathcal{G}bd(A))$. □

Definition 3.36. Let $(X, \mathcal{G}_1)$ and $(Y, \mathcal{G}_2)$ be any two fuzzy rough structure spaces. A function $f : (X, \mathcal{G}_1) \to (Y, \mathcal{G}_2)$ is said to be fuzzy rough $\mathcal{BG}$-irresolute if and only if for each fuzzy rough $\mathcal{B}$-open group W in $\mathcal{G}_2$ the inverse image $f^{-1}(W)$ is a fuzzy rough $\mathcal{B}$-open group in $\mathcal{G}_1$.

Proposition 3.37. Let $(X, \mathcal{G}_1)$ and $(Y, \mathcal{G}_2)$ be any two fuzzy rough structure spaces. Let $f : (X, \mathcal{G}_1) \to (Y, \mathcal{G}_2)$ be a fuzzy rough $\mathcal{BG}$-irresolute function. Then,

$$FR\mathcal{BG}bd(f^{-1}(A)) \subseteq f^{-1}(FR\mathcal{BG}bd(A)).$$

Proof. Let f be a fuzzy rough $\mathcal{BG}$-irresolute function. Let A be any fuzzy rough topological group in $(Y, \mathcal{G}_2)$. Then, $FR\mathcal{BG}cl(A)$ is a fuzzy rough $\mathcal{BG}$-closed group in $(Y, \mathcal{G}_2)$, which implies that $f^{-1}(FR\mathcal{BG}cl(A))$ is a fuzzy rough $\mathcal{BG}$-closed group in

$(X, \mathcal{G}_1)$. Therefore,

$$\begin{aligned} FR\mathcal{BG}bd(f^{-1}(A)) &= FR\mathcal{BG}cl(f^{-1}(A)) \cap FR\mathcal{BG}cl(f^{-1}(A))' \\ &\subseteq FR\mathcal{BG}cl(f^{-1}(FR\mathcal{BG}cl(A))) \cap FR\mathcal{BG}cl(f^{-1}(FR\mathcal{BG}cl(A'))) \\ &= f^{-1}(FR\mathcal{BG}cl(A)) \cap f^{-1}(FR\mathcal{BG}cl(A')) \\ &= f^{-1}(FR\mathcal{BG}cl(A) \cap FR\mathcal{BG}cl(A')) \\ &= f^{-1}(FR\mathcal{BG}bd(A)). \end{aligned}$$

Therefore, $FR\mathcal{BG}bd(f^{-1}(A)) \subseteq f^{-1}(FR\mathcal{BG}bd(A))$. □

4. Characterization of fuzzy rough $\mathcal{BG}$-boundary spaces

Definition 4.1. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let $FR\mathcal{BG}bd(A)$ be the fuzzy rough $\mathcal{BG}$-boundary of A. Then the fuzzy rough $\mathcal{BG}$-interior of $FR\mathcal{BG}bd(A)$ is defined by

$$FR\mathcal{BG}^\circ(FR\mathcal{BG}bd(A)) = \cup\{B : B \text{ is a fuzzy rough } \mathcal{B}\text{-open group and } B \subseteq FR\mathcal{BG}bd(A)\}.$$

Definition 4.2. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let $FR\mathcal{BG}bd(A)$ be the fuzzy rough $\mathcal{BG}$-boundary of A. Then the fuzzy rough $\mathcal{BG}$-closure of $FR\mathcal{BG}bd(A)$ is defined by

$$FR\mathcal{BG}^\neg(FR\mathcal{BG}bd(A)) = \cap\{B : B \text{ is a fuzzy rough } \mathcal{B}\text{-closed group and } B \supseteq FR\mathcal{BG}bd(A)\}.$$

Proposition 4.3. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let $FR\mathcal{BG}bd(A)$ be the fuzzy rough $\mathcal{BG}$-boundary of A. Then the following conditions hold.

(i) $FR\mathcal{BG}^\circ(FR\mathcal{BG}bd(A)) \subseteq FR\mathcal{BG}bd(A) \subseteq FR\mathcal{BG}^\neg(FR\mathcal{BG}bd(A))$.
(ii) $(FR\mathcal{BG}^\circ(FR\mathcal{BG}bd(A)))' = FR\mathcal{BG}^\neg(FR\mathcal{BG}bd(A)')$.
(iii) $(FR\mathcal{BG}^\neg(FR\mathcal{BG}bd(A)))' = FR\mathcal{BG}^\circ(FR\mathcal{BG}bd(A)')$.

Proof. The proof follows from Definition 4.1 and Definition 4.2. □

Definition 4.4. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Then $(X, \mathcal{G})$ is said to be a fuzzy rough $\mathcal{BG}$-boundary space if the fuzzy rough $\mathcal{BG}$-closure of fuzzy rough $\mathcal{BG}$-boundary of each fuzzy rough open group is a fuzzy rough $\mathcal{B}$-open group. That is, $FR\mathcal{BG}^\neg(FR\mathcal{BG}bd(A))$ is fuzzy rough $\mathcal{B}$-open group for every $A \in \mathcal{G}$.

Proposition 4.5. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Then the following statements are equivalent:

(i) $(X, \mathcal{G})$ is a fuzzy rough $\mathcal{BG}$-boundary space.
(ii) Let $FR\mathcal{BG}bd(A)$ be fuzzy rough $\mathcal{BG}$-boundary of A. Then $FR\mathcal{BG}^\circ(FR\mathcal{BG}bd(A))$ is a fuzzy rough $\mathcal{B}$-closed group.
(iii) For each $FR\mathcal{BG}bd(A)$,

$$FR\mathcal{BG}^\neg(FR\mathcal{BG}bd(A)) + FR\mathcal{BG}^\neg(FR\mathcal{BG}^\neg(FR\mathcal{BG}bd(A)))' = \tilde{1}.$$

(iv) For every pair of fuzzy rough $\mathcal{BG}$-boundary sets $FR\mathcal{BG}bd(A)$ and $FR\mathcal{BG}bd(B)$ with $FR\mathcal{BG}^\neg(FR\mathcal{BG}bd(A)) + FR\mathcal{BG}bd(B) = \tilde{1}$, we have $FR\mathcal{BG}^\neg(FR\mathcal{BG}bd(A)) + FR\mathcal{BG}^\neg(FR\mathcal{BG}bd(B)) = \tilde{1}$.

Proof. (i)⇒(ii): Let $FR\mathcal{BG}bd(A)$ be the fuzzy rough $\mathcal{BG}$ boundary of A. Then, $(FR\mathcal{BG}bd(A))'$ is a fuzzy rough boundary complement of $FR\mathcal{BG}bd(A)$. Now,

$$FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A))' = (FR\mathcal{BG}^{\circ}(FR\mathcal{BG}bd(A)))'.$$

By (i), $FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A))'$ is a fuzzy rough $\mathcal{B}$-open group, which implies that $FR\mathcal{BG}^{\circ}(FR\mathcal{BG}bd(A))$ is a fuzzy rough $\mathcal{B}$-closed group.

(ii)⇒(iii): Let $FR\mathcal{BG}bd(A)$ be the fuzzy rough $\mathcal{BG}$-boundary of A. Then,

$$FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A)) + FR\mathcal{BG}^{\neg}(FR\mathcal{G}^{\neg}(FR\mathcal{BG}bd(A)))'$$

$$= FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A)) + FR\mathcal{BG}^{\neg}(FR\mathcal{G}^{\circ}(FR\mathcal{BG}bd(A))') \tag{4.1}$$

Since $FR\mathcal{BG}bd(A)$ is a fuzzy rough $\mathcal{BG}$-boundary of A, $(FR\mathcal{BG}bd(A))'$ is a fuzzy rough $\mathcal{BG}$-boundary complement of $FR\mathcal{BG}bd(A)$. Hence by (ii), $FR\mathcal{BG}^{\circ}(FR\mathcal{BG}bd(A))'$ is fuzzy rough $\mathcal{B}$-closed group. Therefore, by (4.1)

$$\begin{aligned}
&FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A)) + FR\mathcal{BG}^{\neg}(FR\mathcal{G}^{\neg}(FR\mathcal{BG}bd(A)))'\\
&= FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A)) + FR\mathcal{BG}^{\circ}(FR\mathcal{BG}bd(A))'\\
&= FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A)) + (FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A)))'\\
&= \tilde{1}.
\end{aligned}$$

Therefore, $FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bdA) + FR\mathcal{BG}^{\neg}(FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bdA) = \tilde{1}$.

(iii)⇒(iv): Let $FR\mathcal{BG}bd(A)$ and $FR\mathcal{BG}bd(B)$ be any two fuzzy rough $\mathcal{BG}$-boundary of A and B respectively, such that

$$FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A)) + FR\mathcal{BG}bd(B) = \tilde{1}. \tag{4.2}$$

Then by (iii),
$$\begin{aligned}
\tilde{1} &= FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A)) + FR\mathcal{BG}^{\neg}(FR\mathcal{G}^{\neg}(FR\mathcal{BG}bd(A))'\\
&= FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A)) + FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(B)).
\end{aligned}$$

Therefore, $FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A)) + FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(B)) = \tilde{1}$.

(iv)⇒(i): Let $FR\mathcal{BG}bd(A)$ be a fuzzy rough $\mathcal{BG}$-boundary of A. Put $FR\mathcal{BG}bd(B) = (FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A)))' = \tilde{1} - FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A))$. Then, $FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A)) + FR\mathcal{BG}bd(B) = \tilde{1}$. Therefore by (iv), $FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A)) + FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(B)) = \tilde{1}$. This implies that, $FR\mathcal{BG}^{\neg}(FR\mathcal{BG}bd(A))$ is a fuzzy rough $\mathcal{B}$-open group and so $(X, \mathcal{G})$ is a fuzzy rough $\mathcal{BG}$-boundary spaces. □

Definition 4.6. Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let A be any fuzzy rough topological group. Then A is said to be a

(i) fuzzy rough α^*-open group if $FR\mathcal{G}int(A) = FR\mathcal{G}int(FR\mathcal{G}cl(FR\mathcal{G}int(A)))$.
(ii) fuzzy rough C-open group if $A = B \cap D$ where A is a fuzzy rough open set and D is a fuzzy rough α^*-open set.

Remark 4.7. Every fuzzy rough $\mathcal{B}$-open group is a fuzzy rough C-open group.

Proposition 4.8. If $(X, \mathcal{G})$ is a fuzzy rough $\mathcal{BG}$-boundary space then every fuzzy rough $\mathcal{BG}$-closure of $\mathcal{BG}$-boundary of each fuzzy rough open group is a fuzzy rough C-open group.

Proof. The proof follows from Definition 4.4 and Remark 4.7. □

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