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Soft topological subspaces

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Abstract. The first aim of this paper is to study soft topological spaces and soft topological subspaces can be named as soft subtopology. Second is to introduce new properties of soft subtopology. Third is to define and discuss the properties of soft topology and soft subtopology which are fundamental for further research on soft topology and will strengthen the theory of soft topological spaces.

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1. Introduction

It is very difficult to solve problems in engineering, economics, medicine and sociology by general mathematical tools if they contain data with uncertainties. To solve such problems some theories developed such as fuzzy sets, rough sets, vague sets, intuitionistic fuzzy sest and interval mathematics. Recently, Molodtsov [12] added another theory called soft set theory to deal with uncertain, fuzzy, not clearly defined objects. Molodtsov [12] applied it to several concepts like game theory, operations research, theory of probability, Riemann-integration, Perron-integration, smoothness of functions and so on. After Molodtsov, research on the soft set theory has been accelerated rapidly [1, 2, 3, 5, 6, 8, 10, 11, 12, 13, 18, 19].

The topological structures of set theories dealing with uncertainities were first defined by Chang [4]. He introduced fuzzy topology and gave its properties. Lashin et al. [9] studied topological spaces by generalizing rough set theory. The notion of soft topological spaces introduced by Shabir and Naz [17] on an initial universe with a fixed set of parameters. In recent years, the topological space theory has been embedding in the soft set theory to obtain some interesting applications (e.g. [14, 15, 16, 17, 18]).

In this study, we introduce some new concepts in soft topological subspaces such as soft open and soft closed by characterization and give their relation with soft open and soft closed sets in soft topological space. We also observe that relations of soft topological space and soft topological subspace in different cases. Furthermore, we serve a bridge among soft topological space theory and soft topological subspace theory. It shows how soft sets in soft topological space can preserve their properties in soft topological subspaces.

Based on the definition of soft subspace in soft topological space, we introduce soft topological subspace of a soft topological subspace. We then prove that whatever the soft subspaces of the soft spaces might be, their reduced soft spaces are coincident with regard to both of the spaces. Moreover, we obtain significant relations between these soft topological spaces and give examples to compare their properties.

2. Preliminaries

In this section, we recall some basic notions of soft set theory which may be found in [6, 7, 12, 19] for further details.

Throughout this work, U refers to an initial universe, E is a set of parameters and $P(U)$ is the power set of U.

Definition 2.1 ([12])**.** A pair (f, E) is called a soft set (over U) if and only if f is a mapping or E into the set of all subsets of the set U.

From now on, we will use the definitions and operations of soft sets are written with the form of ([7]).

Definition 2.2 ([7])**.** A soft set f on the universe U is defined by the set of ordered pairs

$$f = \Big\{\big(e, f(e)\big) : e \in E\Big\}$$

where $f : E \to \mathcal{P}(U)$ such that $f(e) = \varnothing$ if $e \in E \setminus A$ then $f = f_A$.

Note that the set of all soft sets over U will be denoted by $\mathbb{S}$.

Definition 2.3 ([7])**.** Let $f \in \mathbb{S}$. Then,

If $f(e) = \varnothing$ for all $e \in E$, then f is called an empty set, denoted by Φ.

If $f(e) = U$ for all $e \in E$, then f is called universal soft set, denoted by $\tilde{E}$.

Definition 2.4 ([7])**.** Let $f, g \in \mathbb{S}$. Then,

f is a soft subset of g, denoted by $f \widetilde{\subseteq} g$, if $f \subseteq g$ for all $e \in E$.

f and g are soft equal, denoted by $f = g$, if and only if $f(e) = g(e)$ for all $e \in E$.

Definition 2.5 ([7])**.** Let $f, g \in \mathbb{S}$. Then, soft union and soft intersection of f and g are defined by the soft sets, respectively,

$$f \tilde{\cup} g = \Big\{f(e) \cup g(e) : e \in E\Big\}$$

$$f \tilde{\cap} g = \Big\{f(e) \cap g(e) : e \in E\Big\}$$

and the soft complement of f is defined by,

$$f^{\tilde{c}} = \Big\{f(e)^c : e \in E\Big\}$$

where $f^{\tilde{c}}$ is complement of the set $f(e)$, that is, $f(e)^c = U \setminus f_A(e)$ for all $e \in E$.

It is easy to see that $(f^{\tilde{c}})^{\tilde{c}} = f$ and $\Phi^{\tilde{c}} = \tilde{E}$

Proposition 2.6 ([7]). *Let $f \in \mathbb{S}$. Then,*

i. $f\tilde{\cup}f = f, f\tilde{\cap}f = f$
ii. $f\tilde{\cup}\Phi = f, f\tilde{\cap}\Phi = \Phi$
iii. $f\tilde{\cup}\tilde{E} = \tilde{E}, f\tilde{\cap}\tilde{E} = f$
iv. $f\tilde{\cup}f^{\tilde{c}} = \tilde{E}, f\tilde{\cap}f^{\tilde{c}} = \Phi$

Proposition 2.7 ([7]). *Let $f, g, h \in \mathbb{S}$. Then,*

i. $f\tilde{\cup}g = g\tilde{\cup}f, f\tilde{\cap}g = g\tilde{\cap}f$
ii. $(f\tilde{\cap}g)^{\tilde{c}} = g^{\tilde{c}}\tilde{\cup}f^{\tilde{c}}, (f\tilde{\cup}g)^{\tilde{c}} = g^{\tilde{c}}\tilde{\cap}f^{\tilde{c}}$
iii. $(f\tilde{\cup}g)\tilde{\cup}h = f\tilde{\cup}(g\tilde{\cup}h), (f\tilde{\cap}g)\tilde{\cap}h = f\tilde{\cap}(g\tilde{\cap}h)$
iv. $f\tilde{\cup}(g\tilde{\cap}h) = (f\tilde{\cup}g)\tilde{\cap}(f\tilde{\cup}h)$
$f\tilde{\cap}(g\tilde{\cup}h) = (f\tilde{\cap}g)\tilde{\cup}(f\tilde{\cap}h)$

Definition 2.8 ([6]). Let $f \in \mathbb{S}$. Power soft set of f is defined by

$$\mathcal{P}(f) = \{f_i\tilde{\subseteq}f : i \in I\}$$

and its cardinality is defined by

$$|\mathcal{P}(f)| = 2^{\sum_{e\in E}|f(e)|}$$

where $|f(e)|$ is cardinality of $f(e)$.

Example 2.9 ([6]). Let $U = \{u_1, u_2, u_3\}$ and $E = \{e_1, e_2\}$. $f \in \mathbb{S}$ and

$$f = \big\{(e_1, \{u_1, u_2\}), (e_2, \{u_2, u_3\})\big\}$$

Then,

$$\begin{aligned}
f_1 &= \{(e_1, \{u_1\})\},\\
f_2 &= \{(e_1, \{u_2\})\},\\
f_3 &= \{(e_1, \{u_1, u_2\})\},\\
f_4 &= \{(e_2, \{u_2\})\},\\
f_5 &= \{(e_2, \{u_3\})\},\\
f_6 &= \{(e_2, \{u_2, u_3\})\},\\
f_7 &= \{(e_1, \{u_1\}), (e_2, \{u_2\})\},\\
f_8 &= \{(e_1, \{u_1\}), (e_2, \{u_3\})\},\\
f_9 &= \{(e_1, \{u_1\}), (e_2, \{u_2, u_3\})\},\\
f_{10} &= \{(e_1, \{u_2\}), (e_2, \{u_2\})\},\\
f_{11} &= \{(e_1, \{u_2\}), (e_2, \{u_3\})\},\\
f_{12} &= \{(e_1, \{u_2\}), (e_2, \{u_2, u_3\})\},\\
f_{13} &= \{(e_1, \{u_1, u_2\}), (e_2, \{u_2\})\},\\
f_{14} &= \{(e_1, \{u_1, u_2\}), (e_2, \{u_3\})\},\\
f_{15} &= f,\\
f_{16} &= \Phi
\end{aligned}$$

are all soft subsets of f. So $|\tilde{P}(f)| = 2^4 = 16$.

Definition 2.10 ([19])**.** The soft set $f \in \mathbb{S}$ is called a soft point in $\tilde{E}$, denoted by e_f, if there exists a element $e \in E$ such that $f(e) \neq \varnothing$ and $f(e') = \varnothing$, for all $e' \in E \setminus \{e\}$.

Definition 2.11 ([19])**.** The soft point e_f is said to belong to a soft set $g \in \mathbb{S}$, denoted by $e_f \tilde{\in} g$, if for the element $e \in E$ and $f(e) \subseteq g(e)$.

Theorem 2.12 ([19])**.** *A soft set can be written as the soft union of all its soft points.*

Theorem 2.13 ([19])**.** *Let $f, g \in \mathbb{S}$. Then,*

$$e_{i_f} \tilde{\in} g \Rightarrow f \tilde{\subseteq} g$$

for all $e_{i_f} \tilde{\in} f$.

3. Soft Topology

In this section, we study some basic results of soft topological spaces based on Çağman et al. [6]'s soft topology and define soft topological subspace.

Definition 3.1 ([6])**.** Let $\Phi \neq X \subseteq E$ and $f \in \mathbb{S}$. $\tilde{\tau} = \{g_i\}_{i \in I}$ be the collection of soft subsets over f. Then, $\tilde{\tau}$ is called a soft topology on f if $\tilde{\tau}$ satisfies the following axioms:

i. $\Phi, f \in \tilde{\tau}$,
ii. $\{g_i\}_{i \in I} \subseteq \tilde{\tau} \Rightarrow \tilde{\bigcup}_{i \in I} g_i \in \tilde{\tau}$,
iii. $\{g_i\}_{i=1}^{n} \subseteq \tilde{\tau} \Rightarrow \tilde{\bigcap}_{i=1}^{n} g_i \in \tilde{\tau}$.

The pair $(f, \tilde{\tau})$ is called a soft topological space over f and the members of $\tilde{\tau}$ are said to be soft open in f.

Example 3.2. Let us consider the soft subsets of f that are given in Example 2.9. Then, $\tilde{\tau}^1 = \tilde{\mathcal{P}}(f)$, $\tilde{\tau}^0 = \{\Phi, f\}$ and $\tilde{\tau} = \{\Phi, f, f_2, f_{11}, f_{13}\}$ are some soft topologies on f.

Theorem 3.3 ([18])**.** *Every soft power set of a soft set is a soft topological space over the soft set.*

Proof. Let $\tilde{\mathcal{P}}(f)$ be the soft power set of f. Then,

i. $\Phi, f \in \tilde{\mathcal{P}}(f)$
ii. Since $g_i \subseteq f$ then, $g_i \in \tilde{\mathcal{P}}(f)$, for all $i \in I$. Moreover, $\tilde{\bigcup}_{i \in I} g_i \subseteq f$ it follows that, $\tilde{\bigcup}_{i \in I} g_i \in \tilde{\mathcal{P}}(f)$.
iii. Since $g_i \subseteq f$, for all $i \in I'$, I' finite soft set, so that $\tilde{\bigcap}_{i \in I'} g_i \subseteq f$. Thus $\tilde{\bigcap}_{i \in I'} g_i \in \tilde{\mathcal{P}}(f)$.

□

Hence $(f, \tilde{\mathcal{P}}(f))$ is a soft topological space which is called the soft discrete topology on f and is denoted by $\tilde{\tau}^1$ contains all soft sets over f.

Theorem 3.4 ([18])**.** *$(f, \{\Phi, f\})$ is a soft topological space over f.*

Proof. For satisfing the following axioms:

i. $\Phi, f \in \{\Phi, f\}$

ii. $\Phi \cup f = f \in \{\Phi, f\}$
iii. $\Phi \cap f = \Phi \in \{\Phi, f\}$

□

$(f, \{\Phi, f\})$ is a soft topological space. Then, it is called the indiscrete soft topology denoted by $\tilde{\tau}^0$ contains only Φ and f.

Theorem 3.5. *Let $\{\tilde{\tau}_i\}_{i\in I}$ be the family of all soft topologies on f. "$\leq$" is a partial ordering relation if,*

$$(\tilde{\tau}_i \leq \tilde{\tau}_j) \Leftrightarrow (\forall g_i \in \tilde{\tau}_i \Rightarrow \forall g_i \in \tilde{\tau}_j)$$

for all $i, j \in I$.

Proof. i. Since $\tilde{\tau}_i \subseteq \tilde{\tau}_i$ then, $\tilde{\tau}_i \leq \tilde{\tau}_i$ for all $i \in I$. So, "$\leq$" is reflective.

ii. Suppose that $\tilde{\tau}_i \leq \tilde{\tau}_j$ and $\tilde{\tau}_j \leq \tilde{\tau}_i$ for all $i, j \in I$. Then, $\tilde{\tau}_i \subseteq \tilde{\tau}_j$ and $\tilde{\tau}_j \subseteq \tilde{\tau}_i$. So, $\tilde{\tau}_i = \tilde{\tau}_j$ and "$\leq$" is anti-symmetric.

iii. Suppose that $\tilde{\tau}_i \leq \tilde{\tau}_j$ and $\tilde{\tau}_j \leq \tilde{\tau}_k$ for all $i, j, k \in I$. Then, $\tilde{\tau}_i \subseteq \tilde{\tau}_j$ and $\tilde{\tau}_j \subseteq \tilde{\tau}_k$. So, $\tilde{\tau}_i \subseteq \tilde{\tau}_k$. It implies that $\tilde{\tau}_i \leq \tilde{\tau}_k$ and "$\leq$" has transitive property.

□

Definition 3.6. Let $\{\tilde{\tau}_i\}_{i\in I}$ be the family of all soft topologies on f. Then,

i. If $\tilde{\tau}_i \leq \tilde{\tau}_j$, then $\tilde{\tau}_j$ is soft finer (or stronger) than $\tilde{\tau}_i$. In this case $\tilde{\tau}_i$ is said to be soft coarser (or weaker) than $\tilde{\tau}_j$.
ii. If $\tilde{\tau}_i < \tilde{\tau}_j$, then $\tilde{\tau}_j$is soft strictly finer (or stictly stronger)than $\tilde{\tau}_i$.
iii. If either $\tilde{\tau}_i \leq \tilde{\tau}_j$ or $\tilde{\tau}_j \leq \tilde{\tau}_i$, then $\tilde{\tau}_i$ is comparable with $\tilde{\tau}_j$.

for all $i, j \in I$.

Example 3.7 ([18])**.** Consider soft topologies on f in the Example 3.2. Here, it is easily seen that $\tilde{\tau}^0 \subseteq \tilde{\tau}^1$, $\tilde{\tau}^0 \subseteq \tilde{\tau}$ and $\tilde{\tau} \subseteq \tilde{\tau}^1$. So, $\tilde{\tau}^1$ is soft finer than $\tilde{\tau}^0$ and $\tilde{\tau}$ is soft finer than $\tilde{\tau}^0$.

Definition 3.8. Soft set of all elements in the soft universal set that are not in the soft initial set are said to be soft complement set. The soft complement of a soft set is represented by the symbol $^{\tilde{c}}$.

Definition 3.9 ([6])**.** Let $(f, \tilde{\tau})$ and $g \in \mathbb{S}$. Then, g is soft closed in $\tilde{\tau}$ if $g^{\tilde{c}} \in \tilde{\tau}$.

Throughout this work, the collection of soft closed sets in $(f, \tilde{\tau})$ is denoted by $\tilde{F}$

Theorem 3.10 ([18])**.** *If $\tilde{F}$ is a collection of soft closed sets in a soft topological space $(f, \tilde{\tau})$, then*

i. Universal soft set $\tilde{E}$ is soft closed.
ii. Any intersection of members of $\tilde{F}$ belongs to $\tilde{F}$.
iii. Any finite union of members of $\tilde{F}$ belongs to $\tilde{F}$.

Remark 3.11. *Since $\tilde{E}^{\tilde{c}} = \Phi \in \tilde{\tau}$, $\tilde{E}$ is soft closed. But, Φ and f need not to be soft closed. The following example shows that:*

Example 3.12. Consider the topology $\tilde{\tau} = \{\Phi, f, f_2, f_{11}, f_{13}\}$ is defined in Example 3.2. Here, f and Φ are not soft closed sets because $f^{\tilde{c}} = \Big\{\big(e_1, \{u_3\}\big), \big(e_2, \{u_1\}\big)\Big\} \notin \tilde{\tau}$ and $\Phi^{\tilde{c}} = \tilde{E} \notin \tilde{\tau}$.

Theorem 3.13 ([18]). *Let $(f, \tilde{\tau}_1)$ and $(f, \tilde{\tau}_2)$ be two soft topologies, then $(f, \tilde{\tau}_1 \cap \tilde{\tau}_2)$ is a soft topological space.*

Remark 3.14 ([18]). *Although $(f, \tilde{\tau}_1)$ and $(f, \tilde{\tau}_2)$ be two soft topologies, $(f, \tilde{\tau}_1 \cup \tilde{\tau}_2)$ is not need to be soft topological space. The following example shows that:*

Example 3.15 ([18]). Let $U = \{u_1, u_2, u_3\}$, $E = \{e_1, e_2, e_3\}$ and $\tilde{\tau}_1 = \big\{\Phi, f, g, h, m\big\}$ is a soft topology over f where f, g, h, m are soft sets, defined as follows:

$$\begin{aligned} f &= \Big\{\big(e_1, \{u_1, u_2\}\big), \big(e_2, \{u_2, u_3\}\big)\Big\} \\ g &= \Big\{\big(e_1, \{u_2\}\big)\Big\} \\ h &= \Big\{\big(e_1, \{u_2\}\big), \big(e_2, \{u_3\}\big)\Big\} \\ m &= \Big\{\big(e_1, \{u_1, u_2\}\big), \big(e_2, \{u_2\}\big)\Big\} \end{aligned}$$

If

$$\begin{aligned} n &= \Big\{\big(e_1, \{u_1\}\big)\Big\} \\ p &= \Big\{\big(e_1, \{u_1\}\big), \big(e_2, \{u_2\}\big)\Big\} \\ r &= \Big\{\big(e_1, \{u_1, u_2\}\big), \big(e_2, \{u_3\}\big)\Big\} \end{aligned}$$

then

$$\tilde{\tau}_2 = \big\{\Phi, f, n, p, r\big\}$$

is a soft topology over f, too. But, $(f, \tilde{\tau}_1 \cup \tilde{\tau}_2)$ is not a soft topology because, it can easily seen that

$$g \tilde{\cup} n = \Big\{\big(e_1, \{u_1, u_2\}\big)\Big\} \notin \tilde{\tau}_1 \cup \tilde{\tau}_2$$

Definition 3.16 ([18]). Let $(f, \tilde{\tau})$ be a soft topological space. A subcollection $\tilde{\mathcal{B}}$ of $\tilde{\tau}$ is said to be a soft base for $\tilde{\tau}$ if every member of $\tilde{\tau}$ can be expressed as a union of members of $\tilde{\mathcal{B}}$.

Let $(f, \tilde{\tau})$ be a soft topological space and $\tilde{\mathcal{B}} = \{g_i\}_{i \in I}$ be a soft base for $\tilde{\tau}$. Then,

$$h = \tilde{\bigcup}_{j \in J \subseteq I} g_j$$

for $h \in \tilde{\tau}$.

Example 3.17 ([18]). Consider the topology $\tilde{\tau}$ is defined in example 3.2. Then,

$$\tilde{\mathcal{B}} = \{\Phi, f_2, f_{11}, f_{13}\}$$

is a soft base for $\tilde{\tau}$.

Theorem 3.18 ([18]). *Let $(f, \tilde{\tau}_1)$ and $(f, \tilde{\tau}_2)$ be soft topological spaces. If $\tilde{\mathcal{B}}$ is a soft base for each $\tilde{\tau}_1$ and $\tilde{\tau}_2$ then, $\tilde{\tau}_1 = \tilde{\tau}_2$.*

Theorem 3.19. *Let $(f, \tilde{\tau}_1)$ and $(f, \tilde{\tau}_2)$ be soft topological spaces, and $\tilde{\mathcal{B}}_1$ and $\tilde{\mathcal{B}}_2$ be soft bases for $\tilde{\tau}_1$ and $\tilde{\tau}_2$, respectively. If $\tilde{\mathcal{B}}_1 \tilde{\subseteq} \tilde{\mathcal{B}}_2$ then $\tilde{\tau}_1 \tilde{\subseteq} \tilde{\tau}_2$.*

Remark 3.20. *A soft base can be consider only a soft topological space over a soft topological space. But a soft topological space can have several soft bases can be seen in the following example:*

Example 3.21. Consider the soft subsets of f defined as in Example 2.9. Then, $\tilde{\tau} = \{\Phi, f, f_3, f_4, f_5, f_6, f_{13}, f_{14}\}$ is a soft topological space over f. The collections $\tilde{\mathcal{B}}_1, \tilde{\mathcal{B}}_2, \tilde{\mathcal{B}}_3$ defined below are soft bases for $(f, \tilde{\tau})$. Also here $\tilde{\mathcal{B}}_1 \tilde{\subseteq} \tilde{\mathcal{B}}_2 \tilde{\subseteq} \tilde{\mathcal{B}}_3$.

$$\begin{aligned}\tilde{\mathcal{B}}_1 &= \{\Phi, f, f_3, f_4, f_5\}\\ \tilde{\mathcal{B}}_2 &= \{\Phi, f, f_3, f_4, f_5, f_6\}\\ \tilde{\mathcal{B}}_3 &= \{\Phi, f, f_3, f_4, f_5, f_6, f_{13}\}\end{aligned}$$

Theorem 3.22. *Let $(f, \tilde{\tau})$ be a soft topological space and $g \tilde{\subseteq} f$. Then, the collection*

$$\tilde{\tau}_g = \{h \tilde{\cap} g : h \in \tilde{\tau}\}$$

is a soft topology on g and the pair $(g, \tilde{\tau}_g)$ is a soft topological space.

Proof. Since $\Phi \tilde{\cap} g = \Phi$ and $f \tilde{\cap} g = g$, then $g, \Phi \in \tilde{\tau}_g$.
Moreover,

$$\tilde{\bigcap}_{i=1}^{n} (h_i \tilde{\cap} g) = \left(\tilde{\bigcap}_{i=1}^{n} h_i\right) \tilde{\cap} g$$

and

$$\tilde{\bigcup}_{i\in I} (h_i \tilde{\cap} g) = \left(\tilde{\bigcup}_{i\in I} h_i\right) \tilde{\cap} g$$

for $\tilde{\tau} = \{h_i \tilde{\subseteq} f : i \in I\}$. Thus, the union of any number of soft sets in $\tilde{\tau}_g$ belongs to $\tilde{\tau}_g$ and the finite intersection of soft sets in $\tilde{\tau}_g$ belongs to $\tilde{\tau}_g$. So, $\tilde{\tau}_g$ is a soft topology on g. □

Definition 3.23. Let $(f, \tilde{\tau})$ be a soft topological space and $g \tilde{\subseteq} f$. Then, the collection

$$\tilde{\tau}_g = \{h \tilde{\cap} g : h \in \tilde{\tau}\}$$

is called a soft subspace topology on g and $(g, \tilde{\tau}_g)$ is called a soft topological subspace of $(f, \tilde{\tau})$.

Example 3.24. Let $(f, \tilde{\tau})$ be a soft topological space and $g \tilde{\subseteq} f$.

$$If\ \tilde{\tau} = \tilde{\mathcal{P}}(f) \text{ then } \tilde{\tau}_g = \tilde{\mathcal{P}}(g)$$

and

$$If\ \tilde{\tau} = \{f, \Phi\} \text{ then } \tilde{\tau}_g = \{g, \Phi\}$$

Example 3.25. Let us consider the soft topology on f in the example 3.2. If $g = f_9$, then $\tilde{\tau}_g = \{\Phi, f_5, f_7, f_9\}$, and so $(g, \tilde{\tau}_g)$ is a soft topological subspace of $(f, \tilde{\tau})$.

Definition 3.26. A soft topological property is said to be hereditary if whenever a soft topological space $(f, \tilde{\tau})$ has that property, then so does every soft topological subspace of it.

4. The Relation between Soft Topological Subspace and Soft Topological Space

In this section, we search for the relation between soft topological space and soft topological subspace and discuss their properties with examples.

Definition 4.1. Let $(g, \tilde{\tau}_g)$ be a soft subspace of a soft topological space $(f, \tilde{\tau})$ and $h \tilde{\subseteq} g$. h is called a soft open set in soft subspace g if $h = m \tilde{\cap} g$ for $m \in \tilde{\tau}$.
So, the members of $\tilde{\tau}_g$ are said to be a soft open sets in soft subspace $(g, \tilde{\tau}_g)$.

Theorem 4.2. *Let $(g, \tilde{\tau}_g)$ be a soft subspace of a soft topological space $(f, \tilde{\tau})$ and $h \tilde{\subseteq} g$. If $h \in \tilde{\tau}$ then, $h \in \tilde{\tau}_g$.*

Proof. Suppose that $h \in \tilde{\tau}$. Since $h \tilde{\subseteq} g$, $h = h \tilde{\cap} g$. Then, $h \in \tilde{\tau}_g$ by assumption $h \in \tilde{\tau}$. □

Remark 4.3. *A soft open set in a soft subspace is not need to be soft open in the universal soft set which is given in the following example:*

Example 4.4. Consider the soft subspace $(g, \tilde{\tau}_g)$ defined in the Example 3.25. Here, $f_5 \in \tilde{\tau}_g$ but $f_5 \notin \tilde{\tau}$.

Theorem 4.5. *Let $(g, \tilde{\tau}_g)$ be a soft subspace of a soft topological space $(f, \tilde{\tau})$. Then, the following are equivalent:*

i. $g \in \tilde{\tau}$
ii. $\tilde{\tau}_g \tilde{\subseteq} \tilde{\tau}$.

Proof. $(i) \Rightarrow (ii)$: Let $g \in \tilde{\tau}$. Take as given $\forall h \in \tilde{\tau}_g$. From the definition of soft subspace topology, $h = m \tilde{\cap} g$, where $\exists m \in \tilde{\tau}$. Since $g \in \tilde{\tau}$ and $m \in \tilde{\tau}$ then, $h \in \tilde{\tau}$. Hence $\tilde{\tau}_g \tilde{\subseteq} \tilde{\tau}$.
$(ii) \Rightarrow (i)$: Assume that $\tilde{\tau}_g \tilde{\subseteq} \tilde{\tau}$. Since $g \in \tilde{\tau}_g$ then, $g \in \tilde{\tau}$. □

Definition 4.6. Let $h \tilde{\subseteq} g \tilde{\subseteq} f$, $(g, \tilde{\tau}_g)$ and $(h, \tilde{\tau}_h)$ be soft subspaces of soft topological space $(f, \tilde{\tau})$ and $(h, (\tilde{\tau}_g)_h) \tilde{\subseteq} (g, \tilde{\tau}_g)$. Then, $(h, (\tilde{\tau}_g)_h)$ is called soft topological subspace of a soft topological subspace $(g, \tilde{\tau}_g)$.

Theorem 4.7. *Let $h \tilde{\subseteq} g \tilde{\subseteq} f$, $(g, \tilde{\tau}_g)$ and $(h, \tilde{\tau}_h)$ be soft subspaces of soft topological space $(f, \tilde{\tau})$ and $(h, (\tilde{\tau}_g)_h)$ be a soft subspace of $(g, \tilde{\tau}_g)$. Then,*

$$\tilde{\tau}_h = (\tilde{\tau}_g)_h$$

Proof. Take as given $\forall w \in \tilde{\tau}_h$. From the definition of soft subspace topology, $w = w \tilde{\cap} h$, where $w \in \tilde{\tau}$. We obtain that $w \tilde{\cap} g \in \tilde{\tau}_g$. Then, by choosing $w \tilde{\cap} g = y$, $y \tilde{\cap} h \in (\tilde{\tau}_g)_h$ because of $(h, (\tilde{\tau}_g)_h) \subseteq (g, \tilde{\tau}_g)$ and $y \in \tilde{\tau}_g$.
Since $y = w \tilde{\cap} g$ then, $y \tilde{\cap} h = w \tilde{\cap} g \tilde{\cap} h \in (\tilde{\tau}_g)_h$.
$h \subseteq g \Leftrightarrow h = h \tilde{\cap} g$ then, $y \tilde{\cap} h = w \tilde{\cap} h \in (\tilde{\tau}_g)_h$.
Since $w = w \tilde{\cap} h \in (\tilde{\tau}_g)_h$ then, $w \in (\tilde{\tau}_g)_h$. Hence, we get $\tilde{\tau}_h \subseteq (\tilde{\tau}_g)_h$.
Conversely, assume that $\forall z \in (\tilde{\tau}_g)_h$. From the definition of soft subspace topology, $z = t \tilde{\cap} h$, where $t \in \tilde{\tau}_g$. We obtain that $t = w \tilde{\cap} g$, where $w \in \tilde{\tau}$.
$z = t \tilde{\cap} h = w \tilde{\cap} g \tilde{\cap} h = w \tilde{\cap} h \in \tilde{\tau}_h$, so this completes the proof. □

Theorem 4.8. *$(g, \tilde{\tau}_g)$ and $(h, \tilde{\tau}_h)$ be soft subspaces of soft topological space $(f, \tilde{\tau})$ and $w \subseteq h \tilde{\cap} g$. Then,*

$$\tilde{\tau}_w = (\tilde{\tau}_g)_w = (\tilde{\tau}_h)_w$$

Proof. Since $h \tilde{\cap} g \subseteq h$ then, $\tilde{\tau}_w = (\tilde{\tau}_h)_w$ and $h \tilde{\cap} g \subseteq g$ then, $\tilde{\tau}_w = (\tilde{\tau}_g)_w$. So, we get $\tilde{\tau}_w = (\tilde{\tau}_g)_w = (\tilde{\tau}_h)_w$. □

Remark 4.9. *Whatever the soft subspaces of the soft spaces might be, their reduced soft spaces are coincident with regard to both of the spaces.*

Definition 4.10. Let $(f,\tilde{\tau})$ be a soft topological space, $g\subseteq f$ and $(g,\tilde{\tau}_g)$ be a soft topological subspace of f. Throughout this work, the collection of soft closed sets in $(g,\tilde{\tau}_g)$ is denoted by $\tilde{F}_g$.

Remark 4.11. *The soft complement of soft closed set $h\in\tilde{F}_g$ in $(g,\tilde{\tau}_g)$ will be denoted by $(h)_g^{\tilde{c}}$ and the soft complement in $(f,\tilde{\tau})$ will be denoted by $(h)_f^{\tilde{c}}$ for $h\tilde{\subseteq}g\tilde{\subseteq}f$.*

Definition 4.12. Let $(f,\tilde{\tau})$ be a soft topological space, $g\subseteq f$ and $(g,\tilde{\tau}_g)$ be a soft topological subspace of f. If $h\in\tilde{F}_g$ then, $(h)_g^{\tilde{c}}\in\tilde{\tau}_g$ for $h\tilde{\subseteq}g\tilde{\subseteq}f$.

Theorem 4.13. *Let $(f,\tilde{\tau})$ be a soft topological space, $g\subseteq f$ and $(g,\tilde{\tau}_g)$ be a soft topological subspace of f. For the soft sets $h\tilde{\subseteq}g\tilde{\subseteq}f$, the following statements are equivalent:*

i. $h\in\tilde{F}_g$
ii. $h=k\tilde{\cap}g$ *for* $\exists k\in\tilde{F}$.

Proof. $(i)\Rightarrow(ii)$: Let $h\in\tilde{F}_g$. Then, $(h)_g^{\tilde{c}}\in\tilde{\tau}_g$. So, there exists a $w\in\tilde{\tau}$ such that $(h)_g^{\tilde{c}}=w\tilde{\cap}g$. We can write $h=((h)_g^{\tilde{c}})_g^{\tilde{c}}=(w\tilde{\cap}g)_g^{\tilde{c}}$.

$$\begin{aligned}(w\tilde{\cap}g)_g^{\tilde{c}} &= g\tilde{\setminus}(w\tilde{\cap}g)\\ &= g\tilde{\cap}(w\tilde{\cap}g)_f^{\tilde{c}}\\ &= g\tilde{\cap}((w)_f^{\tilde{c}}\tilde{\cup}(g)_f^{\tilde{c}})\\ &= (g\tilde{\cap}(w)_f^{\tilde{c}})\tilde{\cup}(g\tilde{\cap}(g)_f^{\tilde{c}})\\ &= g\tilde{\cap}(w)_f^{\tilde{c}}\end{aligned}$$

Since $w\in\tilde{\tau}$ then, $(w)_f^{\tilde{c}}\in\tilde{F}$. By choosing $(w)_f^{\tilde{c}}=k$ we obtain $h=k\tilde{\cap}g$ for $k\in\tilde{F}$.
$(ii)\Rightarrow(i)$: Since $h=k\tilde{\cap}g$ then, $(h)_g^{\tilde{c}}=(k\tilde{\cap}g)_g^{\tilde{c}}$. It follows that,

$$\begin{aligned}(k\tilde{\cap}g)_g^{\tilde{c}} &= g\tilde{\setminus}(k\tilde{\cap}g)\\ &= g\tilde{\cap}(k\tilde{\cap}g)_f^{\tilde{c}}\\ &= g\tilde{\cap}((k)_f^{\tilde{c}}\tilde{\cup}(g)_f^{\tilde{c}})\\ &= (g\tilde{\cap}(k)_f^{\tilde{c}})\tilde{\cup}(g\tilde{\cap}(g)_f^{\tilde{c}})\\ &= g\tilde{\cap}(k)_f^{\tilde{c}}\end{aligned}$$

Moreover, by assumption $k\in\tilde{F}$, then $(k)_f^{\tilde{c}}\in\tilde{\tau}$. Hence, we get $(h)_g^{\tilde{c}}\in\tilde{\tau}_g$ and $h\in\tilde{F}_g$, as required. □

Theorem 4.14. *Let $(f,\tilde{\tau})$ be a soft topological space, $g\subseteq f$ and $(g,\tilde{\tau}_g)$ be a soft topological subspace of f. If $h\tilde{\subseteq}g, h\in\tilde{F}$ then, $h\in\tilde{F}_g$.*

Proof. Since $h\tilde{\subseteq}g$, then $h=h\tilde{\cap}g$. By assumption $h\in\tilde{F}$ and we get $h\in\tilde{F}_g$, by Theorem 4.13, as required. □

Remark 4.15. *There exists a soft closed set in soft topological subspace which is not soft closed in soft topological space. The following example shows that:*

Example 4.16. Consider the topology $\tilde{\tau}$ is defined in Example 3.2. Here, $\tilde{F}=$
$\Big\{\Big\{\big(e_1,U\big),\big(e_2,U\big)\Big\},\Big\{\big(e_1,\{u_3\}\big),\big(e_2,\{u_1\}\big)\Big\},\Big\{\big(e_1,\{u_1,u_3\}\big)\Big\},$
$\Big\{\big(e_1,\{u_1,u_3\}\big),\big(e_2,\{u_1,u_2\}\big)\Big\},\Big\{\big(e_1,\{u_3\}\big),\big(e_2,\{u_1,u_3\}\big)\Big\}\Big\}.$
If $g=\Big\{\big(e_1,\{u_1\}\big),\big(e_2,\{u_2,u_3\}\big)\Big\}$ then,

$\tilde{\tau}_g = \Big\{\Phi, g, \Big\{(e_1, \{u_1\}), (e_2, \{u_2\})\Big\}, \Big\{(e_2, \{u_3\})\Big\}\Big\}$ and

$\tilde{F}_g = \Big\{\Big\{(e_1, U), (e_2, U)\Big\}, \Big\{(e_1, \{u_2, u_3\}), (e_2, \{u_1\})\Big\}, \Big\{(e_1, \{u_2, u_3\}), (e_2, \{u_1, u_3\})\Big\}, \Big\{(e_2, \{u_1, u_2\})\Big\}\Big\}$.

Here, $h = \Big\{(e_1, \{u_2, u_3\}), (e_2, \{u_1\})\Big\} \in \tilde{F}_g$ but $h \notin \tilde{F}$.

Theorem 4.17. *Let $(f, \tilde{\tau})$ be a soft topological space, $g \subseteq f$ and $(g, \tilde{\tau}_g)$ be a soft topological subspace of f. The following statements are equivalent:*

i. $g \in \tilde{F}$
ii. $\tilde{F}_g \tilde{\subseteq} \tilde{F}$

Proof. $(i) \Rightarrow (ii)$: Let $g \in \tilde{F}$ and take as given $h \in \tilde{F}_g$. So, there exists a $k \in \tilde{F}$ such that $h = k \tilde{\cap} g$ by Theorem 4.13. Moreover, by assumption $g \in \tilde{F}$ then, $h \in \tilde{F}$. Thus, $\tilde{F}_g \tilde{\subseteq} \tilde{F}$.
$(ii) \Rightarrow (i)$: Since $(g, \tilde{\tau}_g)$ be a soft topology, then $g \in \tilde{F}_g$ and $\tilde{F}_g \tilde{\subseteq} \tilde{F}$ then we get $g \in \tilde{F}$. □

5. Conclusion

In this present work, we studied the properties of soft topological subspaces. We defined soft open and soft closed sets in soft topological subspace by characterization. We then introduced the concept of soft topological subspace of soft topological subspace. We proved that whatever the soft subspaces of the soft spaces might be, their reduced soft spaces are coincident with regard to both of the spaces. We also established several relations of soft topological and soft topological subspace and have compared their properties with some examples. All these results present a bridge among soft topological space theory and soft topological subspace theory. In the future, to extend this study, based on the soft topological subspace theory, soft closure, soft interior points, soft neighborhood of a point and soft separation axioms one can preserve their properties in soft topological subspaces.

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