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# Numerical solution of hybrid fuzzy differential equations by runge-kutta method of order five and the dependency problem

K. Kanagarajan, S. Indrakumar, S. Muthukumar

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Abstract. In this paper, we study the numerical solution of hybrid fuzzy differential equations by using Runge-Kutta method of order five. This method is adopted to solve the dependency problem in fuzzy computation. Numerical examples are presented to illustrate the theory.

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Corresponding Author: S. Indrakumar (indrakumar1729@gmail.com)

## 1. Introduction

Fuzzy set was first introduced by Zadeh [12]. Since then, the theory has been developed and it is now emerged as an independent branch of Applied Mathematics. The elementary fuzzy calculus based on the extension principle was studied by Dubois and Prade[6]. Hybrid systems are devoted to modelling, design and validation of interactive systems of computer programs and continuous systems. That is, control systems that are capable of controlling complex systems which have discrete time dynamics and continuous time dynamics can be modelled by hybrid systems. The differential systems containing fuzzy valued functions and interaction with a discrete time controller are named as hybrid fuzzy differential systems. Seikkala[11] and Kaleva [7] have discussed fuzzy differential equations(FDEs). The numerical solutions of FDEs by Euler's method was studied by Ma et al.[8]. Abbasbandy and Allviranloo [1, 2] proposed the Taylor method and the fourth order Runge-Kutta method for solving FDEs. Pederson and Sambandham[9, 10] used the Euler and Runge-Kutta methods for solving hybrid fuzzy differential equations(HFDEs). Omar and Hasan [4] used the modified fourth order Runge-Kutta method for solving FDEs by considering the dependency problem in fuzzy computation based on

Zadeh's extension principle. In this paper we use the same procedure to solve HFDEs by taking the dependency problem in fuzzy computation.

## 2. Preliminaries

Let $P_K(R^n)$ denote the family of all non-empty compact, convex subsets of $R^n$. If $\alpha, \beta \in R$ and $A,\ B \in P_K(R^n)$, then

$$\alpha(A+B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1A = A$$

and if $\alpha,\ \beta \geq 0$, then $(\alpha+\beta)A = \alpha A + \beta A$. Denote by $E^n$ the set of $u : R^n \to [0,1]$ such that $u$ satisfies (i)-(iv) mentioned below:
(i) $u$ is normal, that is, there exists an $x_0 \in R^n$
such that $u(x_0) = 1$,
(ii) $u$ is fuzzy convex,
(iii) $u$ is upper semicontinuous,
(iv) $[u]^0 = \mathrm{cl}\{x \in R^n : u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, we denote $[u]^\alpha = \{x \in R^n : u(x) \geq \alpha\}$. Then from (i)-(iv), it follows that $\alpha$-level set $[u]^\alpha \in P_K(R^n)$ for $0 < \alpha \leq 1$. An example of a $u \in E^1$ is given by

$$u(x) = \begin{cases} 4x-3, & \text{if } \ x \in (0.75, 1], \\ -2x+3, & \text{if } \ x \in (1, 1.5), \\ 0, & \text{if } \ x \notin (0.75, 1.5). \end{cases}$$

The $\alpha-$level sets are given by

$$[u]^\alpha = [0.75 + 0.25\alpha, 1.5 - 0.5\alpha].$$

Let $I$ be a real interval. A mapping $y : I \to E$ is called a fuzzy process and its $\alpha-$level set is denoted by $[y(t)]^\alpha = [\underline{y}^\alpha(t), \overline{y}^\alpha(t)],\ t \in I,\ \alpha \in (0,1]$.

## 3. The Hybrid Fuzzy Differential Systems

Consider the hybrid fuzzy differential systems

$$\left\{ \begin{array}{ll} x'(t) = f(t, x(t), \lambda_k(x_k)), & t \in [t_k, t_{k+1}], \\ x(t_k) = x_k, & \end{array} \right. \tag{3.1}$$

where $0 \leq t_0 < t_1 < \cdots < t_k < \cdots, t_k \to \infty$, $f \in C[R_+ \times E \times E, E], \lambda_k \in C[E, E]$. To be specific the system look like

$$x'(t) = \begin{cases} x_0'(t) = f(t, x_0(t), \lambda_0(x_0)), \quad x_0(t_0) = x_0, \quad t_0 \leq t \leq t_1, \\ x_1'(t) = f(t, x_1(t), \lambda_1(x_1)), \quad x_1(t_1) = x_1, \quad t_1 \leq t \leq t_2, \\ \ldots \\ x_k'(t) = f(t, x_k(t), \lambda_k(x_k)), \quad x_k(t_k) = x_k, \quad t_k \leq t \leq t_{k+1}, \\ \ldots \end{cases}$$

Assuming that the existence and uniqueness of solution of (3.1) hold for each $[t_k, t_{k+1}]$, by the solution of (3.1) we mean the following function:

$$x(t) = x(t, t_0, x_0) = \begin{cases} x_0(t), & t_0 \le t \le t_1, \\ x_1(t), & t_1 \le t \le t_2, \\ \dots \\ x_k(t), & t_k \le t \le t_{k+1}, \\ \dots \end{cases}$$

We note that the solution of (3.1) is piecewise differentiable in each interval for $t \in [t_k, t_{k+1}]$ for a fixed $x_k \in E$ and $k = 0, 1, 2, \dots$.

## 4. Dependency Problem

The dependency problem arises in fuzzy computation while applying the straight forward fuzzy interval arithmetic and Zadeh's extension principle by computing the intervals separately. This will affect errors in numerical computations and will make the results to deviate from correct results[5].

We now explain the concept of dependency problem:
Consider the real valued function

$$f(x) = 2x^2 + 5x - 3. \tag{4.1}$$

For a fuzzy number $[\widetilde{X}]^\alpha = [\alpha - 1, 2 - \alpha]$, $f(\widetilde{X})$ can be calculated in two ways. First by using the straightforward fuzzy interval arithmetic:

$$\begin{aligned} f([\widetilde{X}]^\alpha) &= 2[\alpha - 1, 2 - \alpha]^2 + 5[\alpha - 1, 2 - \alpha] - 3, \\ &= \big[\min\big\{(\alpha-1)^2, (\alpha-1)(2-\alpha), (2-\alpha)^2\big\}, \\ &\quad \max\big\{(\alpha-1)^2, (\alpha-1)(2-\alpha), (2-\alpha)^2\big\}\big] \\ &\quad +5[\alpha - 1, 2 - \alpha] - 3, \end{aligned} \tag{4.2}$$

when $\alpha = 0$, we get

$$f([\widetilde{X}]^0) = [-12, 15]. \tag{4.3}$$

The second is obtained by applying Zadeh's extension principle alone, that is:

$$f([\widetilde{X}]^\alpha) = 2\widetilde{X}^2 + 5\widetilde{X} - 3. \tag{4.4}$$

For $\alpha = 0$, the solution is :

$$f([\widetilde{X}]^0) = [-8, 15]. \tag{4.5}$$

As the solutions to the equations (4.3) and (4.5) are not the correct range of $f(\widetilde{X})$, the correct range is obtained by assuming the right hand side of equation (4.4) as one expression and then applying Zadeh's extension principle. Hence the correct range is

$$f([\widetilde{X}]^0) = \left[-\frac{49}{8}, 15\right]. \tag{4.6}$$

A new computation method was developed by Ahmad and Hassan [3] which reduces the computational complexity and overestimation in the results. The method is obtained by incorporating the optimization technique into Zadeh's extension principle.

Let $\widetilde{X}$ be a triangular fuzzy number defined by the three numbers $a_1 < a_2 < a_3$, with its graph having a triangular base on the interval $[a_1, a_3]$. For $0 < \alpha \leq 1$, let $[\tilde{x}(t)]^\alpha = [x_1^\alpha(t), x_2^\alpha(t)]$ be the $\alpha$-level interval. To make the partition for this fuzzy interval, we divide this interval into subintervals using the discrete set of points $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$ in [0,1], where $\alpha_i - \alpha_{i-1} = k = \frac{1}{n}$, for $i = 1, \ldots, n$.

It is obvious that for every $\alpha_i,\ i = 0, 1, \ldots, n$ we get

$$(4.7)\ [\widetilde{X}]^{\alpha_i}=[x_1^{\alpha_i}, x_1^{\alpha_{i+1}}]\bigcup[x_1^{\alpha_{i+1}}, x_1^{\alpha_n}]\bigcup[x_1^{\alpha_n}, x_2^{\alpha_n}]\bigcup[x_2^{\alpha_n}, x_2^{\alpha_{i+1}}]\bigcup[x_2^{\alpha_{i+1}}, x_2^{\alpha_i}],$$

for triangular fuzzy number $x_1^{\alpha_n} = x_2^{\alpha_n}$.

In general, we can decompose a fuzzy interval into $n$ compact fuzzy sets $f_1$, $f_2$, $\ldots$, $f_n$, as follows:

(i) Consider a regular partition of the uncertainty interval $x \in [a, b]$ i.e., divide $[a, b]$ into $n$ subintervals $[x_i, x_{i+1}]$, with $x_i = a + (i-1)(b-a)/n$ for $1 \leq i \leq n+1$.

(ii) Consider the fuzzy sets $f_i$ defined by $f_i(x) = f(x)$ if $x \in [x_i, x_{i+1}]$, and $f_i(x) = 0$ otherwise, $1 \leq i \leq n$. Consequently, we have

$$f(x) = \bigvee_{i=1,\ldots,n} f_i(x)\ \forall\ x \in \mathcal{R}.$$

While computing $\widetilde{Y} = f(\widetilde{X})$, where $f$ is a real continuous function and $\widetilde{X}$ is a triangular fuzzy number, we have to do this at each possibility level as follows:

$$(4.8)\qquad y_1^{\alpha_i} = \min\left[\min_{x\in[x_1^{\alpha_i}, x_1^{\alpha_{i+1}}]} f(x), \ldots, \min_{x\in[x_1^{\alpha_n}, x_2^{\alpha_n}]} f(x), \ldots, \min_{x\in[x_2^{\alpha_{i+1}}, x_2^{\alpha_i}]} f(x)\right],$$

$$(4.9)\qquad y_2^{\alpha_i} = \max\left[\max_{x\in[x_1^{\alpha_i}, x_1^{\alpha_{i+1}}]} f(x), \ldots, \max_{x\in[x_1^{\alpha_n}, x_2^{\alpha_n}]} f(x), \ldots, \max_{x\in[x_2^{\alpha_{i+1}}, x_2^{\alpha_i}]} f(x)\right],$$

where $y_1^{\alpha_i}$ and $y_2^{\alpha_i}$ are lower and upper bounds of $\widetilde{Y}$, respectively at $\alpha_i$ for $i = 1, 2, \ldots, n$. In order to interpolate the points $(y_1^{\alpha_i}, \alpha_i)$ and $(y_2^{\alpha_i}, \alpha_i)$ for all $i = 0, 1, \ldots, n$, we use linear spline interpolation. Finnaly, a fuzzy interval $\widetilde{Y}$ is obtained. This process is repeated for all $t_j \in [t_0, T]$ for $j = 1, 2, \ldots, N-1$.

## 5. Runge-Kutta method of order five

Consider the IVP (3.1) with crisp initial condition $x(t_0) = x_0 \in \mathcal{R}$ and $t \in [t_0, T]$. The formula

$$(5.1)\qquad x_{k,j+1} = x_{k,j} + \sum_{j=1}^{n} w_j k_j,$$

is the basis of all classical Runge-Kutta methods, where $w_j$ is constant for $j = 1, 2, \ldots, n$ and

$$(5.2)\qquad k_j = h \cdot f\left(t_{k,r} + \beta_{k,j} h_k,\ x_{k,r} + \sum_{m=1}^{j-1} w_{j,r} k_{j,r}, \lambda_k(x_k)\right).$$

Therefore we get,

$$
\begin{aligned}
k_1 &= h_k \cdot f(t_{k,r},\ x_{k,r},\ \lambda_k(x_k)),\\
k_2 &= h_k \cdot f\left(t_{k,r} + \frac{h_k}{3},\ x_{k,r} + \frac{k_1}{3},\ \lambda_k(x_k)\right),\\
k_3 &= h_k \cdot f\left(t_{k,r} + \frac{h_k}{3},\ x_{k,r} + \frac{k_1}{6} + \frac{k_2}{6},\ \lambda_k(x_k)\right),\\
k_4 &= h_k \cdot f\left(t_{k,r} + \frac{h_k}{2},\ x_{k,r} + \frac{k_1}{8} + \frac{3k_3}{8},\ \lambda_k(x_k)\right)\\
k_5 &= h_k \cdot f\left(t_{k,r} + h_k,\ x_{k,r} + \frac{k_1}{2} - \frac{3k_3}{2} + 2k_4,\ \lambda_k(x_k)\right)
\end{aligned}
$$

and

$$x_{k,r+1} = x_{k,r} + \frac{1}{6}\Big(k_1 + 4k_4 + k_5\Big), \tag{5.3}$$

where $t_0 \leq t_1 \leq \ldots \leq t_N = T$ and $h = \frac{T-t_0}{N} = t_{r+1} - t_r,\ r = 0, 1, \ldots, N$.

For the fuzzy initial condition of equation (3.1), we modify the classical Runge-Kutta method of order five by taking into account the dependency problem in fuzzy computation. We consider the right-hand side of equation (5.3) as one function

(5.4)

$$
\begin{aligned}
&U(t_k, h_k, x_k, \lambda_k) = x_{k,r} + \frac{1}{6}\Big[k_1 + 4k_4 + k_5\Big],\\
&= x_{k,r} + \frac{h_k}{6}\Bigg\{ f(t_{k,r},\ x_{k,r},\ \lambda_k(x_k)) + 4f\Bigg(t_{k,r} + \frac{h_k}{2},\ x_{k,r} + \frac{h_k}{8} f(t_{k,r},\ x_{k,r},\ \lambda_k(x_k))\\
&\quad + \frac{3h_k}{8} f\Bigg(t_{k,r} + \frac{h_k}{3},\ x_{k,r} + \frac{h_k}{6} f(t_{k,r},\ x_{k,r},\ \lambda_k(x_k)) + \frac{h_k}{6} f\Bigg(t_{k,r} + \frac{h_k}{3},\ x_{k,r}\\
&\quad + \frac{h_k}{3} f(t_{k,r},\ x_{k,r},\ \lambda_k(x_k)),\ \lambda_k(x_k)\Bigg),\ \lambda_k(x_k)\Bigg),\ \lambda_k(x_k)\Bigg) + f\Bigg(t_{k,r} + h_k,\ x_{k,r}\\
&\quad + \frac{h_k}{2} f(t_{k,r},\ x_{k,r},\ \lambda_k(x_k)) - \frac{3h_k}{2} f\Bigg(t_{k,r} + \frac{h_k}{3},\ x_{k,r} + \frac{h_k}{6} f(t_{k,r},\ x_{k,r},\ \lambda_k(x_k))\\
&\quad + \frac{h_k}{6} f\Bigg(t_{k,r} + \frac{h_k}{3},\ x_{k,r} + \frac{h_k}{3} f(t_{k,r},\ x_{k,r},\ \lambda_k(x_k)),\ \lambda_k(x_k)\Bigg),\ \lambda_k(x_k)\Bigg)\\
&\quad + 2h_k f\Bigg(t_{k,r} + \frac{h_k}{2},\ x_{k,r} + \frac{h_k}{8} f(t_{k,r},\ x_{k,r},\ \lambda_k(x_k)) + \frac{3h_k}{8} f\Bigg(t_{k,r} + \frac{h_k}{3},\ x_{k,r}\\
&\quad + \frac{h_k}{6} f(t_{k,r},\ x_{k,r},\ \lambda_k(x_k)) + \frac{h_k}{6} f\Bigg(t_{k,r} + \frac{h_k}{3},\ x_{k,r}\\
&\quad + \frac{hk}{3} f(t_{k,r},\ x_{k,r},\ \lambda_k(x_k)),\ \lambda_k(x_k)\Bigg),\ \lambda_k(x_k)\Bigg),\ \lambda_k(x_k)\Bigg),\ \lambda_k(x_k)\Bigg)\Bigg\}.
\end{aligned}
$$

If $\widetilde{X} \in E^n$, then (5.4) can be extended in fuzzy setting as follows:

$$U(t_{k,r}, h_k, \widetilde{X}_{k,r}, \lambda_{k,r})(u_{k,r})$$

$$= \begin{cases} \sup\limits_{x_{k,r} \in U_k^{-1}(t_r, h, u_r, \lambda_k)} \widetilde{X}_{k,r}(x_{k,r}), & \text{if } u_{k,r} \in \text{range(U)}, \\ 0, & \text{if } u_{k,r} \notin \text{range(U)}, \end{cases} \tag{5.5}$$

can extend equation (5.4) in the fuzzy setting.

Let $[\widetilde{X}_{k,r}]^\alpha = [\underline{x}^\alpha_{k,r}, \overline{x}^\alpha_{k,r}]$ be the $\alpha$-cuts of $\widetilde{X}$ for all $\alpha \in (0, 1]$ then (5.5) can be computed as follows

$$\begin{aligned} &U(t_{k,r}, h_k, [\widetilde{X}_{k,r}]^\alpha, \lambda_{k,r}) \\ &= \Big[ \min\left\{ U(t_{k,r}, h_k, x, \lambda_k(x_k)) \middle| \, x \in [\underline{x}^\alpha_{k,r}, \overline{x}^\alpha_{k,r}], x_k \in [\underline{x}^\alpha_{k,0}, \overline{x}^\alpha_{k,0}] \right\}, \\ &\quad \max\left\{ U(t_{k,r}, h_k, x, \lambda_k(x_k)) \middle| \, x \in [\underline{x}^\alpha_{k,r}, \overline{x}^\alpha_{k,r}], x_k \in [\underline{x}^\alpha_{k,0}, \overline{x}^\alpha_{k,0}] \right\} \Big]. \end{aligned} \tag{5.6}$$

By applying equation (5.6) in equation (5.3) we get

$$[\tilde{X}_{k,r+1}]^\alpha = [\underline{x}^\alpha_{k,r+1}, \overline{x}^\alpha_{k,r+1}], \tag{5.7}$$

where

$$\begin{aligned} \underline{x}^\alpha_{k,r+1} &= \min\left\{ U(t_k, h_k, x, \lambda_k(x_k)) \middle| \, x \in [\underline{x}^\alpha_{k,r}, \overline{x}^\alpha_{k,r}], x_k \in [\underline{x}^\alpha_{k,0}, \overline{x}^\alpha_{k,0}] \right\}, \\ \overline{x}^\alpha_{k,r+1} &= \max\left\{ U(t_k, h_k, x, \lambda_k(x_k)) \middle| \, x \in [\underline{x}^\alpha_{k,r}, \overline{x}^\alpha_{k,r}], x_k \in [\underline{x}^\alpha_{k,0}, \overline{x}^\alpha_{k,0}] \right\}. \end{aligned}$$

Therefore

$$\underline{x}^\alpha_{k,r+1} = \min\left\{ x_{k,r} + \frac{h_k}{6}\left(k_1 + 4k_4 + k_5\right) \middle| \, x \in [\underline{x}^\alpha_{k,r}, \overline{x}^\alpha_{k,r}], x_k \in [\underline{x}^\alpha_{k,0}, \overline{x}^\alpha_{k,0}] \right\}, \tag{5.8}$$

$$\overline{x}^\alpha_{k,r+1} = \max\left\{ x_{k,r} + \frac{h_k}{6}\left(k_1 + 4k_4 + k_5\right) \middle| \, x \in [\underline{x}^\alpha_{k,r}, \overline{x}^\alpha_{k,r}], x_k \in [\underline{x}^\alpha_{k,0}, \overline{x}^\alpha_{k,0}] \right\}. \tag{5.9}$$

By using the computational method proposed in [3], we compute the minimum and maximum in equations (5.8), (5.9) as follows:

$$\begin{aligned} \underline{x}^{\alpha_i}_{k,r+1} = \min\Bigg[ &\min_{x \in [\underline{x}^{\alpha_i}_{k,r}, \underline{x}^{\alpha_{i+1}}_{k,r}]} U(t_k, h_k, x, \lambda_k), \ldots, \\ &\min_{x \in [\underline{x}^{\alpha_{i+1}}_{k,r}, \overline{x}^{\alpha_{i+1}}_{k,r}]} U(t_k, h_k, x, \lambda_k), \ldots, \min_{x \in [\overline{x}^{\alpha_{i+1}}_{k,r}, \overline{x}^{\alpha_i}_{k,r}]} U(t_k, h_k, x, \lambda_k) \Bigg], \end{aligned} \tag{5.10}$$

$$\begin{aligned} \overline{x}^{\alpha_i}_{k,r+1} = \max\Bigg[ &\max_{x \in [\underline{x}^{\alpha_i}_{k,r}, \underline{x}^{\alpha_{i+1}}_{k,r}]} U(t_k, h_k, x, \lambda_k), \ldots, \\ &\max_{x \in [\underline{x}^{\alpha_{i+1}}_{k,r}, \overline{x}^{\alpha_{i+1}}_{k,r}]} U(t_k, h_k, x, \lambda_k), \ldots, \max_{x \in [\overline{x}^{\alpha_{i+1}}_{k,r}, \overline{x}^{\alpha_i}_{k,r}]} U(t_k, h_k, x, \lambda_k) \Bigg]. \end{aligned} \tag{5.11}$$

## 6. Numerical Examples

We present some numerical examples to illustrate the above technique.

**Example 6.1.** Consider the following HFDEs

$$(6.1)\quad \begin{cases} x'(t) = x(t) + m(t)\lambda_k(x(t_k)), \quad t \in [t_k, t_{k+1}], \quad t_k = k, \quad k = 0, 1, 2, \cdots, \\ \left[\widetilde{X}\right]_0^\alpha = [0.75 + 0.25\alpha, 1.125 - 0.125\alpha], \quad 0 \le \alpha \le 1, \end{cases}$$

where

$$m(t) = \begin{cases} 2(t(\text{mod } 1)), & \text{if } \ t(\text{mod } 1) \le 0.5, \\ 2(1 - t(\text{mod } 1)), & \text{if } \ t(\text{mod } 1) \ge 0.5, \end{cases}$$

$$\lambda_k(\mu) = \begin{cases} \hat{0}, \ \text{if } k = 0, \\ \mu, \ \text{if } k \in \{1, 2, \ldots\}. \end{cases}$$

The exact solution for $t \in [0, 2]$ is given by

$$x(t) = \begin{cases} [(0.75 + 0.25\alpha)e^t, (1.125 - 0.125\alpha)e^t], & t \in [0, 1], \\ x(1;\alpha)[3e^{t-1} - 2t], & t \in [1, 1.5], \\ x(1;\alpha)[2t - 2 + e^{t-1.5}(3\sqrt{e} - 4)], & t \in [1.5, 2]. \end{cases}$$

**Table 1.**
Numerical values for the exact and the approximate solutions at t=2.

| $\alpha$ | RK4 | | RK5 | | Exact | |
|---|---|---|---|---|---|---|
| | $x_1(t_i;r)$ | $x_2(t_i;r)$ | $x_1(t_i;r)$ | $x_2(t_i;r)$ | $X_1(t_i;r)$ | $X_2(t_i;r)$ |
| 0.0 | 7.2577 | 10.8866 | 7.2577 | 10.8866 | 7.2577 | 10.8866 |
| 0.1 | 7.4996 | 10.7656 | 7.4997 | 10.7656 | 7.4997 | 10.7656 |
| 0.2 | 7.7416 | 10.6447 | 7.7416 | 10.6447 | 7.7416 | 10.6447 |
| 0.3 | 7.9835 | 10.5237 | 7.9835 | 10.5237 | 7.9835 | 10.5237 |
| 0.4 | 8.2254 | 10.4027 | 8.2254 | 10.4027 | 8.2254 | 10.4027 |
| 0.5 | 8.4673 | 10.2818 | 8.4674 | 10.2818 | 8.4673 | 10.2818 |
| 0.6 | 8.7093 | 10.1608 | 8.7093 | 10.1608 | 8.7093 | 10.1608 |
| 0.7 | 8.9512 | 10.0398 | 8.9512 | 10.0399 | 8.9512 | 10.0399 |
| 0.8 | 9.1931 | 9.9189 | 9.1931 | 9.9189 | 9.1931 | 9.9189 |
| 0.9 | 9.4350 | 9.7979 | 9.4350 | 9.7979 | 9.4350 | 9.7979 |
| 1.0 | 9.6770 | 9.6770 | 9.6770 | 9.6770 | 9.6770 | 9.6770 |

**Table 2.**
The errors of the methods at t=2.

| $\alpha$ | RK4 | | RK5 | |
|---|---|---|---|---|
| | $x_1(t_i;r)$ | $x_2(t_i;r)$ | $x_1(t_i;r)$ | $x_2(t_i;r)$ |
| 0.0 | 7.59e-06 | 1.13e-05 | -3.18e-6 | -4.78e-6 |
| 0.1 | 7.84e-06 | 1.12e-05 | -3.29e-6 | -4.73e-6 |
| 0.2 | 8.09e-06 | 1.11e-05 | -3.40e-6 | -4.67e-6 |
| 0.3 | 8.35e-06 | 1.10e-05 | -3.50e-6 | -4.62e-6 |
| 0.4 | 8.60e-06 | 1.08e-05 | -3.61e-6 | -4.57e-6 |
| 0.5 | 8.85e-06 | 1.07e-05 | -3.72e-6 | -4.51e-6 |
| 0.6 | 9.11e-06 | 1.06e-05 | -3.82e-6 | -4.46e-6 |
| 0.7 | 9.36e-06 | 1.05e-05 | -3.93e-6 | -4.41e-6 |
| 0.8 | 9.61e-06 | 1.03e-05 | -4.04e-6 | -4.35e-6 |
| 0.9 | 9.87e-06 | 1.02e-05 | -4.14e-6 | -4.30e-6 |
| 1.0 | 1.01e-05 | 1.01e-05 | -4.25e-6 | -4.25e-6 |

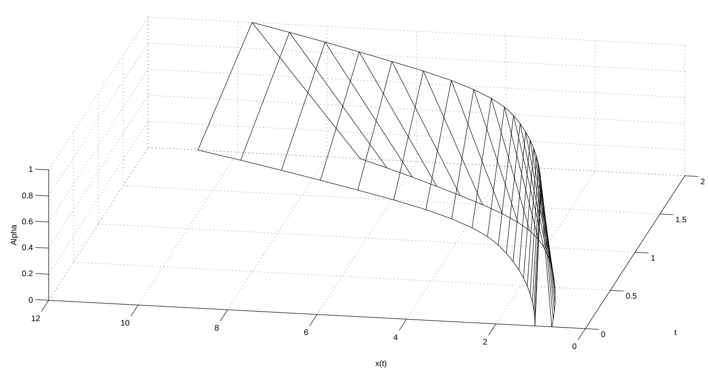


FIGURE 1. The approximation of fuzzy solution by RK5.

**Example 6.2.** Consider the following hybrid FIVP

$$(6.2)\quad \begin{cases} x'(t) = x(t) + m(t)\lambda_k(x(t_k)), \quad t \in [t_k, t_{k+1}], \quad t_k = k, \quad k = 0, 1, 2, \cdots, \\ \left[\widetilde{X}\right]_0^\alpha = [0.75 + 0.25\alpha,\ 1.5 - 0.5\alpha], \quad 0 \le \alpha \le 1, \end{cases}$$

where

$$m(t) = |sin(\pi t)|, \qquad k = 0, 1, 2, \ldots$$

$$\lambda_k(\mu) = \begin{cases} \hat{0}, \text{ if } k = 0, \\ \mu, \text{ if } k \in \{1, 2, \ldots\}. \end{cases}$$

We know (6.2) has a unique solution and the exact solution [0,2] is given by

$$x(t;r) = \begin{cases} [(0.75 + 0.25\alpha)e^t, (1.5 - 0.5\alpha)e^t], & t \in [0, 1], \\ x(1;\alpha)\Big[\dfrac{\pi}{\pi^2+1} + e\Big(1 + \dfrac{\pi}{\pi^2+1}\Big)\Big], & t \in [1, 2]. \end{cases}$$

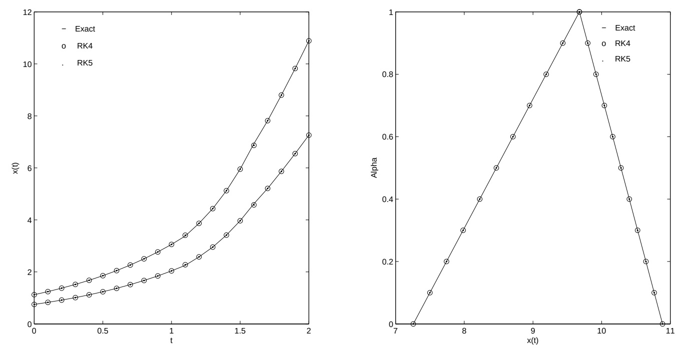


FIGURE 2. Comparison between the exact, RK4 and RK5.

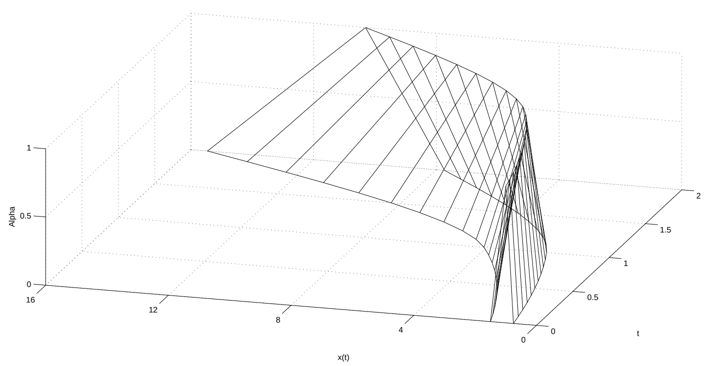


FIGURE 3. The approximate solution by RK5.

**Table 3.**

Numerical values for the exact and the approximate solutions at t=2.

| $\alpha$ | RK4 | | RK5 | | Exact | |
|---|---|---|---|---|---|---|
| | $x_1(t_i;r)$ | $x_2(t_i;r)$ | $x_1(t_i;r)$ | $x_2(t_i;r)$ | $X_1(t_i;r)$ | $X_2(t_i;r)$ |
| 0.0 | 7.7327 | 15.4655 | 7.7327 | 15.4655 | 7.7327 | 15.4655 |
| 0.1 | 7.9905 | 14.9500 | 7.9905 | 14.9500 | 7.9905 | 14.9500 |
| 0.2 | 8.2483 | 14.4345 | 8.2483 | 14.4345 | 8.2483 | 14.4345 |
| 0.3 | 8.5060 | 13.9189 | 8.5060 | 13.9190 | 8.5060 | 13.9189 |
| 0.4 | 8.7638 | 13.4034 | 8.7638 | 13.4035 | 8.7638 | 13.4034 |
| 0.5 | 9.0215 | 12.8879 | 9.0216 | 12.8879 | 9.0215 | 12.8879 |
| 0.6 | 9.2793 | 12.3724 | 9.2793 | 12.3724 | 9.2793 | 12.3724 |
| 0.7 | 9.5370 | 11.8569 | 9.5371 | 11.8569 | 9.5371 | 11.8569 |
| 0.8 | 9.7948 | 11.3414 | 9.7948 | 11.3414 | 9.7948 | 11.3414 |
| 0.9 | 10.0526 | 10.8258 | 10.0526 | 10.8259 | 10.0526 | 10.8258 |
| 1.0 | 10.3103 | 10.3103 | 10.3103 | 10.3103 | 10.3103 | 10.3103 |

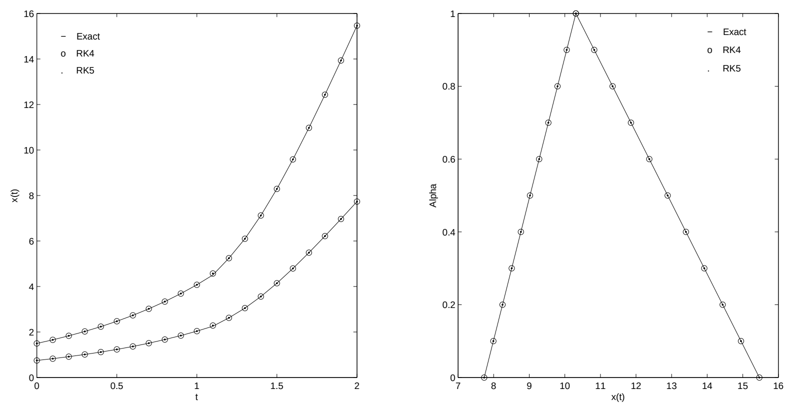


FIGURE 4. Comparison between the exact, RK4 and RK5.

**Table 4.**
The errors of the method at t=2.

| $\alpha$ | RK4 | | RK5 | |
|---|---|---|---|---|
| | $x_1(t_i;r)$ | $x_2(t_i;r)$ | $x_1(t_i;r)$ | $x_2(t_i;r)$ |
| 0.0 | 2.95e-06 | 5.90e-06 | -1.63e-05 | -3.26e-05 |
| 0.1 | 3.05e-06 | 5.71e-06 | -1.68e-05 | -3.15e-05 |
| 0.2 | 3.15e-06 | 5.51e-06 | -1.74e-05 | -3.05e-05 |
| 0.3 | 3.25e-06 | 5.31e-06 | -1.79e-05 | -2.94e-05 |
| 0.4 | 3.34e-06 | 5.12e-06 | -1.85e-05 | -2.83e-05 |
| 0.5 | 3.44e-06 | 4.92e-06 | -1.90e-05 | -2.72e-05 |
| 0.6 | 3.54e-06 | 4.72e-06 | -1.96e-05 | -2.61e-05 |
| 0.7 | 3.64e-06 | 4.53e-06 | -2.01e-05 | -2.50e-05 |
| 0.8 | 3.74e-06 | 4.33e-06 | -2.06e-05 | -2.39e-05 |
| 0.9 | 3.84e-06 | 4.13e-06 | -2.12e-05 | -2.28e-05 |
| 1.0 | 3.93e-06 | 3.93e-06 | -2.17e-05 | -2.17e-05 |

## 7. Conclusion

In this paper, we have developed the numerical method for solving HFDEs by taking into account the dependency problem in fuzzy computation. The convergence order of Runge-Kutta method is O($h^4$) and for Runge-Kutta method is O($h^5$). The comparison of solutions of example (6.1) and (6.2) are analyzed and we see that Runge-Kutta fifth order gives better solution than Runge-Kutta fourth method.

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K. Kanagarajan (kanagarajank@gmail.com)
Muthukumar S (sivagirimuthu@gmail.com)
Indrakumar. S (indrakumar1729@gmail.com)
Department of Mathematics, Sri Ramakrishna mission Vidyalaya College of Arts and Science, Coimbatore - 641020, Tamil Nadu, India