

## Fuzzy $A$ -ideals in $MV$ -modules

FERESHTEH FOROUZESH

Received 19 December 2014; Revised 1 March 2015; Accepted 20 March 2015

---

**ABSTRACT.** In the present paper, we introduce the notion of fuzzy  $A$ -ideals of  $MV$ -modules over  $PMV$ -algebras and several properties of fuzzy  $A$ -ideals are given. Using this concept, a prime fuzzy  $A$ -ideal is defined. Using a level set of a fuzzy set in  $MV$ -modules, we give a characterization of prime fuzzy  $A$ -ideals in  $MV$ -modules. Finally, we prove that the homomorphic image and preimage of prime fuzzy  $A$ -ideals are also prime fuzzy  $A$ -ideals in  $MV$ -modules.

2010 AMS Classification: 03B50, 06D35, 08A72.

Keywords:  $A$ -ideal, fuzzy  $A$ -ideal, Prime fuzzy  $A$ -ideal.

Corresponding Author: Fereshteh Forouzesh ([frouzesh@bam.ac.ir](mailto:frouzesh@bam.ac.ir), [ffrouzesh@yahoo.com](mailto:ffrouzesh@yahoo.com))

---

### 1. INTRODUCTION

The study of  $MV$ -algebras was initiated by C. Chang (1958) [1].  $MV$ -algebras are algebraic counterpart of the Lukasiewicz infinite many valued propositional logic.

Since then, product  $MV$ -algebras, (or  $PMV$ -algebras, for short) were introduced by Di Nola and Dvurecenskij (1998). In fact  $PMV$ -algebras are  $MV$ -algebras with product which are defined on the whole  $MV$ -algebras and are associative and left/right distributive with respect to a partial addition[3].

In 2003, Di Nola. et. al. [4] introduced  $MV$ -modules over  $PMV$ -algebras. These structures naturally correspond to  $lu$ -modules over  $lu$ -ring. They proved the equivalence between the category of  $lu$ -modules over  $(R, v)$  and the category of  $MV$ -modules over  $\Gamma(R, v)$ . They also proved the natural equivalence between  $MV$ -modules and truncated modules [4].

The concept of fuzzy set was formulated by Zadeh [13]. Since then, the theory of fuzzy sets developed by Zadeh and others has evoked tremendous interest among researchers working in different branches of mathematics. Since then fuzzy ideals and fuzzy filters theory have been applied to other algebraic structures (see [5, 9,

[14, 12]). In 1994, Hoo [7] defined fuzzy ideals of  $BCI$ ,  $BCK$ , and  $MV$ -algebras and investigated some properties.

In [6], we introduced and studied the notion of prime  $A$ -ideals in  $MV$ -modules.

In this paper, we study the notion of fuzzy  $A$ -ideals in  $MV$ -modules over  $PMV$ -algebras. We give other characterizations of fuzzy  $A$ -ideals of  $MV$ -modules. We introduce the notion of a fuzzy  $A$ -ideal generated by a fuzzy set. We define and investigate the notion of prime fuzzy  $A$ -ideals of an  $MV$ -module. Also, we establish some properties for a prime fuzzy  $A$ -ideal in  $MV$ -modules.

## 2. PRELIMINARIES

We recollect some definitions and results which will be used in the following:

**Definition 2.1** ([1]). An  $MV$ -algebra is a structure  $(A, \oplus, *, 0)$ , where  $\oplus$  is a binary operation,  $*$ , is a unary operation, and  $0$  is a constant such that the following axioms are satisfied for any  $x, y \in A$ :

(MV1)  $(A, \oplus, 0)$  is an abelian monoid,

(MV2)  $(x^*)^* = x$ ,

(MV3)  $0^* \oplus x = 0^*$ ,

(MV4)  $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$ .

Note that  $1 = 0^*$  and the auxiliary operation  $\odot$  as follow:

$$x \odot y = (x^* \oplus y^*)^*.$$

We recall that the natural order determines a bounded distributive lattice structure such that

$$x \vee y = x \oplus (x^* \odot y) = y \oplus (x \odot y^*) \quad \text{and} \quad x \wedge y = x \odot (x^* \oplus y) = y \odot (y^* \oplus x).$$

**Lemma 2.2** ([2]). In each  $MV$ -algebra, the following relations hold for all  $x, y, z \in A$ :

(1) If  $x \leq y$ , then  $x \oplus z \leq y \oplus z$  and  $x \odot z \leq y \odot z$ ,

(2)  $x \leq y$  if and only if  $x^* \oplus y = 1$  if and only if  $x \odot y^* = 0$  if and only if there exists  $u \in A$  such that  $x \oplus u = y$ ,

(3)  $x, y \leq x \oplus y$  and  $x \odot y \leq x, y$ ,

(4) If  $x \leq y$  and  $z \leq t$ , then  $x \oplus z \leq y \oplus t$ ,

(5)  $(x \odot y^*) \wedge (y \odot x^*) = 0$ ,

(6) If  $x \leq y$ , then  $y^* \leq x^*$ .

**Definition 2.3** ([1, 11]). An ideal of an  $MV$ -algebra  $A$  is a nonempty subset  $I$  of  $A$  satisfying the following conditions:

(I1) If  $x \in I$ ,  $y \in A$  and  $y \leq x$  then  $y \in I$ ,

(I2) If  $x, y \in I$ , then  $x \oplus y \in I$ .

We denote by  $Id(A)$  the set of ideals of an  $MV$ -algebra  $A$ .

**Definition 2.4** ([3]). A product  $MV$ -algebra (or  $PMV$ -algebra, for short) is a structure  $(A, \oplus, *, \cdot, 0)$ , where  $(A, \oplus, *, 0)$  is an  $MV$ -algebra and  $\cdot$  is a binary associative operation on  $A$  such that the following property is satisfied:

if  $x + y$  is defined, then  $x \cdot z + y \cdot z$  and  $z \cdot x + z \cdot y$  are defined and

$$(x + y) \cdot z = x \cdot z + y \cdot z, \quad z \cdot (x + y) = z \cdot x + z \cdot y,$$

where  $+$  is a partial addition on  $A$ , i. e.,  $x + y$  is defined if and only if  $x \leq y^*$  and in this case we put  $x + y := x \oplus y$ .

If  $A$  is a  $PMV$ -algebra, then a unity for the product is an element  $e \in A$  such that  $e \cdot x = x \cdot e = x$  for any  $x \in A$ . A  $PMV$ -algebra that has unity for the product will be called unital.

**Definition 2.5** ([4]). Let  $(A, \oplus, *, \cdot, 0)$  be a  $PMV$ -algebra and  $(M, \oplus, *, 0)$  an  $MV$ -algebra. We say that  $M$  is a (left)  $MV$ -module over  $A$  (or, simply,  $A$ -module) if there is an external operation:

$$\varphi : A \times M \longrightarrow M, \quad \varphi(\alpha, x) = \alpha x,$$

such that the following properties hold for any  $x, y \in M$  and  $\alpha, \beta \in A$ :

(1) If  $x + y$  is defined in  $M$ , then  $\alpha x + \alpha y$  is defined and

$$\alpha(x + y) = \alpha x + \alpha y,$$

(2) If  $\alpha + \beta$  is defined in  $A$  then  $\alpha x + \beta x$  is defined in  $M$  and

$$(\alpha + \beta)x = \alpha x + \beta x,$$

(3)  $(\alpha \cdot \beta)x = \alpha(\beta x)$ .

We say that  $M$  is a unital  $MV$ -module if  $A$  is a unital  $PMV$ -algebra and  $M$  is an  $MV$ -module over  $A$  such that  $1_A x = x$  for any  $x \in M$ .

**Definition 2.6** ([4]). Let  $M$  and  $N$  be two  $MV$ -modules over a  $PMV$ -algebra  $A$ . An  $A$ -module homomorphism is an  $MV$ -algebra homomorphism  $h : M \rightarrow N$  such that  $h(\alpha x) = \alpha h(x)$ , for any  $\alpha \in A$  and  $x \in M$ .

**Definition 2.7** ([4]). Let  $M$  be an  $A$ -module. Then ideal  $I \subseteq M$  is called an  $A$ -ideal if it satisfies the following condition:  
if  $x \in I$  and  $\alpha \in A$ , then  $\alpha x \in I$ .

**Lemma 2.8** ([4]). If  $M$  is an  $A$ -module, then the following properties hold for any  $x, y \in M$  and  $\alpha, \beta \in A$ ,

- (a)  $\alpha x^* \leq (\alpha x)^*$ ,
- (b)  $(\alpha x) \odot (\alpha y)^* \leq \alpha(x \odot y^*)$ ,
- (c)  $\alpha(x \oplus y) \leq \alpha x \oplus \alpha y$ ,
- (d) If  $x \leq y$ , then  $\alpha x \leq \alpha y$ .

**Definition 2.9** ([6]). Let  $M$  be an  $A$ -module. Then an  $A$ -ideal  $P$  of an  $MV$ -module  $M$  is a prime  $A$ -ideal, if (i)  $P \neq M$  (ii) for every  $\alpha \in A$ ,  $x \in M$  if  $\alpha x \in P$ , then  $x \in P$  or  $\alpha \in (P : M)$ , where  $(P : M) = \{r \in A : rM \subseteq P\}$  and  $rM = \{rm | m \in M\}$ .

**Definition 2.10** ([13]). A fuzzy set in  $A$  is a mapping  $\mu : A \rightarrow [0, 1]$ . Let  $\mu$  be a fuzzy set in  $A$ ,  $t \in [0, 1]$ , the set  $\mu_t = \{x \in A : \mu(x) \geq t\}$  is called a level subset of  $\mu$ . For any fuzzy sets  $\mu, \nu$  in  $A$ , we define

$$\mu \subseteq \nu \text{ if and only if } \mu(x) \leq \nu(x) \text{ for all } x \in A.$$

**Definition 2.11** ([13]). Let  $X, Y$  be two sets.  $\mu$  is a fuzzy subset of  $X$ ,  $\mu'$  is a fuzzy subset of  $Y$  and  $f : X \rightarrow Y$  is a homomorphism. The image of  $\mu$  under  $f$  denoted by  $f(\mu)$  is a fuzzy set of  $Y$  defined by:

For all  $y \in Y$ ,  $f(\mu)(y) = \sup_{x \in f^{-1}(y)} \mu(x)$ , if  $f^{-1}(y) \neq \emptyset$  and  $f(\mu)(y) = 0$  if  $f^{-1}(y) = \emptyset$ .

The preimage of  $\mu'$  under  $f$  denoted by  $f^{-1}(\mu')$  is a fuzzy set of  $X$  defined by: For all  $x \in X$ ,  $f^{-1}(\mu')(x) = \mu'(f(x))$ .

**Definition 2.12** ([13]). A fuzzy subset  $\mu$  of  $X$  has sup-property if for any nonempty subset  $Y$  of  $X$ , there exists  $y_0 \in Y$  such that  $\mu(y_0) = \sup_{y \in Y} \mu(y)$ .

**Theorem 2.13** ([7]). Let  $\mu$  be a fuzzy ideal in  $A$ . For any  $x, y, z \in A$ , the following hold:

- (a)  $\mu(x \oplus y) = \mu(x) \wedge \mu(y)$ ,
- (b)  $\mu(x \vee y) = \mu(x) \wedge \mu(y)$ ,
- (c)  $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$ .

### 3. FUZZY $A$ -IDEALS OF $MV$ -MODULES

In this section, we investigate fuzzy  $A$ -ideals of an  $MV$ -module  $M$  over  $PMV$ -algebra  $A$ .

In sequel section,  $M$  is an  $MV$ -module over  $PMV$ -algebra  $A$  and  $A$  is a  $PMV$ -algebra.

**Definition 3.1.** A fuzzy set  $\mu$  in an  $MV$ -module  $M$  is called a fuzzy  $A$ -ideal of  $M$  if it satisfies for all  $x, y \in M$  and  $\alpha \in A$ :

- (d1)  $\mu(0) \geq \mu(x)$ ,
- (d2)  $\mu(y) \geq \mu(x) \wedge \mu(y \odot x^*)$ ,
- (d3)  $\mu(\alpha x) \geq \mu(x)$ .

The following example, shows that fuzzy  $A$ -ideals in  $MV$ -modules exist and a fuzzy set may not be a fuzzy  $A$ -ideal in  $MV$ -module.

**Example 3.2.** Let  $A = \{0, 1, 2\}$  be a linearly ordered set (chain).  $A$  is an  $MV$ -algebra with operations  $\wedge = \min$ ,  $x \oplus y = \min\{2, x+y\}$  and  $x \odot y = \max\{0, x+y-2\}$ , for every  $x, y \in A$  [8]. Also,  $A$  is  $PMV$ -algebra with the following operations:

$\oplus$	0	1	2	$\cdot$	0	1	2	$*$	0	1	2
0	0	1	2	0	0	0	0		2	1	0
1	1	2	2	1	0	0	0				
2	2	2	2	2	0	0	1				

and  $A$  becomes an  $A$ -module over  $A$  with the external operation defined by  $\alpha x = \alpha \cdot x$ , for any  $\alpha \in A$  and  $x \in A$ . (i) Define a fuzzy set  $\mu$  in  $A$  by  $\mu(0) = 0.7$ ,  $\mu(1) = 0.4$  and  $\mu(2) = 0.4$ . It easy checked that  $\mu$  is a fuzzy  $A$ -ideal of  $A$ .

(ii) We define a fuzzy set  $\mu$  in  $A$  as follows:  $\mu(0) = 0.7$ ,  $\mu(1) = 0.4$  and  $\mu(2) = 0.6$ . Since  $\mu(22) = \mu(2 \cdot 2) = \mu(1) = 0.4 \not\geq \mu(2) = 0.6$ ,  $\mu$  is not a fuzzy  $A$ -ideal of  $A$ .

**Example 3.3.** Let  $(R, \cdot, v)$  be an  $lu$ -ring and  $G$  be an  $lu$ -group such that  $R \times G$  is the lexicographic product. Then  $A = \Gamma(R, v) = [0, v]_R$  is a  $PMV$ -algebra. Also,  $M = \Gamma(R \times G, (v, 0)) = [(0, 0), (v, 0)]_{R \times G}$  is an  $MV$ -module over  $A$  with operation

$\circ : [0, v]_R \times M \rightarrow M$  such that  $r \circ (q, x) := (r \cdot q, x)$ , for all  $r, q \in [0, v]_R$ ,  $x \in M$ . Define a fuzzy set  $\mu$  in  $M$  by  $\mu((0, 0)) = \alpha_1$  and  $\mu((x, y)) = \alpha_2$ , for all  $(x, y) \neq (0, 0)$  such that  $0 \leq \alpha_2 < \alpha_1 \leq 1$ . It is clear that  $\mu$  is a fuzzy  $A$ -ideal.

**Example 3.4.** Let  $M_2(\mathbb{R})$  be the ring of square matrices of order 2 with real elements and 0 be the matrix with all element 0. If we define the order relation on components  $A = (a_{ij})_{i,j=1,2} \geq 0$  iff  $a_{ij} \geq 0$  for any  $i, j$ , such that  $v = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ , then  $A = \Gamma(M_2(\mathbb{R}), v)$  is a  $PMV$ -algebra. Let  $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$  be the direct product with the order relation defined on components. If  $M = \Gamma(\mathbb{R}^2, u)$  is an  $MV$ -algebra, where  $u = (1, 1)$ , then  $M$  is an  $A$ -module [4], where the external operation is the usual matrix multiplication

$$(A, (x, y)) \mapsto A \begin{pmatrix} x \\ y \end{pmatrix}.$$

Define a fuzzy set  $\mu$  in  $M$  by

$$\mu((x, y)) = \begin{cases} 1 & \text{if } (x, y) = (0, 0) \\ \alpha_1 & \text{if } (0, 0) < (x, y) < (1/2, 1/2) \\ \alpha_2 & \text{if } (1/2, 1/2) \leq (x, y) < (1, 1) \\ 0 & \text{if } (x, y) = (1, 1) \end{cases}$$

such that  $0 < \alpha_1 < \alpha_2 < 1$ .

$\mu\left(\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}\right) = \mu\left(\begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}\right) = \alpha_1 \not\geq \mu((1/2, 1/2)) = \alpha_2$ , hence  $\mu$  is not fuzzy  $A$ -ideal of  $M$ .

It is not difficult to show the following:

**Remark 3.5.** A fuzzy set  $\mu$  is a fuzzy  $A$ -ideal, if it satisfies for all  $x, y \in M$  and  $\alpha \in A$ :

- (i)  $\mu(x \oplus y) \geq \mu(x) \wedge \mu(y)$ ,
- (ii) if  $y \leq x$ , then  $\mu(y) \geq \mu(x)$ ,
- (iii)  $\mu(\alpha x) \geq \mu(x)$ .

**Theorem 3.6.** If  $\mu$  is a fuzzy  $A$ -ideal of  $M$ , then for all  $x, y, z \in M$ ,  $\alpha \in A$ ,  $z \odot x^* \odot (\alpha y)^* = 0$  implies  $\mu(z) \geq \mu(x) \wedge \mu(\alpha y)$ .

*Proof.* Since  $z \odot x^* \odot (\alpha y)^* = 0$ , so  $z \odot (x \oplus \alpha y)^* = 0$ , by Lemma 2.2 (2),  $z \leq x \oplus \alpha y$ , it follows from Theorem 2.13 (a) that  $\mu(z) \geq \mu(x \oplus \alpha y) = \mu(x) \wedge \mu(\alpha y)$ .  $\square$

In the following lemma, we show that if  $M$  is a unital  $A$ -module, then the converse of the above theorem is true.

**Lemma 3.7.** Let  $M$  be a unital  $A$ -module. If for all  $x, y, z \in M$  and  $\alpha \in A$ ,  $z \odot x^* \odot (\alpha y)^* = 0$ , implies  $\mu(z) \geq \mu(x) \wedge \mu(\alpha y)$ , then  $\mu$  is a fuzzy  $A$ -ideal of  $M$ .

*Proof.* Since  $0 \odot x^* \odot (\alpha x)^* = 0$ , for all  $x \in M$ ,  $\mu(0) \geq \mu(x) \wedge \mu(\alpha x)$ , also, since  $\alpha x \leq 1x = x$ ,  $\mu(\alpha x) \geq \mu(x)$ , hence  $\mu(0) \geq \mu(x)$ . Since  $\alpha x \leq x$ , by Lemma 2.2 (1) and (6), we have  $\alpha x \odot x^* \odot 0^* \leq \alpha x \odot (\alpha x)^* \odot 0^* = 0$ . Hence  $\alpha x \odot x^* \odot 0^* = 0$ . This results by hypothesis,  $\mu(\alpha x) \geq \mu(x) \wedge \mu(0) = \mu(x)$ . Thus  $\mu(\alpha x) \geq \mu(x)$ .

Also, let  $y \leq x$ . Hence by Lemma 2.2 (2), we have  $y \odot x^* \odot 0^* = 0$ . It follows that  $\mu(y) \geq \mu(x)$ .

Finally, since  $0 = (x \oplus y) \odot (x \oplus y)^* = (x \oplus y) \odot x^* \odot (1y)^* = 0$ , we conclude that  $\mu(x \oplus y) \geq \mu(x) \wedge \mu(y)$ . Therefore  $\mu$  is a fuzzy  $A$ -ideal of  $M$ .  $\square$

It is not difficult to show the following:

**Corollary 3.8.** *Let  $\mu$  be a fuzzy set in unital  $A$ -module  $M$ .  $\mu$  is a fuzzy  $A$ -ideal if and only if for all  $x, y, z \in M$ ,  $z \leq x \oplus \alpha y$  implies  $\mu(z) \geq \mu(x) \wedge \mu(\alpha y)$ .*

Now, we describe the transfer principle [10] for fuzzy  $A$ -ideals in terms of level subsets:

**Proposition 3.9.** *Let  $\mu$  be a fuzzy set in  $M$ . Then  $\mu$  is a fuzzy  $A$ -ideal of  $M$  if and only if its level subset  $\mu_t$  is empty or is an  $A$ -ideal of  $M$ , for all  $t \in [0, 1]$ .*

*Proof.* Let  $\mu$  be a fuzzy  $A$ -ideal of  $M$ . Suppose that  $t \in [0, 1]$  and  $x \in \mu_t$ .  $\mu(x) \geq t$ , since  $\mu$  is a fuzzy  $A$ -ideal,  $\mu(0) \geq \mu(x)$ , therefore  $0 \in \mu_t$ . Also, let  $x \in \mu_t$ . We only show that for all  $\alpha \in A$ ,  $\alpha x \in \mu_t$ .

Since  $\mu(\alpha x) \geq \mu(x) \geq t$ , thus  $\mu(\alpha x) \geq t$ . Therefore  $\alpha x \in \mu_t$ . This results  $\mu_t$  is an  $A$ -ideal of  $M$ .

Conversely, let  $\mu_t$  be an  $A$ -ideal of  $M$ . We only show that  $\mu(\alpha x) \geq \mu(x)$ . If not, then there exist  $a \in A$  and  $y \in M$  such that  $\mu(ay) < \mu(y)$ . Setting  $t_0 = 1/2(\mu(ay) + \mu(y))$ . We have  $\mu(ay) < t_0 < \mu(y)$ . We conclude that  $y \in \mu_{t_0}$  while  $ay \notin \mu_{t_0}$ , which is a contradiction. Thus  $\mu$  is a fuzzy  $A$ -ideal of  $M$ .  $\square$

**Corollary 3.10.**  *$J$  is an  $A$ -ideal of  $M$  if and only if  $\chi_J$  is a fuzzy  $A$ -ideal of  $M$ .*

**Corollary 3.11.** *If  $\mu$  is a fuzzy  $A$ -ideal of  $M$ , then  $I = \{x \in M | \mu(x) = \mu(0)\}$  is an  $A$ -ideal of  $M$ .*

The following example shows that the converse of the above corollary does not hold.

**Example 3.12.** Let  $A$  be an  $MV$ -module from Example 3.2. Define a fuzzy set  $\mu$  in  $A$  by  $\mu(x) = 1/3$ , if  $x = 0$  and  $\mu(x) = 3/4$ , if  $x \neq 0$ . Then  $I = \{x \in A | \mu(x) = \mu(0)\} = \{0\}$  is an  $A$ -ideal of  $A$  but  $\mu$  is not a fuzzy  $A$ -ideal of  $A$ .

**Note.** The meet of two fuzzy ideals  $\mu_1$  and  $\mu_2$  of  $M$  is defined as follows:

$$\mu_1 \wedge \mu_2 = \mu_1 \cap \mu_2.$$

Easily, we can show that the following lemma holds.

**Lemma 3.13.** *Let  $\mu_i$ , for all  $i = 1, 2$  be fuzzy  $A$ -ideals of  $M$ . Then  $\mu_1 \wedge \mu_2$  is a fuzzy  $A$ -ideal of  $M$ .*

In general, it is not difficult to see the following:

**Corollary 3.14.** *Let  $\mu_i$ , for all  $i \in I$  be fuzzy  $A$ -ideals of  $M$ . Then  $\bigwedge_{i \in I} \mu_i$  is a fuzzy  $A$ -ideal of  $M$ .*

Now, we introduce the notion of a fuzzy  $A$ -ideal of  $M$  generated by a fuzzy set in  $M$ .

**Definition 3.15.** Let  $f$  be a fuzzy set in  $M$ . A fuzzy  $A$ -ideal  $g$  in  $M$  is said to be generated by  $f$ , if  $f \leq g$  and for any fuzzy  $A$ -ideal  $h$  in  $M$ ,  $f \leq h$  implies  $g \leq h$ . The fuzzy  $A$ -ideal generated by  $f$  will be denoted by  $\langle f \rangle$ .

**Theorem 3.16.** *If  $f$  is a fuzzy set, then*

$$\begin{aligned} \langle f \rangle(x) &= \bigvee \{f(a_1) \wedge \cdots \wedge f(a_n) \wedge f(\alpha_1 b_1) \wedge \cdots \wedge f(\alpha_m b_m) | \\ &a_1, \dots, a_n, b_1, \dots, b_m \in M, \alpha_1, \dots, \alpha_m \in A \text{ and} \\ &x \leq a_1 \oplus \cdots \oplus a_n \oplus \alpha_1 b_1 \oplus \cdots \oplus \alpha_m b_m\}. \end{aligned}$$

*Proof.* Let

$$\begin{aligned} \mu(x) &= \bigvee \{f(a_1) \wedge \cdots \wedge f(a_n) \wedge f(\alpha_1 b_1) \wedge \cdots \wedge f(\alpha_m b_m) | \\ &a_1, \dots, a_n, b_1, \dots, b_m \in M, \alpha_1, \dots, \alpha_m \in A \text{ and} \\ &x \leq a_1 \oplus \cdots \oplus a_n \oplus \alpha_1 b_1 \oplus \cdots \oplus \alpha_m b_m\}. \end{aligned}$$

(1) We show that  $\mu$  is a fuzzy  $A$ -ideal. Obviously,  $\mu(0) \geq \mu(x)$ , for all  $x \in M$ .

(2) Let  $x, y \in M$ . If there exist  $s_1, \dots, s_n, t_1, \dots, t_r, k_1, \dots, k_l, p_1, \dots, p_m \in M$ ,  $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_m \in A$  such that  $x \leq s_1 \oplus \cdots \oplus s_n \oplus \alpha_1 k_1 \oplus \cdots \oplus \alpha_l k_l$  and  $x^* \odot y \leq t_1 \oplus \cdots \oplus t_r \oplus p_1 \beta_1 \oplus \cdots \oplus p_m \beta_m$ . Then  $y \leq x \vee y = x \oplus (x^* \odot y) \leq s_1 \oplus \cdots \oplus s_n \oplus \alpha_1 k_1 \oplus \cdots \oplus \alpha_l k_l \oplus t_1 \oplus \cdots \oplus t_r \oplus p_1 \beta_1 \oplus \cdots \oplus p_m \beta_m$ . By Remark 3.5 (ii) and Theorem 2.13 (a),  $\mu(y) \geq f(s_1) \wedge \cdots \wedge f(s_n) \wedge f(t_1) \wedge \cdots \wedge f(t_r) \wedge f(\alpha_1 k_1) \wedge \cdots \wedge f(\alpha_l k_l) \wedge f(p_1 \beta_1) \wedge \cdots \wedge f(p_m \beta_m)$ .

Denote by  $\Gamma_1 = \{f(a_1) \wedge \cdots \wedge f(a_k) \wedge f(\alpha_1 b_1) \wedge \cdots \wedge f(\alpha_m b_m) | x \leq a_1 \oplus \cdots \oplus a_k \oplus \alpha_1 b_1 \oplus \cdots \oplus \alpha_m b_m\}$

$\Gamma_2 = \{f(d_1) \wedge \cdots \wedge f(d_l) \wedge f(\beta_1 c_1) \wedge \cdots \wedge f(\beta_t c_t) | x^* \odot y \leq d_1 \oplus \cdots \oplus d_l \oplus \beta_1 c_1 \oplus \cdots \oplus \beta_t c_t\}$ , for some  $\beta_1, \dots, \beta_t \in A$ ,  $d_1, \dots, d_l, c_1, \dots, c_t \in M$ .

$\mu(x) \wedge \mu(x^* \odot y) = \bigvee \Gamma_1 \wedge \bigvee \Gamma_2 = \bigvee \{f(a_1) \wedge \cdots \wedge f(a_k) \wedge f(d_1) \wedge \cdots \wedge f(d_l) \wedge f(\alpha_1 b_1) \wedge \cdots \wedge f(\alpha_m b_m) \wedge f(\beta_1 c_1) \wedge \cdots \wedge f(\beta_t c_t) | x \leq a_1 \oplus \cdots \oplus a_k \oplus \alpha_1 b_1 \oplus \cdots \oplus \alpha_m b_m, x^* \odot y \leq d_1 \oplus \cdots \oplus d_l \oplus \beta_1 c_1 \oplus \cdots \oplus \beta_t c_t\}$ . Hence  $\mu(x) \wedge \mu(x^* \odot y) \leq \mu(y)$ .

Now, we show that  $\mu(\alpha x) \geq \mu(x)$ .

Let  $x \in A$ . If there exist  $q_1, \dots, q_n, r_1, \dots, r_m \in M, \gamma_1, \dots, \gamma_m \in A$  such that  $x \leq q_1 \oplus \cdots \oplus q_n \oplus r_1 \gamma_1 \oplus \cdots \oplus r_m \gamma_m$ . Hence by Lemma 2.8 (d), (c), we have  $\alpha x \leq \alpha(q_1 \oplus \cdots \oplus q_n \oplus r_1 \gamma_1 \oplus \cdots \oplus r_m \gamma_m) \leq \alpha q_1 \oplus \cdots \oplus \alpha q_n \oplus r_1(\alpha \cdot \gamma_1) \oplus \cdots \oplus r_m(\alpha \cdot \gamma_m)$ , hence  $\mu(\alpha x) \geq f(z_1) \wedge \cdots \wedge f(z_n) \wedge f(w_1 \gamma_1) \wedge \cdots \wedge f(w_m \gamma_m)$ , where  $\alpha q_i = z_i \in M$  and  $\alpha \cdot \gamma_i = w_i \in A$ .

Denote by  $\Gamma = \{f(a_1) \wedge \cdots \wedge f(a_k) \wedge f(\alpha_1 b_1) \wedge \cdots \wedge f(\alpha_m b_m) | x \leq a_1 \oplus \cdots \oplus a_k \oplus \alpha_1 b_1 \oplus \cdots \oplus \alpha_m b_m\}$ .

We have  $\mu(x) = \bigvee \Gamma = \bigvee \{f(a_1) \wedge \cdots \wedge f(a_k) \wedge f(\alpha_1 b_1) \wedge \cdots \wedge f(\alpha_m b_m)\}$ . Hence  $\mu(\alpha x) \geq \mu(x)$ . It follows that  $\mu$  is a fuzzy  $A$ -ideal of  $M$ .

Now, note that  $f \leq \mu$ . Since  $x \leq x \oplus x$ , we get  $\mu(x) \geq f(x) \wedge f(x) = f(x)$ , for all  $x \in A$ . Let  $\nu$  be a fuzzy  $A$ -ideal such that  $f \leq \nu$ . Then for any  $x \in M$ ,  $\mu(x) = \bigvee \{f(a_1) \wedge \cdots \wedge f(a_n) \wedge f(b_1 \alpha_1) \wedge \cdots \wedge f(b_m \alpha_m) | x \leq a_1 \oplus \cdots \oplus a_n \oplus \alpha_1 b_1 \oplus \cdots \oplus \alpha_m b_m\} \leq \bigvee \{\nu(a_1) \wedge \cdots \wedge \nu(a_n) \wedge \nu(\alpha_1 b_1) \wedge \cdots \wedge \nu(\alpha_m b_m)\} \leq \nu(x)$ , because by Remark 3.5 (ii) and Theorem 2.13 (a). Hence  $\mu \leq \nu$ . Therefore  $\mu$  is the fuzzy  $A$ -ideal generated by  $f$ , that is  $\mu = \langle f \rangle$ .  $\square$

**Theorem 3.17.** *Let  $f$  and  $g$  be fuzzy sets in  $M$ . The following properties hold:*

- (a) *if  $f$  is a fuzzy  $A$ -ideal of  $M$ , then  $\langle f \rangle = f$ ,*
- (b) *if  $f \leq g$ , then  $\langle f \rangle \leq \langle g \rangle$ ,*
- (c)  $\langle 0 \rangle = 0_A$ ,
- (d)  $\langle 1_M \rangle = 1_M$ .

**Example 3.18.** Let  $A$  be the  $A$ -module in Example 3.2. Define a fuzzy set  $f$  in  $A$  by  $f(2) = 0.5$ ,  $f(1) = 0.5$  and  $f(0) = 0.8$ . We can check that the fuzzy  $A$ -ideal  $\langle f \rangle$  generated by  $f$  is  $\langle f \rangle(0) = 0.8$ ,  $\langle f \rangle(1) = 0.5$  and  $\langle f \rangle(2) = 0.5$ .

#### 4. PRIME FUZZY $A$ -IDEALS IN $MV$ -MODULES

**Definition 4.1.**  $\mu$  is called prime fuzzy  $A$ -ideal of an  $MV$ -module  $M$  over  $PMV$ -algebra  $A$  if,  $\mu(rx) > \mu(x)$ ,  $r \in A$ ,  $x \in M$  implies  $\mu(rx) \leq \mu(rm)$ , for all  $m \in M$ .

The following example shows that a fuzzy  $A$ -ideal may not be a prime fuzzy  $A$ -ideal of  $M$ .

**Example 4.2.** Let  $\Omega = \{1, 2\}$  and  $\mathcal{A} = \mathcal{P}(\Omega)$ . Which is a  $PMV$ -algebra with  $\oplus = \cup$  and  $\odot = \cdot = \cap$ . If we consider  $\mathcal{M} = \mathcal{P}(\Omega) = \{\{1\}, \{2\}, \{1, 2\}, \emptyset\}$ , then  $\mathcal{M}$  becomes an  $MV$ -module over  $\mathcal{A}$  with the external operation defined by  $AX := A \cap X$  for any  $A \in \mathcal{A}$  and  $X \in \mathcal{M}$ . Define a fuzzy set  $\mu$  with  $\mu(\emptyset) = 1$  and  $\mu(\{1\}) = \mu(\{2\}) = \mu(\{1, 2\}) = 0.5$ . Obviously,  $\mu$  is a fuzzy  $A$ -ideal of  $\mathcal{M}$ . Consider  $r = \{2\}$  and  $x = \{1\}$ , we have  $\mu(\{2\}\{1\}) = \mu(\emptyset) = 1 > \mu(\{1\})$ , while  $1 = \mu(\emptyset) = \mu(\{2\}\{1\}) \not\leq \mu(\{2\}\{1, 2\}) = \mu(\{2\}) = 0.5$ . Hence  $\mu$  is not a prime fuzzy  $A$ -ideal of  $\mathcal{M}$ .

**Example 4.3.** Let  $l_3 = \{0, 1/2, 1\}$  and  $M = l_3 \times l_3$  be an  $A = \Gamma(\mathbb{Z}, 1)$ -module with operations  $(x, y) \odot (z, t) = [(x, y) - (1, 1) + (z, t)] \vee (0, 0)$ ,  $(x, y)^* = (1, 1) - (x, y)$  and  $\alpha(x, y) = (\alpha x, \alpha y)$ , for any  $\alpha \in A$  ( $\alpha = 0$  or  $\alpha = 1$ ),  $(x, y) \in M$ .

Define a fuzzy set  $\mu$  by  $\mu((0, 0)) = 0.8$  and  $\mu((x, y)) = 0.3$ , if  $(x, y) \neq (0, 0)$ .

Consider  $(x, y) \neq (0, 0)$  and  $r = 0$ , we have  $0.8 = \mu(0(x, y)) = \mu((0, 0)) > \mu((x, y)) = 0.3$  implies  $\mu(0(x, y)) \leq \mu(0(z, t))$ , for all  $(z, t) \in l_3 \times l_3$ .

Hence  $\mu$  is a prime fuzzy  $A$ -ideal of  $M$ .

**Theorem 4.4.** Let  $f : X \rightarrow Y$  be onto  $MV$ -module homomorphism. Then the preimage of a prime fuzzy  $A$ -ideal  $\mu$  under  $f$  is also a prime fuzzy  $A$ -ideal of  $X$ .

*Proof.* Suppose that  $\mu$  is a prime fuzzy  $A$  ideal of  $Y$ .

(i) First, we show that  $f^{-1}(\mu)$  is a fuzzy  $A$ -ideal on  $X$ . Since  $0 = f(0) \leq f(x)$  and  $f^{-1}(\mu)(x) = \mu(f(x)) \leq \mu(f(0)) = f^{-1}(\mu)(0)$ . Also, we have  $f^{-1}(\mu)(rx) = \mu(f(rx)) = \mu(rf(x)) \geq \mu(f(x)) = f^{-1}(\mu)(x)$ .

Now, suppose that  $f^{-1}(\mu)(rx) > f^{-1}(\mu)(x)$ , then  $\mu(rf(x)) = \mu(f(rx)) > \mu(f(x))$ . Hence since  $\mu$  is a prime fuzzy  $A$ -ideal of  $Y$ ,  $\mu(rf(x)) \leq \mu(rm)$ , for all  $m \in Y$ . Since  $f$  is onto, for all  $m \in Y$ , there exists  $t \in X$  such that  $f(t) = m$ . This results  $\mu(f(rx)) = \mu(rf(x)) \leq \mu(rf(t)) = \mu(f(rt))$ . Thus  $f^{-1}(\mu)(rx) \leq f^{-1}(\mu)(rt)$ .

Therefore  $f^{-1}(\mu)$  is a prime fuzzy  $A$ -ideal of  $X$ .  $\square$

**Proposition 4.5.** Let  $f : X \rightarrow Y$  be an onto  $MV$ -module homomorphism. The image  $f(\mu)$  of a prime fuzzy  $A$ -ideal  $\mu$  with a sup-property is also a prime fuzzy  $A$ -ideal of  $Y$ .

*Proof.* Suppose that for all  $y \in Y$ ,  $f(\mu)(ry) > f(\mu)(y)$ . We show that

$$f(\mu)(ry) \leq f(\mu)(rs), \text{ for all } s \in Y.$$

Let  $y \in Y$  and  $x \in f^{-1}(y)$ . Hence  $f(x) = y$ , so  $rf(x) = ry$ , this results  $f(rx) = ry$ . Hence  $rx \in f^{-1}(ry)$ . By hypothesis,  $\mu(rx) = \sup_{t \in f^{-1}(ry)} \mu(t)$ . Also, we have



$f(\mu)(ry) = \sup_{t \in f^{-1}(ry)} \mu(t) = \mu(rx) \leq \mu(rm)$ , for all  $m \in X$ . We get  $\mu(rm) = \sup_{t \in f^{-1}(rs)} \mu(t) = f(\mu)(rs)$ , for all  $s = f(m) \in Y$ . It follows that  $f(\mu)(ry) \leq f(\mu)(rs)$ , for all  $s \in Y$ . Thus  $f(\mu)$  is a prime fuzzy  $A$ -ideal of  $Y$ .  $\square$

**Theorem 4.6.** *A non-empty subset  $I$  of  $M$  is a prime  $A$ -ideal if and only if the characteristic function  $\chi_I$  is a prime fuzzy  $A$ -ideal of  $M$ .*

*Proof.* Assume that  $I$  is a prime  $A$ -ideal of  $M$ . We will prove that  $\chi_I$  is a prime fuzzy  $A$ -ideal of  $M$ .

Let  $x \in M$ ,  $r \in A$  and  $\chi_I(rx) > \chi_I(x)$ . We show that

$$\chi_I(rx) \leq \chi_I(rm), \text{ for all } m \in M$$

If  $x \in I$ , then  $rx \in I$  and we have  $1 = \chi_I(rx) \not> \chi_I(x) = 1$ .

If  $x \notin I$ , then  $\chi_I(x) = 0$ , since  $\chi_I(rx) > \chi_I(x) = 0$ , we obtain  $\chi_I(rx) = 1$ , hence  $rx \in I$ . Since  $I$  is a prime  $A$ -ideal of  $M$ , we get  $rM \subseteq I$ . Hence  $rm \in I$ , for all  $m \in M$ . Thus  $\chi_I(rm) = 1$ . Therefore  $1 = \chi_I(rx) = \chi_I(rm)$ .

Conversely, assume that  $\chi_I$  is a prime fuzzy  $A$ -ideal of  $M$ . We prove that  $I$  is a prime  $A$ -ideal of  $M$ . Let  $rx \in I$  but  $x \notin I$ , for  $x \in M$ ,  $r \in A$ . Hence  $1 = \chi_I(rx) > \chi_I(x) = 0$ . By hypothesis, we get  $1 = \chi_I(rx) \leq \chi_I(rm)$ , for all  $m \in M$ . Hence  $\chi_I(rm) = 1$ , so  $rm \in I$ , for all  $m \in M$ . Thus  $rM \subseteq I$ . This results  $I$  is a prime  $A$ -ideal of  $M$ .  $\square$

Now, we describe the transfer principle [10] for prime fuzzy  $A$ -ideals in terms of level subsets:

**Theorem 4.7.** *A fuzzy subset  $\mu$  of an  $MV$ -module  $M$  is a prime fuzzy  $A$ -ideal of  $M$ , if and only if  $\mu_t = \{x \in A : \mu(x) \geq t\}$  is either empty or a prime  $A$ -ideal for every  $t \in [0, 1]$ .*

*Proof.* (i) Assume that  $\mu$  is a prime fuzzy  $A$ -ideal of  $M$ . Let  $rx \in \mu_t$ ,  $x \notin \mu_t$ , for  $r \in A$ ,  $x \in M$ . We show that  $rM \subseteq \mu_t$ .

Since  $rx \in \mu_t$  and  $x \notin \mu_t$ ,  $\mu(rx) \geq t$  and  $\mu(x) < t$ . Hence  $\mu(rx) > \mu(x)$ . By hypothesis, we conclude that  $t \leq \mu(rx) \leq \mu(rm)$ , for all  $m \in M$ . Thus  $t \leq \mu(rm)$ , for all  $m \in M$ . This results  $rm \in \mu_t$ , for all  $m \in M$ . Therefore  $rM \subseteq \mu_t$ .

Conversely, let  $\mu_t$  is a prime  $A$ -ideal of  $M$ . Suppose that  $\mu(rx) > \mu(x)$ ,  $r \in A$ ,  $x \in M$ . We show that  $\mu(rx) \leq \mu(rm)$ , for all  $m \in M$ . If not, then  $\mu(rx) > \mu(rn)$ , for some  $n \in M$ . Hence  $\mu(x) \vee \mu(rn) < \mu(rx)$ . So there exists  $t_0 \in [0, 1]$  such that  $t_0 = (\mu(x) \vee \mu(rn) + \mu(rx))/2$ . Hence  $\mu(x) \vee \mu(rn) < t_0 < \mu(rx)$ . It follows that  $rx \in \mu_{t_0}$ , for  $r \in A$ ,  $x \in M$ . Since  $\mu_{t_0}$  is a prime  $A$ -ideal of  $M$ ,  $x \in \mu_{t_0}$  or  $rM \subseteq \mu_{t_0}$ .

Hence  $\mu(x) \geq t_0$  or  $\mu(ry) \geq t_0$ , for all  $y \in M$ , since  $\mu(x) < t_0$  and  $\mu(rn) < t_0$ , they are contradictions, thus  $\mu$  is a prime fuzzy  $A$ -ideal of  $M$ .  $\square$

**Corollary 4.8.** *Let  $\mu$  be a fuzzy  $A$ -ideal of an  $MV$ -module  $M$ . The level ideal  $I = \{x \in A : \mu(x) = \mu(0)\}$  is a prime  $A$ -ideal of  $M$  if  $\mu$  is a prime fuzzy  $A$ -ideal of  $M$ .*

## 5. CONCLUSIONS

$MV$ -algebras were introduced by C. Chang [1] in 1958 in order to provide an algebraic proof for the completeness theorem of the Łukasiewicz infinite valued propositional logic.

Since then, Di Nola and Dvurecenskij in (1998) introduced the concept of *PMV*-algebras. In 2003, Di Nola. et. al. [4] introduced *MV*-modules over *PMV*-algebras. In this paper, we defined and studied fuzzy *A*-ideals and introduced the notion of a fuzzy *A*-ideal generated by a fuzzy set. We introduced the concept of prime fuzzy *A*-ideals of an *MV*-module. We described the transfer principle for prime fuzzy *A*-ideals in terms of level subsets.

Finally, we proved that the homomorphic image and preimage of prime fuzzy *A*-ideals are also prime fuzzy *A*-ideals in *MV*-modules.

**Acknowledgements.** The authors thank the referees for their valuable comments and suggestions.

#### REFERENCES

- [1] C. C. Chang, Algebraic analysis of many valued logic, Trans. Amer. Math. Soc. 88 (1958) 467–490.
- [2] R. Cignoli, I. M. L. D'Ottaviano and D. Mundici, Algebraic Foundations of Many-Valued Reasoning, Kluwer Academic, Dordrecht, Boston, London 2000.
- [3] A. Di Nola and A. Dvurecenskij, Product *MV*-algebras, Mult.-Valued Logic 6 (2001) 193–215.
- [4] A. Di Nola, P. Flondor and I. Leustean, *MV*-modules, J. Algebra 267 (2003) 21–40.
- [5] G. Dymek, Fuzzy prime ideals of Pseudo-*MV*-algebras, Soft comput. 12 (2008) 365–372.
- [6] F. Forouzesh, E. Eslami and A. Borumand Saeid, On prime *A*-ideals in *MV*-modules, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 87 (2014) 181–198.
- [7] C. S. Hoo, Fuzzy ideals of BCI and *MV*-algebras, Fuzzy Sets and Systems 62 (1994) 111–114.
- [8] A. Iorgulescu, Algebras of logic as BCK algebras, Academy of Economic Studies Bucharest, Romania 2008.
- [9] Y. B. Jun, Y. Xu and X. H. Zhang, Fuzzy filters of MTL-algebras, Inform. Sci. 175 (2005) 120–138.
- [10] M. Kondo and W. A. Dudek, On the transfer principle in fuzzy theory, Mathware. Soft Comput. 12 (2005) 4–55.
- [11] D. Piciu, Algebras of fuzzy logic, Ed. Universitaria Craiova 2007.
- [12] M. Sambasiva Rao, On fuzzy filters of BE-algebras, Ann. Fuzzy Math. Inform. 7 (2) (2014) 229–238.
- [13] A. Zadeh, Fuzzy set, Information and Control 8 (1965) 338–353.
- [14] J. Zhan, Y. Yin and Y. Xu, A new view of fuzzy ideals in rings, Ann. Fuzzy Math. Inform. 2 (1) (2011) 99–115.

FERESHTEH FOROUZESH (frouzesh@bam.ac.ir, ffrouzesh@yahoo.com)

Assistant Professor, Faculty of Mathematics and computing, Higher Education complex of Bam, Kerman, Iran